Research Article

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Error Estimates for the Numerical Approximation of Unregularized Sparse Parabolic Control Problems

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Abstract: We study the numerical approximation of a control problem governed by a semilinear parabolic problem, where the usual Tikhonov regularization term in the cost functional is replaced by a non-differentiable sparsity-promoting term.

Keywords: Optimal Control, Sparse Controls, Semilinear Parabolic Equations, Finite Element Approximation

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Dedicated to Professor Thomas Apel on his 60th birthday

1 Introduction

Let us consider a convex domain $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, with boundary Γ. We will assume that Ω is polygonal if $d = 2$ or polyhedral if $d = 3$. Given $T > 0$, we denote $I = (0, T)$, $Q = \Omega \times I$ and $\Sigma = \Gamma \times I$. In this paper, we investigate the numerical approximation of

$$\min_{u \in U_{\mathrm{ad}}} J(u) := \int_Q (y_u(x,t) - y_d(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t + \mu \|u\|_{L^1(Q)}, \tag{P}$$

where $\mu \geq 0$ and

$$U_{\mathrm{ad}} = \{u \in L^\infty(Q) : u_{\min} \leq u(x,t) \leq u_{\max} \text{ for a.a. } (x,t) \in Q\}$$

with $-\infty < u_{\min} < u_{\max} < +\infty$. For $\mu > 0$, we will further suppose that $u_{\min} < 0 < u_{\max}$.

Above, y_u denotes the state associated with the control u related by the following semilinear parabolic state equation:

$$\begin{cases} \dfrac{\partial y_u}{\partial t} + Ay_u + f(x,t,y_u) = u & \text{in } Q, \\ y_u = 0 & \text{on } \Sigma, \\ y_u(0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Assumptions on the data A, f, y_0 and the target state y_d are specified in Section 2.

The obtention of error estimates for the numerical approximation of optimal control problems that do not include a Tikhonov regularization term is a challenging problem. The only references we are aware of are [12, 21, 23, 28] and [16]. In the first four ones, problems governed by elliptic equations are studied. In [23], the equation is elliptic and linear, the variational discretization is used and the control is assumed to

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be of bang-bang type. In [21], the authors deal with a bilinear control problem governed by a semilinear elliptic equation; a structural assumption on the adjoint state is done so that error estimates can be obtained for bang-bang controls. In our paper [12], we deal with a control problem governed by a semilinear elliptic equation; a sparsity promoting term is included in the objective functional, and error estimates are obtained for the state even in the case of an optimal control with singular arcs. In [28], similar results are obtained for a Dirichlet control problem governed by a linear elliptic equation. One of the main ingredients in the obtention of error estimates for control problems governed by nonlinear equations is the use of appropriate and realistic second-order sufficient conditions for strong local minima (see Definition 1). In [16], second-order sufficient conditions for problems governed by a semilinear parabolic equation in which the controls are only functions of time are obtained before getting the error estimates. For the work at hand, we will use the second-order sufficient conditions obtained in [11].

This paper continues a series of previous works [11, 12, 14–16] where related problems are studied.

In [14, 15], the objective functional has a Tikhonov regularization term and includes a term promoting directional sparsity. Specifically, the numerical approximation of problems with objective function

$$J(u) = \frac{1}{2}\int_Q (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\kappa}{2}\int_Q u^2 \,\mathrm{d}x\,\mathrm{d}t + \mu\|u\|_{L^1(\Omega;L^2(I))}, \quad \kappa > 0,$$

is studied.

In [16], the Tikhonov regularization term is removed; no sparsity-promoting term is included yet in this work, and error estimates rely on second-order sufficient optimality conditions formulated on the cone $D^\tau_{\bar u}$ (see Section 2 below for a definition of this cone). The numerical approximation of that problem is studied, and error estimates are obtained.

In [11], it is shown that the smaller cone $C^\tau_{\bar u}$ is enough to obtain second-order sufficient optimality conditions in problems with no Tikhonov term. No numerical discretization is studied in [11].

The paper [12] deals with error estimates for a problem governed by a semilinear *elliptic* equation where the Tikhonov regularization term is replaced by a non-differentiable sparsity-promoting term. The results of this paper are obtained assuming a condition on the corresponding cone $C^\tau_{\bar u}$.

In the paper at hand, we discretize problem (P) and take advantage of the second-order conditions provided in [11] to obtain error estimates for the state variable assuming again only conditions on the smaller cone $C^\tau_{\bar u}$. The novelty with respect to [14, 15] is that we are able to drop the Tikhonov term; with respect to [16], it is that we include a sparsity-promoting term, the control variable depends on both time and space, and we use a smaller cone; and with respect to [12], it is that we deal with a parabolic equation.

2 Results about the Continuous Problem

On the partial differential equation (1.1), we make the following assumptions.

(A1) A denotes the elliptic operator

$$Ay = -\sum_{i,j=1}^{d} \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

where $a_{i,j} \in C^{0,1}(\bar\Omega)$, and satisfies the following uniform ellipticity condition:

$$\text{there exists } \lambda_A > 0 \text{ such that } \lambda_A|\xi|^2 \le \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \text{ for all } \xi \in \mathbb{R}^d \text{ and a.a. } x \in \Omega.$$

(A2) We assume that $f\colon Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable satisfying the following properties:

$$\text{there exists } C_f \in \mathbb{R} \text{ such that } \frac{\partial f}{\partial y}(x, t, y) \ge C_f \text{ for all } y \in \mathbb{R},$$

$$f(\cdot\,,\cdot\,,0) \in L^\infty(Q),$$

for all $M > 0$, there exists $C_{f,M} > 0$ such that $\left|\frac{\partial^j f}{\partial y^j}(x, t, y)\right| \leq C_{f,M}$ for all $|y| \leq M$ and $j = 1, 2$,

for all $\rho > 0$ and all $M > 0$, there exists $\varepsilon > 0$

$$\text{such that } \left|\frac{\partial^2 f}{\partial y^2}(x, t, y_1) - \frac{\partial^2 f}{\partial y^2}(x, t, y_2)\right| < \rho \text{ for all } |y_1|, |y_2| \leq M \text{ with } |y_1 - y_2| < \varepsilon,$$

for almost all $(x, t) \in Q$.

(A3) For the initial state, we suppose that $y_0 \in C^\alpha(\bar\Omega) \cap H_0^1(\Omega)$, where $C^\alpha(\bar\Omega)$ denotes the space of α-Hölder continuous functions in $\bar\Omega$ with $\alpha \in (0, 1]$.

(A4) $y_d \in L^{\hat p}(0, T; L^{\hat q}(\Omega))$ for some $\hat p, \hat q \geq 2$ with $\frac{1}{\hat p} + \frac{d}{2\hat q} < 1$.

We denote $H^{2,1}(Q) = L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$, and $C^{\beta,\beta/2}(\bar Q)$ for $0 < \beta \leq 1$ is the space of Hölder functions with exponent β in x and $\beta/2$ in t; see [27, pp. 7 and 8]. We have the following result concerning the regularity of the state.

Theorem 2.1. *Suppose assumptions* (A1)–(A3) *hold. Then, for every $u \in L^r(0, T; L^p(\Omega))$ with $\frac{1}{r} + \frac{d}{2p} < 1$ and $r, p \geq 2$, there exists a unique solution y_u of* (1.1) *in the space $H^{2,1}(Q) \cap C^{\beta,\beta/2}(\bar Q)$ for some $\beta \in (0, \alpha]$. Moreover, the following estimate holds:*

$$\|y_u\|_{C^{\beta,\beta/2}(\bar Q)} + \|y_u\|_{H^{2,1}(Q)} \leq \eta(\|u\|_{L^r(0,T;L^p(\Omega))} + M_0)$$

for a monotone non-decreasing function $\eta: [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ independent of u, and

$$M_0 = \|f(\cdot, \cdot, 0)\|_{L^\infty(Q)} + \|y_0\|_{C^\alpha(\bar\Omega)} + \|y_0\|_{H_0^1(\Omega)}.$$

Further, if $u_k \rightharpoonup u$ weakly in $L^\infty(Q)$, then the strong convergence $\|y_{u_k} - y_u\|_{C(\bar Q)} \to 0$ holds.*

The existence of a unique solution of (1.1) in the space $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ as well as the estimates in $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H_0^1(\Omega))$ were proved in [8]. The $H^{2,1}(Q)$ regularity is a well-known consequence of the convexity of Ω, the Lipschitz regularity of the coefficients $a_{i,j}$, and the $H_0^1(\Omega)$ regularity of y_0. The reader is referred to [27, Chapter III, § 10] or [24] for the Hölder regularity. The convergence $y_{u_k} \to y_u$ in $C(\bar Q)$ follows easily from the estimates for y_{u_k} in $H^{2,1}(Q) \cap C^{\beta,\beta/2}(\bar Q)$.

Notice that Theorem 2.1 implies that there exists $M_\infty > 0$ such that

$$\|y_u\|_{C^{\beta,\beta/2}(\bar Q)} + \|y_u\|_{H^{2,1}(Q)} \leq M_\infty \quad \text{for all } u \in U_{\mathrm{ad}}. \tag{2.1}$$

We denote $Y = H^{2,1}(Q) \cap C^{\beta,\beta/2}(\bar Q)$ and $G: L^\infty(Q) \to Y$ as the mapping associating to each control the corresponding state $G(u) = y_u$.

Theorem 2.2. *The mapping G is of class C^2. Moreover, for every $u, v, v_1, v_2 \in L^\infty(Q)$, we have that $z_v = G'(u)v$ is the solution of*

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \dfrac{\partial f}{\partial y}(x, t, y_u)z = v & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

and $z_{v_1,v_2} = G''(u)(v_1, v_2)$ solves the equation

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \dfrac{\partial f}{\partial y}(x, t, y_u)z = -\dfrac{\partial^2 f}{\partial y^2}(x, t, y_u)z_{v_1}z_{v_2} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } \Omega, \end{cases}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$. Moreover, z_v and z_{v_1,v_2} are continuous functions in $\bar Q$.

The proof of this result can be obtained by using the implicit function theorem; see e.g. [18, Theorem 5.1].

For any $u \in L^\infty(Q)$ and any $v \in L^1(Q)$, we will denote z_v the solution of (2.2).

Lemma 2.3. *Let $u \in U_{\mathrm{ad}}$ be arbitrary. Then there exists $C_{q,s} > 0$ independent of u such that*

$$\|z_v\|_{L^q(Q)} \le C_{q,s}\|v\|_{L^s(Q)} \quad \text{for all } v \in L^s(Q)$$

if q and s satisfy any of the following conditions:
(A) $s = 1$ *and* $q < \frac{d+2}{d}$,
(B) $s > 1 + \frac{d}{2}$ *and* $q = +\infty$,
(C) $s \in (1,2)$ *and* $q < r_d$, *where* $r_2 = \frac{2s}{2-s}$ *and* $r_3 = \frac{5s}{5-2s}$,
(D) $s = 2$ *and* $q < +\infty$ *if* $d = 2$ *or* $s = 2$ *and* $q = 10$ *if* $d = 3$.

Proof. (A) From [3, Theorem 6.3] or [7, Theorem 2.2], we know that, for $v \in L^1(Q)$, $z_v \in L^r(I; W^{1,p}(\Omega))$ for all $r, p \in [1,2)$ such that $\frac{2}{r} + \frac{d}{p} > d + 1$ and $\|z_v\|_{L^r(I;W^{1,p}(\Omega))} \le C\|v\|_{L^1(Q)}$. Using the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p = \frac{qd}{q+d}$ and setting $q = r$, we obtain the condition

$$\frac{2}{q} + \frac{d}{p} = \frac{2}{q} + \frac{q+d}{q} > d + 1,$$

which is equivalent to $q < \frac{d+2}{d}$. This proves the estimate in case (A).

(B) In this case, the estimate can be deduced from [27, Theorem III.7.1].

(C) In this case, the estimate follows by the Riesz–Thorin interpolation theorem [1, Theorem 1.1.1] between the corresponding estimates for cases (A) and (D). Indeed, for every $s \in (1,2)$, we have

$$\frac{1}{s} = \frac{\frac{2}{s} - 1}{1} + \frac{2 - \frac{2}{s}}{2} \quad \text{and} \quad \frac{2}{s} - 1 \in (0,1).$$

Hence, the estimate follows for r given by the relation

$$\frac{1}{r} = \frac{\frac{2}{s} - 1}{q} + \frac{2 - \frac{2}{s}}{p}$$

for all $q < +\infty$ if $d = 2$ and $q = 10$ if $d = 3$. This implies that $r < \frac{2s}{2-s}$ if $d = 2$ and $r < \frac{5s}{5-2s}$ if $d = 3$.

(D) This result is proved in [20, Theorem 2.4]. □

Corollary 2.4. *For all $u, v \in U_{\mathrm{ad}}$ and all $q < \frac{d+2}{d}$, there exists a constant $C_{q,1} > 0$ independent of u and v such that $\|y_u - y_v\|_{L^q(Q)} \le C_{q,1}\|u - v\|_{L^1(Q)}$.*

Proof. Denote $\zeta = y_u - y_v$. Subtracting the equations satisfied by u and v and using the mean value theorem, we get

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} + A\zeta + \dfrac{\partial f}{\partial y}(x,t,y_\theta)\zeta = u - v & \text{in } Q, \\ \zeta = 0 & \text{on } \Sigma, \\ \zeta(0) = 0 & \text{in } \Omega, \end{cases}$$

where $y_\theta = y_u + \theta(y_u - y_v)$ and $\theta\colon Q \to [0,1]$ is a measurable function. Then the result follows from Lemma 2.3 along with (A2) and (2.1). □

Let us denote

$$F(u) = \frac{1}{2}\int_Q (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad j(u) = \|u\|_{L^1(Q)}.$$

Before computing the derivatives of F, we define for every $u \in U_{\mathrm{ad}}$ its related adjoint state φ_u as the unique solution of

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial f}{\partial y}(x,t,y_u)\varphi = y_u - y_d & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(\cdot\,, T) = 0 & \text{in } \Omega, \end{cases}$$

and where A^* is the adjoint of A given by

$$A^*\psi = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{j,i}(x)\partial_{x_i}\psi).$$

From Theorem 2.1, we have that, for every $u \in U_{\mathrm{ad}}$, $\varphi_u \in H^{2,1}(Q) \cap C^{\beta,\beta/2}(Q)$ for some $\beta \in (0,1]$ and there exists a constant C independent of u such that

$$\|\varphi_u\|_{C^{\beta,\beta/2}(\bar{Q})} + \|\varphi_u\|_{H^{2,1}(Q)} \le C \quad \text{for all } u \in U_{\mathrm{ad}}. \tag{2.3}$$

The next theorem follows from the chain rule, Theorem 2.2 and assumptions (A2) and (A4).

Theorem 2.5. *The functional $F\colon L^\infty(Q) \to \mathbb{R}$ is of class C^2, and for every $u, v, v_1, v_2 \in L^\infty(Q)$,*

$$F'(u)v = \int_Q \varphi_u v \,\mathrm{d}x\,\mathrm{d}t, \quad F''(u)(v_1, v_2) = \int_Q \Big(1 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big) z_{v_1} z_{v_2} \,\mathrm{d}x\,\mathrm{d}t,$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

From (A2), (2.1), Lemma 2.3 and (2.3), we infer that the forms $F'(u)$ and $F''(u)$ can be extended through the same formulas to continuous linear and bilinear forms, respectively, in $L^2(Q)$. The following result is proved in [19, Lemma 6] for the case $\theta \equiv 1$. See [16, Lemma 3.5] for the adaptation of the proof for any measurable function $\theta\colon Q \to [0,1]$.

Lemma 2.6. *Suppose that $\bar{u}$ and u belong to U_{ad} and $\theta\colon Q \to [0,1]$ is a measurable function. Then, for every $\gamma > 0$, there exists $\varepsilon > 0$ such that*

$$|(F''(u_\theta) - F''(\bar{u}))v^2| \le \gamma \|z_v\|^2_{L^2(Q)} \quad \textit{for all } v \in L^2(Q) \textit{ if } \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon, \tag{2.4}$$

where $u_\theta = \bar{u} + \theta(u - \bar{u})$ and $z_v = G'(\bar{u})v$.

The functional $j\colon L^1(Q) \to \mathbb{R}$ is convex and Lipschitz. The directional derivative of j at u in the direction v can be computed as

$$j'(u; v) = \int_{u>0} v \,\mathrm{d}x\,\mathrm{d}t + \int_{u=0} |v| \,\mathrm{d}x\,\mathrm{d}t - \int_{u<0} v \,\mathrm{d}x\,\mathrm{d}t.$$

In what follows, we will write $J'(u; v) = F'(u)v + \mu j'(u; v)$. Moreover, denoting by $\partial j(u)$ the subdifferential of j at u in the sense of convex analysis, we have that $\lambda \in \partial j(u)$ if and only if $\lambda(x,t) \in \operatorname{sign}(u(x,t))$ for a.a. $(x,t) \in Q$, where $\operatorname{sign}(u)$ denotes the multi-valued function $\operatorname{sign}(u) = \{1\}$ if $u > 0$, $\operatorname{sign}(u) = \{-1\}$ if $u < 0$, and $\operatorname{sign}(u) = [-1, 1]$ if $u = 0$.

Existence of a solution of (P) follows in a standard way using Theorem 2.1; see e.g. [10]. Since (P) is not a convex problem, we consider local solutions as well. Let us state precisely the different concepts of local solution.

Definition 1. We say that $\bar{u} \in U_{\mathrm{ad}}$ is an $L^r(Q)$-weak local solution of (P), with $r \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

$$J(\bar{u}) \le J(u) \quad \text{for all } u \in U_{\mathrm{ad}} \text{ with } \|\bar{u} - u\|_{L^r(Q)} \le \varepsilon.$$

An element $\bar{u} \in U_{\mathrm{ad}}$ is said to be a strong local solution of (P) if there exists some $\varepsilon > 0$ such that

$$J(\bar{u}) \le J(u) \quad \text{for all } u \in U_{\mathrm{ad}} \text{ with } \|y_{\bar{u}} - y_u\|_{L^\infty(Q)} \le \varepsilon.$$

We say that $\bar{u} \in U_{\mathrm{ad}}$ is a strict (weak or strong) local solution if the above inequalities are strict for $u \ne \bar{u}$.

Next we state first-order optimality conditions; see [11, Theorem 2.9].

Theorem 2.7. *Suppose $\bar{u}$ is a local solution of* (P) *in any of the senses given in Definition 1. Then*

$$J'(\bar{u}; u - \bar{u}) \ge 0 \quad \textit{for all } u \in U_{\mathrm{ad}}$$

holds. Moreover, there exist $\bar{y}$ and $\bar{\varphi}$ in Y and $\bar{\lambda} \in \partial j(\bar{u})$ such that

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} + A\bar{y} + f(x,t,\bar{y}) = \bar{u} & \textit{in } Q,\\ \bar{y} = 0 & \textit{on } \Sigma,\\ \bar{y}(\cdot, 0) = y_0 & \textit{in } \Omega, \end{cases} \tag{2.5a}$$

$$\begin{cases} -\dfrac{\partial\bar\varphi}{\partial t} + A^*\bar\varphi + \dfrac{\partial f}{\partial y}(x,t,\bar y)\bar\varphi = \bar y - y_d & \textit{in } Q,\\ \bar\varphi = 0 & \textit{on } \Sigma,\\ \bar\varphi(\cdot\,,T) = 0 & \textit{in } \Omega,\end{cases} \tag{2.5b}$$

$$\int_Q (\bar\varphi + \mu\bar\lambda)(u - \bar u)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \textit{for all } u \in U_{\mathrm{ad}}. \tag{2.5c}$$

Corollary 2.8. *Under the assumptions of Theorem 2.7, the following properties are fulfilled:*

$$\begin{aligned} &\textit{if } \bar\varphi(x,t) > +\mu, \textit{ then } \bar u(x,t) = u_{\min},\\ &\textit{if } \bar\varphi(x,t) < -\mu, \textit{ then } \bar u(x,t) = u_{\max}.\end{aligned}$$

In addition, if $\mu > 0$, then the following properties:

$$\begin{aligned} &\textit{if } \bar\varphi(x,t) = +\mu, \textit{ then } \bar u(x,t) \leq 0,\\ &\textit{if } \bar\varphi(x,t) = -\mu, \textit{ then } \bar u(x,t) \geq 0,\\ &\textit{if } |\bar\varphi(x,t)| < \mu, \textit{ then } \bar u(x,t) = 0,\\ &\bar\lambda(x,t) = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\bar\varphi(x,t)\Big),\end{aligned}$$

and $\bar\lambda \in H^1(Q) \cap C^{\beta,\beta/2}(\bar Q)$ hold.

The reader is referred to [4] for the proof of the above result. In this reference, the result is proved for the elliptic case, the changes for the parabolic case being obvious.

Given $\bar u \in U_{\mathrm{ad}}$ satisfying (2.5a)–(2.5c), we say that $v \in L^2(Q)$ satisfies the sign condition if

$$v(x,t)\begin{cases} \geq 0 & \text{if } \bar u(x,t) = u_{\min},\\ \leq 0 & \text{if } \bar u(x,t) = u_{\max}.\end{cases} \tag{2.6}$$

We also introduce the cone

$$C_{\bar u} = \{v \in L^2(Q) \text{ satisfying (2.6) and } J'(\bar u; v) = 0\}.$$

If $\mu = 0$, we deduce from Corollary 2.8 that $\bar\varphi(x,t)v(x,t) = |\bar\varphi(x,t)v(x,t)|$ for every $v \in L^2(Q)$ satisfying the sign condition (2.6). Consequently, the following identity holds:

$$C_{\bar u} = \{v \in L^2(Q) \text{ satisfying (2.6) and } v(x,t) = 0 \text{ if } |\bar\varphi(x,t)| > 0\}.$$

For $\mu > 0$, from Corollary 2.8, we also infer that

$$C_{\bar u} = \left\{ v \in L^2(Q) \text{ satisfying (2.6) and } v(x,t)\begin{cases} \geq 0 & \text{if } \bar\varphi(x,t) = -\mu \text{ and } \bar u(x,t) = 0,\\ \leq 0 & \text{if } \bar\varphi(x,t) = +\mu \text{ and } \bar u(x,t) = 0,\\ = 0 & \text{if } \big||\bar\varphi(x,t)| - \mu\big| > 0.\end{cases}\right\}$$

We formulate the second-order necessary optimality conditions on the critical cone $C_{\bar u}$.

Theorem 2.9. *Suppose $\bar u$ is a local solution of* (P) *in any of the senses given in Definition 1. Then we have $F''(\bar u)v^2 \geq 0$ for all $v \in C_{\bar u}$.*

The proof of this theorem is exactly as the one of [5, Theorem 3.7].

To formulate second-order sufficient conditions, we define the extended cone. For $\varsigma \geq 0$, we define

$$C_{\bar u}^{\varsigma} = D_{\bar u}^{\varsigma} \cap G_{\bar u}^{\varsigma},$$

where

$$G_{\bar u}^{\varsigma} = \{v \in L^2(Q) \text{ satisfying (2.6) and } J'(\bar u; v) \leq \varsigma\|z_v\|_{L^1(Q)}\},$$

and

$$\text{if } \mu = 0,\ D^{\varsigma}_{\bar u} = \{v \in L^2(Q) \text{ satisfying (2.6) and } v(x,t) = 0 \text{ if } |\bar\varphi(x,t)| > \varsigma\},$$

$$\text{if } \mu > 0,\ D^{\varsigma}_{\bar u} = \left\{ v \in L^2(Q) \text{ satisfying (2.6) and } v(x,t) \begin{cases} \geq 0 & \text{if } \bar\varphi(x,t) = -\mu \text{ and } \bar u(x,t) = 0, \\ \leq 0 & \text{if } \bar\varphi(x,t) = +\mu \text{ and } \bar u(x,t) = 0, \\ = 0 & \text{if } \big||\bar\varphi(x,t)| - \mu\big| > \varsigma. \end{cases} \right\}$$

Notice that $C_{\bar u} = C^0_{\bar u} \subset C^{\varsigma}_{\bar u}$ for all $\varsigma > 0$. The following theorem on second-order sufficient optimality conditions was proved in [11, Theorem 3.1].

Theorem 2.10. *Let $\bar u \in U_{\mathrm{ad}}$ satisfy the first-order optimality conditions* (2.5a)–(2.5c). *Suppose in addition that there exist $\delta > 0$ and $\varsigma > 0$ such that*

$$F''(\bar u)v^2 \geq \delta \|z_v\|^2_{L^2(Q)} \quad \text{for all } v \in C^{\varsigma}_{\bar u}, \tag{2.7}$$

where $z_v = G'(\bar u)v$. Then there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$J(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(Q)} \leq J(u) \quad \text{for all } u \in U_{\mathrm{ad}} \text{ such that } \|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon. \tag{2.8}$$

Note that if $\varsigma < \varsigma'$, then $C^{\varsigma}_{\bar u} \subseteq C^{\varsigma'}_{\bar u}$, and hence, without loss of generality, we can suppose that $\varsigma < \mu$ in the case $\mu > 0$.

The purpose of this work is to obtain error estimates for the approximations described in Section 3, of the state and control variables. As we will show later, the second-order sufficient optimality conditions are enough to deduce some error estimates for the approximation of the states. Nevertheless, to get error estimates for the control, it is well known that an extra assumption is needed. To this end, we introduce the following assumption:

$$\text{there exist } \Lambda > 0 \text{ and } \gamma \in (0,1] \text{ such that } \operatorname{meas}\{(x,t) \in Q : \big||\bar\varphi(x,t)| - \mu\big| \leq \varepsilon\} < \Lambda \varepsilon^{\gamma}. \tag{H}$$

The second-order sufficient optimality condition combined with this assumption lead to a stronger inequality than (2.8). We prove two lemmas to derive this new inequality.

Lemma 2.11. *Let $\bar u \in U_{\mathrm{ad}}$ satisfy the first-order condition* (2.5a)–(2.5c) *and the structural assumption* (H). *Then*

$$F'(\bar u)(u - \bar u) + \mu j(u) - \mu j(\bar u) \geq \nu \|u - \bar u\|^{1+1/\gamma}_{L^1(Q)} \quad \text{for all } u \in U_{\mathrm{ad}} \tag{2.9}$$

holds, where $\nu = \frac{1}{2}[2(u_{\max} - u_{\min})]^{-1/\gamma}$.

Proof. The inequality $F'(\bar u)(u - \bar u) + \mu j'(\bar u; u - \bar u) \geq \nu\|u - \bar u\|^{1+1/\gamma}_{L^1(Q)}$ was proved in [30, Lemma 6.3]. Then it is enough to use that $j(u) - j(\bar u) \geq j'(\bar u; u - \bar u)$ to obtain (2.9). □

Lemma 2.12. *Let $\bar u \in U_{\mathrm{ad}}$ satisfy* (2.5a)–(2.5c), *and suppose that there exist $\varsigma > 0$ and $\delta > 0$ such that the second-order condition* (2.7) *holds. Then there exists $\kappa > 0$ such that, for all $\rho > 0$, a number $\varepsilon_\rho > 0$ can be found such that*

$$\rho[F'(\bar u)(u - \bar u) + \mu j(u) - \mu j(\bar u)] + F''(u_\theta)(u - \bar u)^2 \geq \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(Q)}$$

for all measurable functions $\theta\colon Q \to [0,1]$ and all $u \in U_{\mathrm{ad}}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon_\rho$, and $u_\theta = \bar u + \theta(u - \bar u)$.

Proof. We split the proof in two steps.

Step 1. Consider a measurable function $\theta\colon Q \to [0,1]$. Let us prove that there exists a constant $C > 0$ such that, for all $\rho > 0$, a number $\varepsilon_\rho > 0$ can be found so that

$$\rho J'(\bar u; u - \bar u) + F''(u_\theta)(u - \bar u)^2 \geq C\|z_{u-\bar u}\|^2_{L^2(Q)} \tag{2.10}$$

for all $u \in U_{\mathrm{ad}}$ satisfying $\|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon_\rho$.

As in [11, Theorem 3.1], we distinguish three cases.

Case 1: $u - \bar{u} \in C^{\varsigma}_{\bar{u}}$. On one hand, since $\bar{u}$ satisfies (2.5a)–(2.5c) and $u \in U_{\rm ad}$, we have that $J'(\bar{u}; u - \bar{u}) \geq 0$. Given $\gamma = \frac{\delta}{2}$, from Lemma 2.6, we deduce the existence of $\varepsilon_0 > 0$ such that, if $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0$, we have

$$|(F''(u_\theta) - F''(\bar{u}))v^2| \leq \frac{\delta}{2}\|z_v\|^2_{L^2(Q)} \quad \text{for all } v \in L^2(Q).$$

This inequality and the second-order condition (2.7) imply

$$F''(u_\theta)v^2 \geq \frac{\delta}{2}\|z_v\|^2_{L^2(Q)} \quad \text{for all } v \in L^2(Q).$$

Therefore, (2.10) follows with $C = \frac{\delta}{2}$.

Case 2: $u - \bar{u} \notin G^{\varsigma}_{\bar{u}}$. Arguing as in [11, (3.8)], we know that there exists $\varepsilon_1 > 0$ such that, if $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon_1$, we have

$$J'(\bar{u}; u - \bar{u}) \geq \frac{\varsigma}{2\varepsilon_1}\|z_{u-\bar{u}}\|^2_{L^2(Q)}.$$

Using [11, Remark 2.6], we have that there exists $M > 0$ such that

$$F''(u_\theta)(u - \bar{u})^2 \leq M\|z_{u-\bar{u}}\|^2_{L^2(Q)}.$$

Selecting $\varepsilon_1 < \frac{\rho\varsigma}{2M+\delta}$, (2.10) holds again with $C = \frac{\delta}{2}$.

Case 3: $u - \bar{u} \notin D^{\varsigma}_{\bar{u}}$ and $u - \bar{u} \in G^{\varsigma}_{\bar{u}}$. Let us define $\varsigma^* = \varsigma/\max\{1, C_{1,1}\}$, where $C_{1,1}$ is defined in Lemma 2.3. If $u - \bar{u} \notin G^{\varsigma^*}_{\bar{u}}$, then Case 2 applies. Otherwise, we define

$$\begin{aligned}
\text{if } \mu = 0, \quad & W = \{(x, t) \in Q : |\bar{\varphi}(x, t)| > \varsigma \text{ and } u(x, t) - \bar{u}(x, t) \neq 0\},\\
\text{if } \mu > 0, \quad & W = \{(x, t) \in Q : \bar{\varphi}(x, t) = -\mu \text{ and } \bar{u}(x, t) = 0 \text{ and } u(x, t) < 0,\\
& \qquad \text{or } \bar{\varphi}(x, t) = +\mu \text{ and } \bar{u}(x, t) = 0 \text{ and } u(x, t) > 0,\\
& \qquad \text{or } \big||\bar{\varphi}(x, t)| - \mu\big| > \varsigma \text{ and } u(x, t) \neq \bar{u}(x, t)\},
\end{aligned}$$

and denote $V = Q \setminus W$. Define the functions $v = (u - \bar{u})\chi_V$ and $w = (u - \bar{u})\chi_W$. It is proved in [17, Proposition 3.6] that

$$J'(\bar{u}; u - \bar{u}) \geq \varsigma\|w\|_{L^1(Q)} \tag{2.11}$$

and also that $v \in C^{\varsigma}_{\bar{u}}$. Using now that $u - \bar{u} = v + w$, (2.4) and (2.7), we deduce the existence of a constant $c > 0$ such that

$$F''(u_\theta)(u - \bar{u})^2 \geq \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(Q)} - c\|z_w\|^2_{L^2(Q)}. \tag{2.12}$$

In [13], it is proved that there exists $\varepsilon > 0$ such that, for all $\varepsilon_2 \in (0, \varepsilon)$,

$$\text{if } \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_2, \text{ then } \|z_w\|_{L^\infty(Q)} < 2\varepsilon_2.$$

This yields

$$\|z_w\|^2_{L^2(Q)} \leq 2\varepsilon_2\|z_w\|_{L^1(Q)} \leq 2\varepsilon_2 C_{1,1}\|w\|_{L^1(Q)}.$$

Using (2.11), (2.12) and this last inequality, we get, for $\varepsilon_2 < \min\{\varepsilon, \frac{\rho\varsigma}{2C_{1,1}c}\}$,

$$\begin{aligned}
\rho J'(\bar{u}; u - \bar{u}) + F''(u_\theta)(u - \bar{u})^2 &\geq \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(Q)} - c\|z_w\|^2_{L^2(Q)} + \rho\varsigma\|w\|_{L^1(Q)}\\
&\geq \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(Q)} + \Big(\frac{\rho\varsigma}{2\varepsilon_2}C_{1,1} - c\Big)\|z_w\|^2_{L^2(Q)} \geq \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(Q)}.
\end{aligned}$$

Thus, taking $\varepsilon_\rho = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ and $C = \frac{\delta}{8}$, (2.10) holds in all cases.

Step 2. Arguing as in [11, Lemma 2.4], we obtain the existence of $\varepsilon_3 > 0$ such that

$$\|z_{u-\bar{u}}\|_{L^2(Q)} \geq \frac{1}{2}\|y_u - \bar{y}\|_{L^2(Q)} \text{ if } \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_3.$$

Therefore, (2.10) implies that there exists some constant $C > 0$ such that

$$\rho J'(\bar{u}; u-\bar{u}) + F''(u_\theta)(u-\bar{u})^2 \ge C\|y_u - \bar{y}\|^2_{L^2(Q)}$$

if $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_\rho = \min_{0\le i\le 3} \varepsilon_i$. Finally, using the convexity of j, we know that

$$F'(\bar{u})(u-\bar{u}) + \mu j(u) - \mu j(\bar{u}) \ge J'(\bar{u}; u-\bar{u}),$$

and the proof is complete. □

Finally, we obtain sufficient conditions for a strong local solution that will allow us to obtain control error estimates.

Theorem 2.13. *Let $\bar{u} \in U_{\mathrm{ad}}$ be a control satisfying the first-order optimality conditions* (2.5a)–(2.5c) *and the structural assumption* (H). *Suppose further that the second-order condition* (2.7) *is fulfilled for some $\varsigma > 0$ and $\delta > 0$. Then there exists $\varepsilon > 0$ such that the following inequality holds:*

$$J(\bar{u}) + \frac{\nu}{2}\|u-\bar{u}\|^{1+1/\gamma}_{L^1(Q)} + \frac{\kappa}{4}\|y_u - \bar{y}\|^2_{L^2(Q)} \le J(u) \quad \text{for all } u \in U_{\mathrm{ad}} \text{ such that } \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon,$$

where ν is the constant given in Lemma 2.11 and κ is the constant given in Lemma 2.12.

Proof. Performing a Taylor expansion and invoking Lemma 2.11, we infer

$$\begin{aligned} J(u) = F(u) + \mu j(u) &= F(\bar{u}) + \mu j(\bar{u}) + F'(\bar{u})(u-\bar{u}) + \mu j(u) - \mu j(\bar{u}) + \frac{1}{2}F''(u_\theta)(u-\bar{u})^2 \\ &= J(\bar{u}) + \frac{1}{2}[F'(\bar{u})(u-\bar{u}) + \mu j(u) - \mu j(\bar{u})] + \frac{1}{2}[F'(\bar{u})(u-\bar{u}) + \mu j(u) - \mu j(\bar{u}) + F''(u_\theta)(u-\bar{u})^2] \\ &\ge J(\bar{u}) + \frac{\nu}{2}\|u-\bar{u}\|^{1+1/\gamma}_{L^1(Q)} + \frac{1}{2}[F'(\bar{u})(u-\bar{u}) + \mu j(u) - \mu j(\bar{u}) + F''(u_\theta)(u-\bar{u})^2]. \end{aligned}$$

The result now follows from Lemma 2.12, taking $\rho = 1$. □

3 Numerical Approximation

Consider a regular family of triangulations $\{\mathbb{K}_h\}_{h>0}$ of $\bar{\Omega}$, cf. [2, Definition 4.4.13, pp. 107, 108] or [22, p. 131], and a set of points $\{t_j\}_{j=0}^{N_\tau} \subset [0, T]$ with $0 = t_0 < t_1 < \cdots < t_{N_\tau} = T$. We will denote by N_h and $N_{I,h}$ the number of nodes and interior nodes of the triangulation $\mathbb{K}_h$, h_K is the diameter of K for all $K \in \mathbb{K}_h$ and $h = \max h_k$. As usual, $\{e_i\}_{i=1}^{N_h}$ denotes the nodal basis associated with the nodes $\{x_i\}_{i=1}^{N_h}$ of the triangulation, and $\{e_i\}_{i=1}^{N_{I,h}}$ is the nodal basis associated to the interior nodes. We also set $I_j = (t_{j-1}, t_j)$, $\tau_j = t_j - t_{j-1}$, and $\tau = \max_{1\le j\le N_\tau} \tau_j$. We denote χ_K and χ_j the characteristic functions of the element K and the interval I_j, respectively. We also use the notation $\sigma = (h, \tau)$.

We further assume that $\tau \le \frac{T}{4}$ and there exist $\nu > 0$ and $\rho > 0$ independent of h and τ, respectively, such that

$$h \le \nu h_K \text{ for all } K \in \mathbb{K}_h \quad \text{and} \quad \tau \le \rho\tau_j \text{ for all } j = 1, \ldots, N_\tau.$$

Finally, following [29], we will also suppose that
(A5) $\tau|C_f| < 1$ if $C_f < 0$, where C_f is given in (A2);
(A6) there exist $C > 0$ and $\theta > 0$ independent of τ and h such that $\tau \le Ch^\theta$.
Notice that, in particular, the hypotheses on the temporal mesh in [29, Assumptions 3.1 and 5.2] are satisfied.

3.1 Approximation of the State Equation

Now we define the finite-dimensional spaces

$$\begin{aligned} Y_h &= \{y_h \in C(\bar{\Omega}) : y_{h|K} \in P_1(K) \text{ for all } K \in \mathbb{K}_h \text{ and } y_h = 0 \text{ on } \Gamma\}, \\ \mathcal{Y}_\sigma &= \{y_\sigma \in L^2(I; Y_h) : y_{\sigma|I_j} \in Y_h \text{ for all } j = 1, \ldots, N_\tau\}. \end{aligned}$$

The elements of $\mathcal{Y}_\sigma$ can be written as

$$y_\sigma = \sum_{j=1}^{N_\tau} y_{h,j}\chi_j = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_{I,h}} y_{i,j}e_i\chi_j,$$

where $y_{h,j} \in Y_h$ for $j = 1, \dots, N_\tau$, $y_{i,j} \in \mathbb{R}$ for $i = 1, \dots, N_{I,h}$, and $j = 1, \dots, N_\tau$.

We denote by $I_h\colon C_0(\bar{\Omega}) \to Y_h$ the usual continuous piecewise linear interpolation operator given by $I_h y = \sum_{j=1}^{N_{I,h}} y(x_j)e_j$. For every $w \in L^1(I)$, we define

$$P_\tau w = \sum_{j=1}^{N_\tau} \frac{1}{\tau_j}\int_{I_j} w \,\mathrm{d}t\,\chi_j.$$

Finally, we set $I_\sigma = P_\tau \circ I_h = I_h \circ P_\tau$.

For every $u \in L^2(Q)$, we define its associated discrete state as the unique element $y_\sigma(u) \in \mathcal{Y}_\sigma$ such that, for $j = 1, \dots, N_\tau$,

$$\begin{cases} \displaystyle\int_\Omega (y_{h,j} - y_{h,j-1})z_h \,\mathrm{d}x + \tau_j \mathrm{a}(y_{h,j}, z_h) \\ \qquad\qquad \displaystyle + \int_{I_j}\int_\Omega f(x,t,y_{h,j})z_h \,\mathrm{d}x\,\mathrm{d}t = \int_{I_j}\int_\Omega u z_h \,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } z_h \in Y_h, \\ y_{h,0} = P_h y_0, \end{cases} \tag{3.1}$$

where $P_h\colon L^2(\Omega) \to Y_h$ denotes the L^2 projection operator, and $\mathrm{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$\mathrm{a}(y,z) = \int_\Omega \sum_{i,k=1}^{n} a_{ik}\partial_{x_i} y \partial_{x_k} z \,\mathrm{d}x \quad \text{for all } y, z \in H^1(\Omega).$$

Notice that this can be seen as an implicit Euler scheme; see [29]. Existence and uniqueness of a solution of (3.1) is deduced by a straightforward application of Brouwer's fixed-point theorem along with the assumption $\tau|C_f| < 1$. The following result follows from [29, Theorem 5.4], [9, Theorem 3.1], and the fact that U_{ad} is bounded in $L^\infty(Q)$.

Lemma 3.1. *Consider $u \in U_{\mathrm{ad}}$. Under the assumptions* (A1), (A2), *and* (A4), *there exist $h_0 > 0$, $\tau_0 > 0$, $\delta_0 > 0$, $C > 0$, independent of u such that, for every $\tau < \tau_0$ and $h < h_0$,*

$$\|y_\sigma(u)\|_{L^\infty(Q)} \le C, \tag{3.2}$$

$$\|y_u - y_\sigma(u)\|_{L^2(Q)} \le C(h^2 + \tau), \tag{3.3}$$

$$\|y_u - y_\sigma(u)\|_{L^\infty(Q)} \le C|\log h| \log\Big(\frac{T}{\tau}\Big)^2 (h^\beta + \tau^{\beta/2}). \tag{3.4}$$

Proof. The stability estimate (3.2) is proved in [29, Theorem 5.4] and the error estimate (3.3) in [29, Corollary 6.2]. To establish the third inequality, we use [29, Theorem 6.5] to deduce

$$\|y_u - y_\sigma(u)\|_{L^\infty(Q)} \le C|\log h|\Big(\log\frac{T}{\tau}\Big)^2 \|y_u - \eta_\sigma\|_{L^\infty(Q)} \quad \text{for all } \eta_\sigma \in \mathcal{Y}_\sigma.$$

By choosing $\eta_\sigma = I_\sigma y_u$ and taking advantage of the Hölder regularity of y_u, we obtain the claimed estimate. □

3.2 Approximation of the Control Problem

We will consider piecewise constant approximations of the control as follows:

$$U_h = \{u_h \in L^\infty(\Omega) : u_{h|_K} \in \mathcal{P}_0(K) \text{ for all } K \in \mathbb{K}_h\},$$

$$\mathbb{U}_\sigma = \Big\{u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j : u_{h,j} \in U_h \text{ for } j = 1, \dots, N_\tau\Big\},$$

$$\mathbb{U}_{\sigma,\mathrm{ad}} = \{u_\sigma \in \mathbb{U}_\sigma : u_{\min} \le u_\sigma(x,t) \le u_{\max} \text{ for a.e. } (x,t) \in Q\}.$$

Notice that $\mathbb{U}_{\sigma,\text{ad}} \subset U_{\text{ad}}$ and every element $u_\sigma \in \mathbb{U}_\sigma$ can be written in the form

$$u_\sigma = \sum_{j=1}^{N_\tau} u_{h,j}\chi_j = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} u_{K,j}\chi_K\chi_j.$$

We define $\Pi_h : L^1(\Omega) \to U_h$ by

$$\Pi_h v = \sum_{K\in\mathbb{K}_h} \frac{1}{|K|} \int_K v \,\mathrm{d}x\, \chi_K.$$

We also define $\pi_\sigma = P_\tau \circ \Pi_h = \Pi_h \circ P_\tau$.

The discrete problem reads

$$\min_{u_\sigma\in\mathbb{U}_{\sigma,\text{ad}}} J_\sigma(u_\sigma), \tag{P$_\sigma$}$$

where

$$J_\sigma(u) = F_\sigma(u) + \mu j(u) \quad \text{with } F_\sigma(u) = \frac{1}{2}\int_Q (y_\sigma(u) - y_d)^2 \,\mathrm{d}x.$$

We observe that

$$j(u_\sigma) = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \tau_j |K| |u_{K,j}|.$$

In the next three lemmas, we collect and prove some properties that will be needed later.

Lemma 3.2. *For every $u \in L^1(Q)$ and every σ, the inequality $j(\pi_\sigma u) \le j(u)$ holds.*

Proof. Notice that

$$\pi_\sigma u = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \frac{1}{\tau_j|K|} \int_{I_j}\int_K u \,\mathrm{d}x\,\mathrm{d}t\, \chi_K\chi_j = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} u_{K,j}\chi_K\chi_j.$$

Then we have

$$j(\pi_\sigma u) = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \tau_j|K||u_{K,j}| = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \left|\int_{I_j}\int_K u \,\mathrm{d}x\,\mathrm{d}t\right| \le \int_Q |u| \,\mathrm{d}x\,\mathrm{d}t = j(u). \qquad \square$$

Lemma 3.3. *The following properties are fulfilled:*

$$\lim_{\sigma\to(0,0)} \|\pi_\sigma u - u\|_{L^1(Q)} = 0 \text{ and } \lim_{\sigma\to(0,0)} j(\pi_\sigma u) = j(u) \quad \text{for all } u \in L^1(Q), \tag{3.5}$$

$$\pi_\sigma u \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(Q) \text{ as } \sigma \to (0,0) \quad \text{for all } u \in L^\infty(Q), \tag{3.6}$$

$$\|\pi_\sigma u - u\|_{H^1(Q)^*} \le C(h+\tau)\|u\|_{L^2(Q)} \quad \text{for all } u \in L^2(Q). \tag{3.7}$$

Proof. To prove (3.5), we assume first that $u \in C(\bar{Q})$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u(x_2,t_2) - u(x_1,t_1)| < \frac{\varepsilon}{|Q|} \quad \text{if } |x_2 - x_1| + |t_2 - t_1| < \delta.$$

Taking $\hat{\sigma} = (\hat{h}, \hat{\tau})$ with $\hat{h} + \hat{\tau} \le \delta$, we infer, for $\sigma = (h,\tau)$ with $|\sigma| \le |\hat{\sigma}|$,

$$\|u - \pi_\sigma u\|_{L^1(Q)} = \sum_{j=1}^{N_\tau} \sum_{K\in\mathbb{K}_h} \int_{I_j}\int_K \left| u(x,t) - \frac{1}{\tau_j|K|}\int_{I_j}\int_K u(\xi,s)\,\mathrm{d}\xi\,\mathrm{d}s \right| \mathrm{d}x\,\mathrm{d}t \le \varepsilon.$$

This proves the first identity of (3.5) for continuous functions in $\bar{Q}$. The proof for arbitrary elements of $L^1(Q)$ follows from the density of $C(\bar{Q})$ in $L^1(Q)$ and the stability property established in Lemma 3.2.

Using that $|j(u) - j(\pi_\sigma u)| \le \|u - \pi_\sigma u\|_{L^1(Q)}$, the second convergence of (3.5) is obtained. To prove (3.6), we observe that $\|\pi_\sigma u\|_{L^\infty(Q)} \le \|u\|_{L^\infty(Q)}$. Therefore, the existence of subsequences converging weakly* in $L^\infty(Q)$ follows. But (3.5) implies that all the subsequences must converge to u, and consequently, (3.6) is satisfied.

The third statement follows by duality. Using the well-known inequality $\|\pi_\sigma v - v\|_{L^2(Q)} \le C(h+\tau)\|v\|_{H^1(Q)}$ and denoting $S = \{v \in H^1(Q) : \|v\|_{H^1(Q)} = 1\}$, we have

$$\begin{aligned}
\|u - \pi_\sigma u\|_{H^1(Q)^*} &= \sup_{v\in S}\langle u - \pi_\sigma u, v\rangle_{H^1(Q)^*,H^1(Q)} \\
&= \sup_{v\in S}\int_Q (u-\pi_\sigma u)v \,dx\,dt = \sup_{v\in S}\int_Q (u-\pi_\sigma u)(v-\pi_\sigma v)\,dx\,dt \\
&= \sup_{v\in S}\int_Q u(v-\pi_\sigma v)\,dx\,dt \le \sup_{v\in S}\|u\|_{L^2(Q)}\|v-\pi_\sigma v\|_{L^2(Q)} \\
&\le \sup_{v\in S} C(h+\tau)\|v\|_{H^1(Q)}\|u\|_{L^2(Q)} = C(h+\tau)\|u\|_{L^2(Q)}. \qquad \square
\end{aligned}$$

Lemma 3.4. *Let u be an element of U_{ad} and let $\{u_\sigma\}_\sigma$ be a sequence with each $u_\sigma \in \mathbb{U}_{\sigma,\mathrm{ad}}$ such that $u_\sigma \stackrel{*}{\rightharpoonup} u$ in $L^\infty(Q)$. Then the following convergence properties are fulfilled:*

$$\lim_{\sigma\to(0,0)}\|y_\sigma(u_\sigma) - y_u\|_{L^\infty(Q)} = 0, \tag{3.8}$$

$$\lim_{\sigma\to(0,0)}\|y_\sigma(\pi_\sigma u) - y_\sigma(u_\sigma)\|_{L^\infty(Q)} = 0, \tag{3.9}$$

$$\lim_{\sigma\to(0,0)} F_\sigma(u_\sigma) = F(u), \tag{3.10}$$

$$j(u) \le \liminf_{\sigma\to(0,0)} j(u_\sigma). \tag{3.11}$$

Proof. By the triangle inequality, we can write

$$\|y_\sigma(u_\sigma) - y_u\|_{L^\infty(Q)} \le \|y_\sigma(u_\sigma) - y_{u_\sigma}\|_{L^\infty(Q)} + \|y_{u_\sigma} - y_u\|_{L^\infty(Q)}.$$

The first term converges to 0 thanks to the finite element error estimate (3.4). The convergence to 0 of the second term can be deduced from Theorem 2.1 and the assumption $u_\sigma \stackrel{*}{\rightharpoonup} u$.

For the second statement, we use

$$\|y_\sigma(\pi_\sigma u) - y_\sigma(u_\sigma)\|_{L^\infty(Q)} \le \|y_\sigma(\pi_\sigma u) - y_{\pi_\sigma u}\|_{L^\infty(Q)} + \|y_{\pi_\sigma u} - y_u\|_{L^\infty(Q)} + \|y_u - y_\sigma(u_\sigma)\|_{L^\infty(Q)}.$$

The first terms converge to 0 thanks to the finite element error estimate (3.4). The convergence to 0 of the second term is consequence of Theorem 2.1 and (3.6). The convergence to 0 of the last term follows from (3.8).

The convergence (3.10) is a straightforward consequence of (3.8). Finally, (3.11) follows from the convexity of j. $\square$

3.3 First-Order Optimality Conditions and Sparsity Properties of the Discrete Control Problem

For every $u \in L^2(Q)$, we define its associated discrete adjoint state $\varphi_\sigma(u) \in \mathcal{Y}_\sigma$ such that, for $j = N_\tau, \dots, 1$,

$$\begin{cases}
\displaystyle\int_\Omega (\varphi_{h,j} - \varphi_{h,j+1}) z_h \,dx + \tau_j \mathrm{a}(z_h, \varphi_{h,j}) + \int_{I_j}\int_\Omega \frac{\partial f}{\partial y}(x,t,y_{h,j}(u))\varphi_{h,j} z_h \,dx\,dt \\
\qquad\displaystyle = \int_{I_j}\int_\Omega (y_{h,j}(u) - y_d) z_h \,dx\,dt \quad \text{for all } z_h \in Y_h, \\
\varphi_{h,N_\tau+1} = 0.
\end{cases}$$

The next identity is well known:

$$F'_\sigma(u_\sigma)v_\sigma = \int_Q \varphi_\sigma(u_\sigma)v_\sigma \,dx\,dt \quad \text{for all } u_\sigma, v_\sigma \in \mathbb{U}_\sigma.$$

Let us describe now the subdifferential of j in $\mathbb{U}_\sigma$. Consider $\lambda_\sigma = \sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\lambda_{K,j}\chi_K\chi_j \in \mathbb{U}_\sigma$. We have that

$$\lambda_\sigma \in \partial j(u_\sigma) \iff \lambda_{K,j} \in \operatorname{sign}(u_{K,j}) \quad \text{for all } K \in \mathbb{K}_h \text{ and all } j = 1,\dots,N_\tau. \tag{3.12}$$

In the next lemma, we state the first-order necessary optimality conditions for problem (P_σ).

Lemma 3.5. *Suppose that $\bar u_\sigma$ is a local solution of (P_σ). Then the following inequality holds:*

$$F'_\sigma(\bar u_\sigma)(u_\sigma - \bar u_\sigma) + \mu[j(u_\sigma) - j(\bar u_\sigma)] \geq J'_\sigma(\bar u_\sigma; u_\sigma - \bar u_\sigma) \geq 0 \quad \text{for all } u_\sigma \in \mathbb{U}_{\sigma,\mathrm{ad}}. \tag{3.13}$$

Further, there exists $\bar\lambda_\sigma \in \partial j_\sigma(\bar u_\sigma)$ such that

$$\int_Q (\bar\varphi_\sigma + \mu\bar\lambda_\sigma)(u_\sigma - \bar u_\sigma)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \text{for all } u_\sigma \in \mathbb{U}_{\sigma,\mathrm{ad}}, \tag{3.14}$$

where $\bar\varphi_\sigma = \varphi_\sigma(\bar u_\sigma)$.

The proof of this result is the same as the corresponding one for the continuous problem and can be found, e.g., in [11, Theorem 2.9]. The first inequality in (3.13) follows from the convexity of j.

Let us denote $\bar\phi_\sigma = \pi_\sigma\bar\varphi_\sigma = \sum_{j=1}^{N_\tau}\sum_{K\in\mathbb{K}_h}\bar\phi_{K,j}\chi_K\chi_j$. From (3.12) and (3.14), we have that

$$(\bar\phi_{K,j} + \mu\bar\lambda_{K,j})(s - \bar u_{K,j}) \geq 0 \quad \text{for all } s \in [u_{\min}, u_{\max}].$$

Using these inequalities, we obtain the sparsity properties of the discrete solution. The proof is as the corresponding one for the continuous problem, which can be found in [4].

Corollary 3.6. *Let $\bar u_\sigma$ and $\bar\lambda_\sigma$ as in Lemma 3.5; then we have, for every $j = 1,\dots,N_\tau$ and $K \in \mathbb{K}_h$,*

$$\begin{aligned} &\text{if } \bar\phi_{K,j} > +\mu, \text{ then } \bar u_{K,j} = u_{\min},\\ &\text{if } \bar\phi_{K,j} < -\mu, \text{ then } \bar u_{K,j} = u_{\max}. \end{aligned}$$

In addition, if $\mu > 0$, then the following properties hold:

$$\text{if } |\bar\phi_{K,j}| < \mu, \text{ then } \bar u_{K,j} = 0, \quad \bar\lambda_{K,j} = \operatorname{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\bar\phi_{K,j}\Big).$$

4 Convergence and Error Estimates

Theorem 4.1. *The following statements hold.*

(A) *Let $\{\bar u_\sigma\}_\sigma$ be a sequence of global solutions of (P_σ). Then every weak-* limit $\bar u$ of a subsequence of $\{\bar u_\sigma\}_\sigma$ is a global solution of (P).*

(B) *Let $\bar u$ be a strict strong local solution of (P) in the sense of Definition 1. Then there exists a sequence $\{\bar u_\sigma\}_\sigma$ of local minimizers of (P_σ) such that $\bar u_\sigma \overset{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$. Moreover, there exists σ_0 such that*

$$J_\sigma(\bar u_\sigma) \leq J_\sigma(\pi_\sigma\bar u) \quad \text{for all } |\sigma| \leq |\sigma_0|. \tag{4.1}$$

Proof. (A) Take any $u \in U_{\mathrm{ad}}$, and consider $\pi_\sigma u \in \mathbb{U}_{\sigma,\mathrm{ad}}$. Using (3.11) and (3.10), the optimality of $\bar u_\sigma$, and (3.10) and (3.5), we get

$$J(\bar u) \leq \liminf_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma) \leq \limsup_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma) \leq \limsup_{\sigma\to(0,0)} J_\sigma(\pi_\sigma u) = J(u),$$

which proves the first statement.

(B) Since $\bar u$ is a strict local solution of (P), there exists $\varepsilon > 0$ such that

$$J(\bar u) < J(u) \quad \text{for all } u \in U_{\mathrm{ad}} \text{ with } \|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon,\ u \neq \bar u, \tag{4.2}$$

where $\bar y = y_{\bar u}$. Consider the auxiliary problems

$$\min_{u_\sigma \in V_{\sigma,\mathrm{ad},\varepsilon}} J_\sigma(u_\sigma), \tag{P$_{\sigma,\varepsilon}$}$$

where

$$V_{\sigma,\mathrm{ad},\varepsilon} = \{u_\sigma \in \mathbb{U}_{\sigma,\mathrm{ad}} : \|y_\sigma(u_\sigma) - \bar y\|_{L^\infty(Q)} \le \varepsilon\}.$$

From (3.6) and (3.8), we deduce the existence of σ_0 such that $\pi_\sigma \bar u \in V_{\sigma,\mathrm{ad},\varepsilon}$ for $|\sigma| < |\sigma_0|$. Therefore, $V_{\sigma,\mathrm{ad},\varepsilon}$ is a nonempty compact set for every $|\sigma| < |\sigma_0|$. Since J_σ is a continuous function, (P$_{\sigma,\varepsilon}$) has a solution $\bar u_\sigma$ and (4.1) holds.

Since $\{\bar u_\sigma\}_\sigma$ is bounded in $L^\infty(Q)$, we can extract a subsequence, still indexed by σ, such that $\bar u_\sigma \stackrel{*}{\rightharpoonup} u^*$. Since $\mathbb{U}_{\sigma,\mathrm{ad}} \subset U_{\mathrm{ad}}$ and U_{ad} is weakly⋆ closed in $L^\infty(Q)$, we have that $u^* \in U_{\mathrm{ad}}$.

Using that $\bar u_\sigma \in V_{\sigma,\mathrm{ad},\varepsilon}$ and (3.9), we obtain

$$\|y_{u^*} - \bar y\|_{L^\infty(Q)} \le \|y_{u^*} - \bar y_\sigma\|_{L^\infty(Q)} + \|\bar y_\sigma - \bar y\|_{L^\infty(Q)} \le \|y_{u^*} - \bar y_\sigma\|_{L^\infty(Q)} + \varepsilon \to \varepsilon,$$

and hence $\|y_{u^*} - \bar y\|_{L^\infty(Q)} \le \varepsilon$. Using (3.11) and (3.10), the local optimality of $\bar u_\sigma$, and (3.10) and (3.5), we get

$$J(u^*) \le \liminf_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma) \le \limsup_{\sigma\to(0,0)} J_\sigma(\pi_\sigma \bar u) = J(\bar u).$$

From (4.2), we deduce that $u^* = \bar u$, and the second statement is proved. □

4.1 State Error Estimates

In the next result, we obtain error estimates for the optimal state when the solution satisfies a second-order sufficient optimality condition.

Theorem 4.2. *Let $\bar u$ be a local solution of* (P) *satisfying the second-order condition* (2.7), *and let $\{\bar u_\sigma\}$ be a sequence of local solutions of* (P$_\sigma$) *such that $\bar u_\sigma \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$ and* (4.1) *holds. Then there exists a constant $C > 0$ independent of σ such that*

$$\|\bar y_\sigma - \bar y\|_{L^2(Q)} \le C\sqrt{h + \tau},$$

where $\bar y = y_{\bar u}$ and $\bar y_\sigma = y_\sigma(\bar u_\sigma)$.

Remark 1. From Theorems 2.10 and 4.1 (B), we deduce the existence of a sequence $\{\bar u_\sigma\}$ satisfying the assumptions of Theorem 4.2. If every (P$_\sigma$) has only a local solution in a neighborhood of $\bar u$ for $|\sigma| \le |\sigma_0|$, then these solutions satisfy (4.1). Although existence of an infinite number of solutions of (P$_\sigma$) in every neighborhood of $\bar u$ cannot be discarded, it is a rather pathological case.

Proof of Theorem 4.2. By the triangle inequality, we have that

$$\|\bar y_\sigma - \bar y\|_{L^2(Q)} \le \|\bar y_\sigma - y_{\bar u_\sigma}\|_{L^2(Q)} + \|\bar y_{\bar u_\sigma} - \bar y\|_{L^2(Q)}.$$

Since $\{\bar u_\sigma\}_\sigma$ is bounded in $L^\infty(Q)$, then we can apply (3.3) to the first term to have

$$\|\bar y_\sigma - y_{\bar u_\sigma}\|_{L^2(Q)} \le C(h^2 + \tau). \tag{4.3}$$

For the second term, we first notice that the weak-⋆ convergence of $\bar u_\sigma$ to $\bar u$ and the last statement of Theorem 2.1 imply that $\|\bar y_{\bar u_\sigma} - \bar y\|_{L^\infty(Q)} \to 0$. Therefore, for $|\sigma|$ small enough, $\|\bar y_{\bar u_\sigma} - \bar y\|_{L^\infty(Q)} < \varepsilon$, where ε is the one given in Theorem 2.10. Using (2.8), we have that

$$\frac{\kappa}{2}\|\bar y_{\bar u_\sigma} - \bar y\|^2_{L^2(Q)} \le J(\bar u_\sigma) - J(\bar u).$$

From Lemma 3.2, we deduce that

$$\begin{aligned} J(\bar u_\sigma) - J(\bar u) &= \big(J(\bar u_\sigma) - J_\sigma(\bar u_\sigma)\big) + \big(J_\sigma(\bar u_\sigma) - J_\sigma(\pi_\sigma \bar u)\big) + \big(J_\sigma(\pi_\sigma \bar u) - J(\pi_\sigma \bar u)\big) + \big(J(\pi_\sigma \bar u) - J(\bar u)\big) \\ &\le \big(F(\bar u_\sigma) - F_\sigma(\bar u_\sigma)\big) + \big(J_\sigma(\bar u_\sigma) - J_\sigma(\pi_\sigma \bar u)\big) + \big(F_\sigma(\pi_\sigma \bar u) - F(\pi_\sigma \bar u)\big) + \big(F(\pi_\sigma \bar u) - F(\bar u)\big) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

From assumption (4.1), we infer that $\mathrm{II} = J_\sigma(\bar u_\sigma) - J_\sigma(\pi_\sigma \bar u) \le 0$. Using the $L^\infty(Q)$ bounds (2.1) and (3.2), and the finite element error estimate (3.3), we get

$$\begin{aligned} F(\bar u_\sigma) - F_\sigma(\bar u_\sigma) &= \frac{1}{2}\int_Q \big((y_{\bar u_\sigma} - y_d)^2 - (\bar y_\sigma - y_d)^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\int_Q (y_{\bar u_\sigma} - \bar y_\sigma)(y_{\bar u_\sigma} + \bar y_\sigma + 2y_d)\,\mathrm{d}x\,\mathrm{d}t \le C\|y_{\bar u_\sigma} - \bar y_\sigma\|_{L^2(Q)} \le C(h^2 + \tau). \end{aligned}$$

Analogously, we obtain $F_\sigma(\pi_\sigma \bar u) - F(\pi_\sigma \bar u) \le C\|y_\sigma(\pi_\sigma \bar u) - y_{\pi_\sigma \bar u}\|_{L^2(Q)} \le C(h^2 + \tau)$. To estimate the last term, we use the mean value theorem to deduce the existence of $u_\theta = \bar u + \theta(\pi_\sigma \bar u - \bar u)$, where θ is a measurable function such that $0 \le \theta(x, t) \le 1$ for a.e. $(x, t) \in Q$, Theorem 2.5, the regularity of the adjoint state, (2.3), and (3.7) to obtain

$$F(\pi_\sigma \bar u) - F(\bar u) = F'(u_\theta)(\pi_\sigma \bar u - \bar u) = \int_Q \varphi_{u_\theta}(\pi_\sigma \bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t \le \|\varphi_{u_\theta}\|_{H^1(Q)}\|\pi_\sigma \bar u - \bar u\|_{H^1(Q)^*} \le C(\tau + h).$$

Gathering all the estimates, we obtain the desired result. □

4.2 Bang-Off-Bang Controls and Control Error Estimates

The goal of this section is to obtain error estimates for $\|\bar u - \bar u_\sigma\|_{L^1(Q)}$. We recall that $\beta \in (0, 1]$ is the exponent introduced in Theorem 2.1 and $\gamma \in (0, 1]$ is given in the structural assumption (H).

Lemma 4.3. *Let $\bar u \in U_{\mathrm{ad}}$ satisfy the first-order conditions* (2.5a)–(2.5c) *and the structural assumption* (H). *Then, for all $s \ge 1$, there exists $C > 0$ independent of σ, s, and γ such that*

$$\|\bar u - \pi_\sigma \bar u\|_{L^s(Q)} \le C(h^\beta + \tau^{\beta/2})^{\gamma/s}, \tag{4.4}$$

$$|F'(\bar u)(\pi_\sigma \bar u - \bar u) + \mu j(\pi_\sigma \bar u) - \mu j(\bar u)| \le C(h^\beta + \tau^{\beta/2})^{1+\gamma}. \tag{4.5}$$

Proof. We will write the proof for the case $\mu > 0$. For $\mu = 0$, the proof follows the same sketch, but in a slightly simplified way; see [12, Lemma 7].

Denote $\mathbb{I}_\tau = \{I_j,\ j = 1, \dots, N_\tau\}$, and consider an element $K \in \mathbb{K}_h$ and an interval $I \in \mathbb{I}_\tau$ where $|\bar\varphi(x, t)| - \mu$ changes sign, i.e.,

$$\text{there exist } (x^-, t^-),\ (x^o, t^o), (x^+, t^+) \in K \times I \text{ such that } \begin{cases} |\bar\varphi(x^-, t^-)| - \mu < 0, \\ |\bar\varphi(x^o, t^o)| - \mu = 0, \\ |\bar\varphi(x^+, t^+)| - \mu > 0. \end{cases} \tag{4.6}$$

Denote $\mathbb{S}_\sigma = \{(K, I) \in \mathbb{K}_h \times \mathbb{I}_\tau : \text{satisfying (4.6)}\}$. If $(x, t) \in K \times I$, with $(K, I) \in \mathbb{S}_\sigma$, then

$$\big||\bar\varphi(x, t)| - \mu\big| = \big||\bar\varphi(x, t)| - \mu - (|\bar\varphi(x^o, t^o)| - \mu)\big| = \big||\bar\varphi(x, t)| - |\bar\varphi(x^o, t^o)|\big| \le M(h^\beta + \tau^{\beta/2}), \tag{4.7}$$

where $M = \max\{1, \|\bar\varphi\|_{C^{\beta,\beta/2}(\bar Q)}\}$. Denote $S = \bigcup\{K \times I : (K, I) \in \mathbb{S}_\sigma\}$. We have proved that

$$S \subset \big\{(x, t) \in Q : \big||\bar\varphi(x, t)| - \mu\big| \le M(h^\beta + \tau^{\beta/2})\big\}.$$

By the structural assumption (H), we have that $\operatorname{meas} S \le \Lambda M^\gamma (h^\beta + \tau^{\beta/2})^\gamma \le \Lambda M (h^\beta + \tau^{\beta/2})^\gamma$.

Now consider a pair $(K, I) \in (\mathbb{K}_h \times \mathbb{I}_\tau) \setminus \mathbb{S}_\sigma$. From Corollary 2.8, we have that $\bar u(x, t)$ is constant in $K \times I$, so $\pi_\sigma \bar u \equiv \bar u$ in $K \times I$. This implies that

$$\|\bar u - \pi_\sigma \bar u\|_{L^s(Q)} = \|\bar u - \pi_\sigma \bar u\|_{L^s(S)} \le (u_{\max} - u_{\min})(\Lambda M)^{1/s}(h^\beta + \tau^{\beta/2})^{\gamma/s},$$

and (4.4) follows for $C = (u_{\max} - u_{\min})\max\{\Lambda M, 1\}$.

To prove the last statement, we use that, for h, τ small enough so that

$$2M(h^\beta + \tau^{\beta/2}) < \frac{\mu}{2},$$

we have that, for every $(x,t) \in S$, $|\bar\varphi(x,t)| > \frac{\mu}{2}$. Since $\bar\varphi \in C(\bar Q)$, we have that either $\bar\varphi > \frac{\mu}{2}$ or $\bar\varphi < -\frac{\mu}{2}$ on every set $K \times I$ such that $(K, I) \in \mathbb{S}_\sigma$. From Corollary 2.8, we deduce that, in the first case, $\bar u \le 0$ and, in the second case, $\bar u \ge 0$. This property is shared by $\pi_\sigma \bar u$. Using these properties on the signs in S, the fact that $\bar u = \pi_\sigma \bar u$ in $Q \setminus S$, (4.7), and (4.4), we infer

$$\begin{aligned}
&|F'(\bar u)(\pi_\sigma\bar u - \bar u) + \mu j(\pi_\sigma \bar u) - \mu j(\bar u)|\\
&= \Big|\iint_Q \bar\varphi(\pi_\sigma\bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t + \mu\int_Q |\pi_\sigma\bar u|\,\mathrm{d}x\,\mathrm{d}t - \mu\int_Q |\bar u|\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&= \Big|\iint_S \bar\varphi(\pi_\sigma\bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t + \mu\int_S |\pi_\sigma\bar u|\,\mathrm{d}x\,\mathrm{d}t - \mu\int_S |\bar u|\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&= \Big|\int_{S\cap\{\bar\varphi>0\}} \bar\varphi(\pi_\sigma\bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t - \mu\int_{S\cap\{\bar\varphi>0\}} \pi_\sigma\bar u\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{S\cap\{\bar\varphi>0\}} \bar u\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_{S\cap\{\bar\varphi<0\}} \bar\varphi(\pi_\sigma\bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{S\cap\{\bar\varphi<0\}} \pi_\sigma\bar u\,\mathrm{d}x\,\mathrm{d}t - \mu\int_{S\cap\{\bar\varphi<0\}} \bar u\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&= \Big|\int_{S\cap\{\bar\varphi>0\}} (\bar\varphi - \mu)(\pi_\sigma\bar u - \bar u)\,\mathrm{d}x\,\mathrm{d}t + \int_{S\cap\{\bar\varphi<0\}} (|\bar\varphi| - \mu)(\bar u - \pi_\sigma\bar u)\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\le \||\bar\varphi| - \mu\|_{L^\infty(S)}\|\bar u - \pi_\sigma\bar u\|_{L^1(S)} \le M(h^\beta + \tau^{\beta/2})C(h^\beta + \tau^{\beta/2})^\gamma.
\end{aligned}$$

Thus, (4.5) is satisfied. □

Lemma 4.4. *For every $u \in U_{\mathrm{ad}}$, there exists a constant $C > 0$, independent of u, such that*

$$\|\varphi_u - \varphi_\sigma(u)\|_{L^\infty(Q)} \le C|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2}).$$

Proof. By the triangle inequality, we have that

$$\|\varphi_u - \varphi_\sigma(u)\|_{L^\infty(Q)} \le \|\varphi_u - \varphi^\sigma\|_{L^\infty(Q)} + \|\varphi^\sigma - \varphi_\sigma(u)\|_{L^\infty(Q)}, \tag{4.8}$$

where $\varphi^\sigma \in C^{\beta,\beta/2}(\bar Q)$ is the solution of

$$\begin{cases} -\dfrac{\partial\varphi^\sigma}{\partial t} + A^*\varphi^\sigma + \dfrac{\partial f}{\partial y}(x,t,y_\sigma(u))\varphi^\sigma = y_\sigma(u) - y_d & \text{in } Q,\\ \varphi^\sigma = 0 & \text{on } \Sigma,\\ \varphi^\sigma(\cdot\,,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.9}$$

For later use, we notice that assumption (A2), estimate (3.2), and Lemma 2.3 imply that there exists $C^* > 0$ independent of u and σ such that $\|\varphi^\sigma\|_{L^\infty(Q)} \le C^*$. Let us denote $\xi^\sigma = \varphi_u - \varphi^\sigma$. We have that

$$\begin{cases} -\dfrac{\partial\xi^\sigma}{\partial t} + A^*\xi^\sigma + \dfrac{\partial f}{\partial y}(x,t,y_u)\xi^\sigma = \Big(\dfrac{\partial f}{\partial y}(x,t,y_\sigma(u)) - \dfrac{\partial f}{\partial y}(x,t,y_u)\Big)\varphi^\sigma + y_u - y_\sigma(u) & \text{in } Q,\\ \xi^\sigma = 0 & \text{on } \Sigma,\\ \xi^\sigma(\cdot\,,T) = 0 & \text{in } \Omega. \end{cases}$$

By the mean value theorem, we know that there exists a measurable function $\theta\colon Q \to [0,1]$ such that, denoting $y_{\sigma,\theta} = y_\sigma(u) + \theta(y_u - y_\sigma(u))$, we have that

$$\begin{cases} -\dfrac{\partial\xi^\sigma}{\partial t} + A^*\xi^\sigma + \dfrac{\partial f}{\partial y}(x,t,y_u)\xi^\sigma = \Big(\dfrac{\partial f^2}{\partial y^2}(x,t,y_{\sigma,\theta})\varphi^\sigma + 1\Big)(y_u - y_\sigma(u)) & \text{in } Q,\\ \xi^\sigma = 0 & \text{on } \Sigma,\\ \xi^\sigma(\cdot\,,T) = 0 & \text{in } \Omega. \end{cases}$$

From (2.1) and (3.2), we know that there exists a constant $M_\infty > 0$ such that $\|y_{\sigma,\theta}\|_{L^\infty(Q)} \le M_\infty$. Using Lemma 2.3, assumption (A2), the estimate $\|\varphi^\sigma\|_{L^\infty(Q)} \le C^*$, and the finite element error estimate (3.4), we have

$$\|\varphi_u - \varphi^\sigma\|_{L^\infty(Q)} \le C\|y_u - y_\sigma(u)\|_{L^\infty(Q)} \le C|\log h| \log\Big(\frac{T}{\tau}\Big)^2 (h^\beta + \tau^{\beta/2}).$$

To estimate the second term in the right-hand side of (4.8), we have that, from [29, Theorem 6.5],

$$\|\varphi^\sigma - \varphi_\sigma(u)\|_{L^\infty(Q)} \le C|\log h| \Big(\log\frac{T}{\tau}\Big)^2 \|\varphi^\sigma - \eta_\sigma\|_{L^\infty(Q)} \quad \text{for all } \eta_\sigma \in \mathcal{Y}_\sigma.$$

By choosing $\eta_\sigma = I_\sigma \varphi^\sigma$ and taking advantage of the Hölder regularity of φ^σ, we obtain the claimed estimate. □

Theorem 4.5. *Let $\bar u$ be a local solution of* (P) *satisfying the structural assumption* (H) *and the second-order condition* (2.7). *Let $\{\bar u_\sigma\}_\sigma$ be a sequence of local solutions of* (P_σ) *such that* (4.1) *holds and $\bar u_\sigma \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$. Then there exists σ_0 such that the following holds.*

If $\gamma < 1$ or $d > 1$, there exists a constant $C_s > 0$ independent of σ such that, for all $|\sigma| < |\sigma_0|$,

$$\|\bar u_\sigma - \bar u\|_{L^1(Q)} \le C_s (h^\beta + \tau^{\beta/2})^{\frac{\gamma^2}{s}},$$
$$\|\bar y_\sigma - \bar y\|_{L^2(Q)} \le C_s (h^\beta + \tau^{\beta/2})^{\frac{\gamma(\gamma+1)}{2s}}$$

for $s = 1$ if $d = 1$, for all $s > 1$ if $d = 2$, and all $s > \frac{5}{4}$ if $d = 3$.

If $d = 1$ and $\gamma = 1$, there exists a constant $C > 0$ independent of σ such that

$$\|\bar u_\sigma - \bar u\|_{L^1(Q)} + \|\bar y_\sigma - \bar y\|_{L^2(Q)} \le C|\log h| \log\Big(\frac{T}{\tau}\Big)^2 (h^\beta + \tau^{\beta/2}) \quad \textit{for all } |\sigma| < |\sigma_0|. \tag{4.10}$$

Proof. First, we use the enhanced first-order optimality condition in Lemma 2.11 and the first-order optimality condition for the discrete problem (3.13) to obtain

$$\begin{aligned}
&\frac{\nu}{2}\|\bar u_\sigma - \bar u\|_{L^1(Q)}^{1+1/\gamma} + \frac{1}{2}[F'(\bar u)(\bar u_\sigma - \bar u) + \mu j(\bar u_\sigma) - \mu j(\bar u)]\\
&\quad\le F'(\bar u)(\bar u_\sigma - \bar u) + \mu j(\bar u_\sigma) - \mu j(\bar u)\\
&\quad\le [F'(\bar u)(\bar u_\sigma - \bar u) + \mu j(\bar u_\sigma) - \mu j(\bar u)] + [F'_\sigma(\bar u_\sigma)(\pi_\sigma \bar u - \bar u_\sigma) + \mu j(\pi_\sigma \bar u) - \mu j(\bar u_\sigma)]\\
&\quad\le [F'(\bar u) - F'_\sigma(\bar u_\sigma)](\bar u_\sigma - \pi_\sigma \bar u) + [F'(\bar u)(\pi_\sigma \bar u - \bar u) + \mu j(\pi_\sigma \bar u) - \mu j(\bar u)].
\end{aligned}$$

Using this and the second estimate in Lemma 4.3, we have that

$$\begin{aligned}
&\frac{\nu}{2}\|\bar u_\sigma - \bar u\|_{L^1(Q)}^{1+1/\gamma} + \frac{1}{2}[F'(\bar u)(\bar u_\sigma - \bar u) + \mu j(\bar u_\sigma) - \mu j(\bar u)]\\
&\quad\le [F'(\bar u) - F'_\sigma(\bar u_\sigma)](\bar u_\sigma - \pi_\sigma \bar u) + C(h^\beta + \tau^{\beta/2})^{1+\gamma}\\
&\quad= [F'(\bar u) - F'(\bar u_\sigma)](\bar u_\sigma - \pi_\sigma \bar u) + [F'(\bar u_\sigma) - F'_\sigma(\bar u_\sigma)](\bar u_\sigma - \pi_\sigma \bar u) + C(h^\beta + \tau^{\beta/2})^{1+\gamma}\\
&\quad= \mathrm{I} + \mathrm{II} + C(h^\beta + \tau^{\beta/2})^{1+\gamma}.
\end{aligned}\tag{4.11}$$

Let us estimate I:

$$\begin{aligned}
I &= [F'(\bar u) - F'(\bar u_\sigma)](\bar u_\sigma - \bar u) + [F'(\bar u) - F'(\bar u_\sigma)](\bar u - \pi_\sigma \bar u)\\
&= -F''(u_\theta)(\bar u_\sigma - \bar u)^2 + \int_Q (\bar\varphi - \varphi_{\bar u_\sigma})(\bar u - \pi_\sigma \bar u)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

If $d = 1$, we deduce from Corollary 2.4 and Lemma 2.3 that

$$\|\bar\varphi - \varphi_{\bar u_\sigma}\|_{L^\infty(Q)} \le C_{\infty,2}\|\bar y - y_{\bar u_\sigma}\|_{L^2(Q)} \le C_{\infty,2} C_{2,1}\|\bar u - \bar u_\sigma\|_{L^1(Q)}.$$

Using (4.4) and Young's inequality, we infer the existence of $C > 0$ such that

$$\begin{aligned}
\int_Q (\bar\varphi - \varphi_{\bar u_\sigma})(\bar u - \pi_\sigma \bar u)\,\mathrm{d}x\,\mathrm{d}t &\le \|\bar\varphi - \varphi_{\bar u_\sigma}\|_{L^\infty(Q)}\|\bar u - \pi_\sigma \bar u\|_{L^1(Q)}\\
&\le C\|\bar u - \bar u_\sigma\|_{L^1(Q)} C(h^\beta + \tau^{\beta/2})^\gamma\\
&\le \frac{\nu}{8}\|\bar u - \bar u_\sigma\|_{L^1(Q)}^{1+1/\gamma} + C(h^\beta + \tau^{\beta/2})^{\gamma(\gamma+1)}.
\end{aligned}$$

If $d = 2$ or $d = 3$, we proceed as follows. Using Corollary 2.4, the same technique as in the proof of Lemma 4.4, and Lemma 2.3, we obtain

$$\|\bar\varphi - \varphi_{\bar u_\sigma}\|_{L^r(Q)} \le C_{r,q}\|\bar y - y_{\bar u_\sigma}\|_{L^q(Q)} \le C_{r,q}C_{q,1}\|\bar u - \bar u_\sigma\|_{L^1(Q)}$$

for all $q < \frac{d+2}{d}$ and for all $r < r_d$, where $r_d = \frac{q}{2-q}$ if $d = 2$ and $r_d = \frac{5q}{5-2q}$ if $d = 3$; see Lemma 2.3. We observe that $r_d \to +\infty$ if $d = 2$ and $r_d \nearrow 5$ if $d = 3$ as $q \nearrow \frac{d+2}{d}$. Therefore, taking $s = \frac{r}{r-1}$, using (4.4) and Young's inequality, we have, for all $s > 1$ if $d = 2$ and $s > \frac{5}{4}$ if $d = 3$,

$$\begin{aligned}\int_Q(\bar\varphi - \varphi_{\bar u_\sigma})(\bar u - \pi_\sigma\bar u)\,\mathrm{d}x\,\mathrm{d}t &\le \|\bar\varphi - \varphi_{\bar u_\sigma}\|_{L^r(Q)}\|\bar u - \pi_\sigma\bar u\|_{L^s(Q)}\\ &\le C_{r,q}C_{q,1}\|\bar u - \bar u_\sigma\|_{L^1(Q)}C(h^\beta + \tau^{\beta/2})^{\gamma/s}\\ &\le \frac{\nu}{8}\|\bar u - \bar u_\sigma\|^{1+1/\gamma}_{L^1(Q)} + C_s(h^\beta + \tau^{\beta/2})^{\gamma(\gamma+1)/s},\end{aligned}$$

where C_s depends on s. We have proved the following inequality for I:

$$\mathrm{I} \le -F''(u_\theta)(\bar u_\sigma - \bar u)^2 + \frac{\nu}{8}\|\bar u - \bar u_\sigma\|^{1+1/\gamma}_{L^1(Q)} + C_s(h^\beta + \tau^{\beta/2})^{\gamma(\gamma+1)/s} \tag{4.12}$$

for $s = 1$ if $d = 1$, for all $s > 1$ if $d = 2$, and all $s > \frac{5}{4}$ if $d = 3$. To estimate II, we use Lemma 4.4, (4.4), and Young's inequality as follows:

$$\begin{aligned}\mathrm{II} &= [F'(\bar u_\sigma) - F'_\sigma(\bar u_\sigma)](\bar u_\sigma - \pi_\sigma\bar u) = \int_Q(\varphi_{\bar u_\sigma} - \bar\varphi_\sigma)(\bar u_\sigma - \pi_\sigma\bar u)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2})(\|\bar u_\sigma - \bar u\|_{L^1(Q)} + \|\bar u - \pi_\sigma\bar u\|_{L^1(Q)})\\ &\le C\Big(|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2})\Big)^{\gamma+1} + \frac{\nu}{8}\|\bar u_\sigma - \bar u\|^{1+1/\gamma}_{L^1(Q)} + C|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2})^{\gamma+1}.\end{aligned} \tag{4.13}$$

Gathering estimates (4.11), (4.12), (4.13), and taking into account only the lowest-order terms, we have that

$$\begin{aligned}&\frac{\nu}{4}\|\bar u_\sigma - \bar u\|^{1+1/\gamma}_{L^1(Q)} + \frac{1}{2}[F'(\bar u)(\bar u_\sigma - \bar u) + \mu j(\bar u_\sigma) - \mu j(\bar u)] + F''(u_\theta)(\bar u_\sigma - \bar u)^2\\ &\quad\le C_s(h^\beta + \tau^{\beta/2})^{\gamma(\gamma+1)/s} + C(|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2}))^{\gamma+1}\end{aligned}$$

for $s = 1$ if $d = 1$, for all $s > 1$ if $d = 2$, and all $s > \frac{5}{4}$ if $d = 3$.

Now we use Lemma 2.12 for $\rho = \frac{1}{2}$. The weak-$\star$ convergence of $\bar u_\sigma$ to $\bar u$ implies the strong convergence in $L^\infty(Q)$ of $y_{\bar u_\sigma}$ to $\bar y$, so there exists σ_0 such that $\|y_{\bar u_\sigma} - \bar y\|_{L^\infty(Q)} < \varepsilon_\rho$ for $|\sigma| < |\sigma_0|$. Hence, the above inequality yields

$$\begin{aligned}&\frac{\nu}{4}\|\bar u_\sigma - \bar u\|^{1+1/\gamma}_{L^1(Q)} + \frac{\kappa}{2}\|y_{\bar u_\sigma} - \bar y\|^2_{L^2(Q)}\\ &\quad\le C_s(h^\beta + \tau^{\beta/2})^{\gamma(\gamma+1)/s} + C\Big(|\log h|\log\Big(\frac{T}{\tau}\Big)^2(h^\beta + \tau^{\beta/2})\Big)^{\gamma+1}\quad\text{for all } |\sigma| < |\sigma_0|.\end{aligned}$$

Finally, combining this estimate and (4.3), the result follows. □

5 Numerical Example

We modify slightly the example presented in [6, Remark 2.11]. As in that reference, we define $\Omega = (0, 1) \subset \mathbb{R}$, $T = 1$, $\mu = 4 \times 10^{-3}$, $u_{\min} = -10$, $u_{\max} = 20$, $y_0 \equiv 0$, and

$$y_d(x, t) = \exp(20[(x - 0.2)^2 + (t - 0.2)^2]) + \exp(20[(x - 0.7)^2 + (t - 0.9)^2]).$$

We further take the nonlinearity $f(x, t, y) = |y|y^3$.

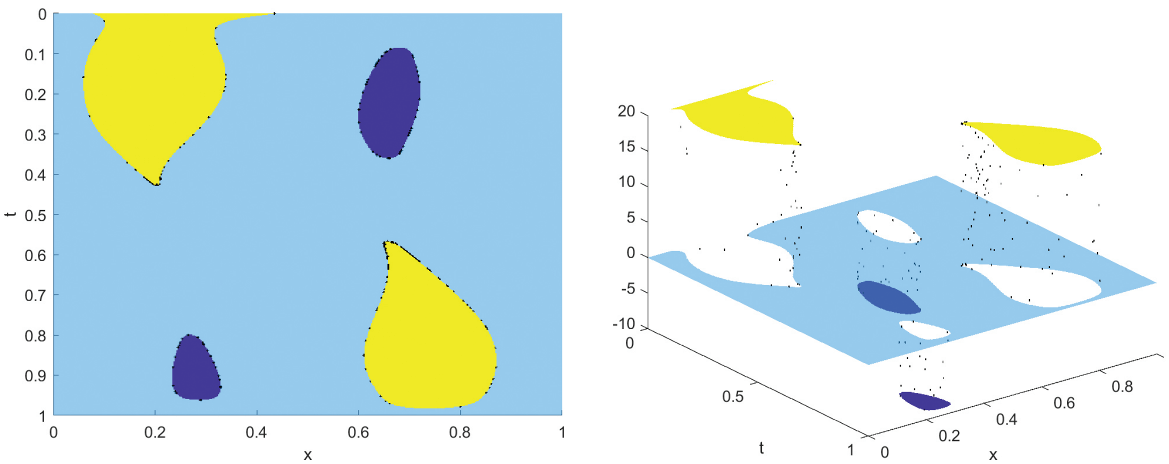

Figure 1: The optimal control exhibits a typical bang-off-bang behavior. Black dots have been marked in the few space-time patches where the numerical approximation exhibits a singular behavior.

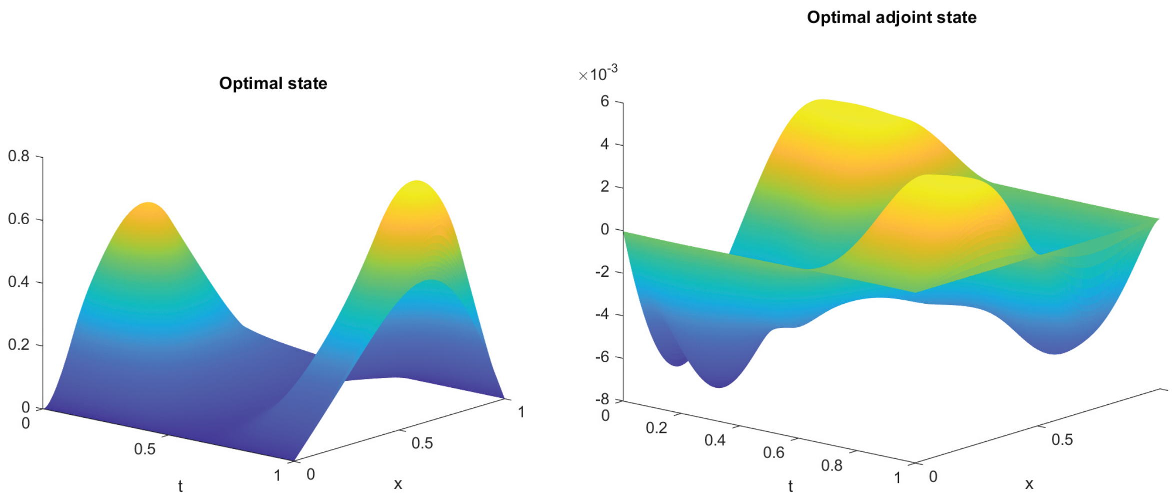

Figure 2: Optimal state and adjoint state.

To discretize the problem, we use meshes uniform in both time and space of size $\sigma_{i,j} = (h_i, \tau_j)$, where $h_i = 2^{-i}$ and $\tau_j = 2^{-j}$. The problem is solved via a Tikhonov regularization approach: for a decreasing sequence of values of $\kappa > 0$, we look for a solution of

$$\min_{u_\sigma \in \mathbb{U}_{\sigma,\mathrm{ad}}} J_{\sigma,\kappa}(u_\sigma) = J_\sigma(u_\sigma) + \frac{\kappa}{2} \int_Q u_\sigma^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Each of these problems is solved using a semismooth Newton method, globalized with the help of a merit function (see [26]), taking as initial guess an approximation of the solution for the previous value of κ. We stop when we find the same solution for three consecutive values of κ. Since we do not have the reference solution, we compare with the solution for $\sigma_{I,I}$ for a big enough index I.

For the final solution, $I = 9$, $\kappa \approx 3.5 \times 10^{-8}$, and approximately 99.9 % of the components of $\bar{u}_{\sigma_{I,I}}$ belong to $\{u_{\min}, 0, u_{\max}\}$; see Figure 1. In Figure 2, the optimal state and adjoint state are shown.

Three tests are carried out. In the first test, we take $h_i = \tau_i$, $i = 5, 6, 7$; in the second one, we take a fixed fine discretization in time given by τ_I, $I = 9$, and solve for h_i, $i = 5, 6, 7$; finally, we fix the discretization parameter in space to h_I, $I = 9$, and solve for τ_i, $i = 5, 6, 7$. For the first test, we measure the experimental order of convergence (EOC) between two consecutive simultaneous refinement levels by setting

$$\mathrm{EOC}_{y,i} = \log_2\Big(\frac{e_{y,i-1}}{e_{y,i}}\Big), \quad \mathrm{EOC}_{u,i} = \log_2\Big(\frac{e_{u,i-1}}{e_{u,i}}\Big),$$

$h_i = \tau_i$	$\|\bar y - \bar y_{\sigma_{i,i}}\|_{L^2(Q)}$	$\mathrm{EOC}_{y,i}$	$\|\bar u - \bar u_{\sigma_{i,i}}\|_{L^1(Q)}$	$\mathrm{EOC}_{u,i}$
2^{-5}	2.52E−2	—	4.93E−1	—
2^{-6}	1.25E−2	1.0	2.40E−1	1.0
2^{-7}	5.85E−3	1.1	1.19E−1	1.0
		1.1		1.0

Table 1: Experimental order of convergence. Simultaneous refinement.

h_i	$\|\bar y - \bar y_{\sigma_{i,I}}\|_{L^2(Q)}$	$\mathrm{EOC}_{y,i}$	$\|\bar u - \bar u_{\sigma_{i,I}}\|_{L^1(Q)}$	$\mathrm{EOC}_{u,i}$
2^{-5}	3.30E−3	—	3.29E−1	—
2^{-6}	1.39E−3	1.2	1.90E−1	0.8
2^{-7}	5.86E−4	1.2	8.30E−2	1.2
		1.2		1.0

Table 2: Experimental order of convergence. Refinement in space for $\tau_I = 2^{-9}$.

τ_i	$\|\bar y - \bar y_{\sigma_{I,i}}\|_{L^2(Q)}$	$\mathrm{EOC}_{y,i}$	$\|\bar u - \bar u_{\sigma_{I,i}}\|_{L^1(Q)}$	$\mathrm{EOC}_{u,i}$
2^{-5}	2.49E−2	—	3.50E−1	—
2^{-6}	1.25E−2	1.0	1.64E−1	1.1
2^{-7}	5.82E−3	1.1	7.46E−2	1.1
		1.0		1.1

Table 3: Experimental order of convergence. Refinement in time for $h_I = 2^{-9}$.

where

$$e_{y,i} = \|\bar y - \bar y_{\sigma_{i,i}}\|_{L^2(Q)} \approx \|\bar y_{\sigma_{I,I}} - \bar y_{\sigma_{i,i}}\|_{L^2(Q)}, \quad e_{u,i} = \|\bar u - \bar u_{\sigma_{i,i}}\|_{L^1(Q)} \approx \|\bar u_{\sigma_{I,I}} - \bar u_{\sigma_{i,i}}\|_{L^1(Q)}.$$

Analogous notation is used for the refinements in space and in time, respectively.

We obtain the results summarized in Table 1 for simultaneous refinement, Table 2 for refinement in space, and Table 3 for refinement in time. The observed experimental order of convergence is $O(h + \tau)$. Since the problem is set in dimension $d = 1$ and numerically it seems that assumption (H) holds for $\gamma = 1$, the order of convergence expected from estimate (4.10) should be, nevertheless, at most close to $O(h + \tau^{1/2})$. In this example, the observed order of convergence in τ can be explained using the same technique of proof and taking into account the regularity of the optimal solution. Let us see how.

Using Theorem 2.1 and assumption (A2), we have that, for all $u \in U_{\mathrm{ad}}$, $\partial_t y_u + Ay_u \in L^\infty(Q)$. Since $y_0 = 0$, using maximal parabolic regularity, see e.g. [25, Theorem 5.3], we have that

$$y_u \in W^{1,p}(0, T, L^p(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega)) \quad \text{for all } p < +\infty.$$

Since the embedding $W^{1,p}(Q) \hookrightarrow C^{1-2/p}(\bar Q)$ holds, then $y_u \in C^\beta(\bar Q)$ for all $\beta < 1$. Using the same argument and observing that $y_d \in L^\infty(Q)$, we also deduce that $\varphi_u \in C^\beta(\bar Q)$ for all $\beta < 1$. Having this regularity for the state and the adjoint state, the proof of estimate (4.10) can be rewritten to obtain

$$\|\bar u_\sigma - \bar u\|_{L^1(Q)} + \|\bar y_\sigma - \bar y\|_{L^2(Q)} \le C_\beta |\log h| \log\Big(\frac{T}{\tau}\Big)^2 (h + \tau)^\beta \quad \text{for all } |\sigma| < |\sigma_0|.$$

To obtain this estimate, we notice that the term $h^\beta + \tau^{\beta/2}$ in estimate (4.10) comes from Lemma 4.3 and the finite element error estimate in the $L^\infty(Q)$ norm for the state variable, estimate (3.4), and the adjoint state, Lemma 4.4.

In Lemma 4.3, the factor $h^\beta + \tau^{\beta/2}$ appears in estimate (4.7). Using the $C^\beta(\bar Q)$-regularity of the optimal adjoint state, this estimate can be replaced by

$$\big||\bar\varphi(x,t)| - \mu\big| = \big||\bar\varphi(x,t)| - \mu - (|\bar\varphi(x^o,t^o)| - \mu)\big| = \big||\bar\varphi(x,t)| - |\bar\varphi(x^o,t^o)|\big| \le M_\beta (h + \tau)^\beta,$$

where $M_\beta = \max\{1, \|\bar\varphi\|_{C^\beta(\bar Q)}\}$.

In Lemma 4.4 and estimate (3.4), the term $h^\beta + \tau^{\beta/2}$ corresponds, respectively, to the approximation errors $\|y_u - I_\sigma y_u\|_{L^\infty(Q)}$ and $\|\varphi^\sigma - I_\sigma\varphi^\sigma\|_{L^\infty(Q)}$, where φ^σ is the solution of equation (4.9). Taking into account the $C^\beta(\bar{Q})$-regularity of both y_u and φ^σ, we have that

$$\|y_u - I_\sigma y_u\|_{L^\infty(Q)} + \|\varphi^\sigma - I_\sigma\varphi^\sigma\|_{L^\infty(Q)} \leq C_\beta(h+\tau)^\beta.$$

So, in this setting, the proof of Theorem 4.5 can be repeated verbatim replacing in all places $h^\beta + \tau^{\beta/2}$ by $(h+\tau)^\beta$, and the result follows.

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