GLOBAL SOLUTIONS FOR A HYPERBOLIC-PARABOLIC SYSTEM OF CHEMOTAXIS

RAFAEL GRANERO-BELINCHÓN

ABSTRACT. We study a hyperbolic-parabolic model of chemotaxis in dimensions one and two. In particular, we prove the global existence of classical solutions in certain dissipation regimes.

CONTENTS

1. INTRODUCTION

In this note we study the following system of partial differential equations

$$\partial_t u = -\Lambda^\alpha u + \nabla \cdot (uq), \text{ for } x \in \mathbb{T}^d,\ t \geq 0, \tag{1}$$

$$\partial_t q = \nabla f(u), \text{ for } x \in \mathbb{T}^d,\ t \geq 0, \tag{2}$$

where u is a non-negative scalar function, q is a vector in $\mathbb{R}^d$, $\mathbb{T}^d$ denotes the domain $[-\pi, \pi]^d$ with periodic boundary conditions, $d = 1, 2$ is the dimension, $f(u) = u^2/2$, $0 < \alpha \leq 2$ and $(-\Delta)^{\alpha/2} = \Lambda^\alpha$ is the fractional Laplacian.

This system was proposed by Othmers & Stevens [21] based on biological considerations as a model of tumor angiogenesis. In particular, in the previous system, u is the density of vascular endothelial cells and $q = \nabla \log(v)$ where v is the concentration of the signal protein known as vascular endothelial growth factor (VEGF) (see Bellomo, Li, & Maini [1] for more details on tumor modelling). Similar hyperbolic-dissipative systems arise also in the study of compressible viscous fluids or magnetohydrodynamics (see S. Kawashima [8] and the references therein).

Equation (1) appears as a singular limit of the following Keller-Segel model of aggregation of the slime mold *Dictyostelium discoideum* [9] (see also Patlak [20])

$$\begin{cases} \partial_t u = -\Lambda^\alpha u - \chi\nabla\cdot(u\nabla G(v)), \\ \partial_t v = \nu\Delta v + (f(u)+\lambda)v, \end{cases} \tag{3}$$

when $G(v) = \log(v)$ and the diffusion of the chemical is negligible, *i.e.* $\nu \to 0$.

Similar equations arising in different context are the Majda-Biello model of Rossby waves [18] or the magnetohydrodynamic-Burgers system proposed by Fleischer & Diamond [3].

Most of the results for (1) corresponds to the case where $d = 1$. Then, when the diffusion is local *i.e.* $\alpha = 2$, (1) has been studied by many different research groups. In particular, Fan & Zhao [2], Li & Zhao [13], Mei, Peng & Wang [19], Li, Pan & Zhao [12], Jun, Jixiong, Huijiang & Changjiang [7] Li & Wang [16] and Zhang & Zhu [25] studied the system (1) when $\alpha = 2$ and $f(u) = u$ under different boundary conditions (see also the works by Jin, Li & Wang [6], Li, Li & Wang [14], Wang & Hillen [22] and Wang, Xiang & Yu [23]). The case with general $f(u)$ was studied by Zhang, Tan & Sun [26] and Li & Wang [17].

Equation (1) in several dimensions has been studied by Li, Li & Zhao [11], Hau [5] and Li, Pan & Zhao [15]. There, among other results, the global existence for small initial data in H^s, $s > 2$ is proved.

To the best of our knowledge, the only result when the diffusion is non-local, *i.e.* $0 < \alpha < 2$, is [4]. In that paper we obtained appropriate lower bounds for the fractional Fisher information and, among other results, we proved the global existence of weak solution for $f(u) = u^r/r$ and $1 \leq r \leq 2$.

In this note, we address the existence of classical solutions in the case $0 < \alpha \leq 2$. This is a challenging issue due to the hyperbolic character of the equation for q. In particular, u verifies a transport equation where the velocity q is one derivative more singular than u (so $\nabla\cdot(uq)$ is two derivatives less regular than u).

2. Statement of the results

For the sake of clarity, let us state some notation: we define the mean as

$$\langle g\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(x)dx.$$

Also, from this point onwards, we write H^s for the L^2-based Sobolev space of order s endowed with the norm

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|u\|_{\dot{H}^s}^2, \quad \|u\|_{\dot{H}^s} = \|\Lambda^s u\|_{L^2}.$$

For $\beta \geq 0$, we consider the following energies E_β and dissipations D_β,

$$E_\beta(t) = \|u\|_{\dot{H}^\beta}^2 + \|q\|_{\dot{H}^\beta}^2, \quad D_\beta(t) = \|u\|_{\dot{H}^{\beta+\alpha/2}}^2. \tag{4}$$

Recall that the lower order norms verify the following energy balance [4]

$$\frac{1}{2}\left(\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2\right) + \int_0^t \|u(s)\|_{\dot{H}^{\alpha/2}}^2 ds = \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|q_0\|_{L^2}^2\right). \tag{5}$$

2.1. **On the scaling invariance.** Notice that the equations (1)-(2) verify the following scaling symmetry: for every $\lambda > 0$

$$u_\lambda(x,t) = \lambda^{\alpha-1} u\left(\lambda x, \lambda^\alpha t\right), \quad q_\lambda(x,t) = \lambda^{\alpha-1} q\left(\lambda x, \lambda^\alpha t\right).$$

This scaling serves as a *zoom in* towards the small scales. We also know that

$$\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$$

is the strongest (known) quantity verifying a global-in-time bound. Then, in the one dimensional case, the L^2 norms of u and q are invariant under the scaling of the equations when $\alpha = 1.5$. That makes $\alpha = 1.5$ the critical exponent for the global estimates known. Equivalently, if we define the rescaled (according to the scaling of the strongest conserved quantity $\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$) functions

$$u_\gamma(x,t) = \gamma^{0.5} u\left(\gamma x, \gamma^\alpha t\right), \quad q_\gamma(x,t) = \gamma^{0.5} q\left(\gamma x, \gamma^\alpha t\right).$$

we have that u_γ and q_γ solve

$$\begin{aligned}
\partial_t u_\gamma &= -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5}\partial_x(u_\gamma q_\gamma),\\
\partial_t q_\gamma &= \gamma^{\alpha-1.5} u_\gamma \partial_x u_\gamma.
\end{aligned}$$

Larger values of α form the subcritical regime where the diffusion dominates the drift in small scales. Smaller values of α form the supercritical regime where the drift might be dominant at small scales.

Similarly, the two dimensional case has critical exponent $\alpha = 2$.

Remark 1. *Notice that the equations (1)-(2) where $f(u) = u$ have a different scaling symmetry but the same critical exponent $\alpha = 1.5$. In this case, the scaling symmetry is given by*

$$u_\lambda(x,t) = \lambda^{2\alpha-2} u\left(\lambda x, \lambda^\alpha t\right), \quad q_\lambda(x,t) = \lambda^{\alpha-1} q\left(\lambda x, \lambda^\alpha t\right),$$

while the conserved quantity is $\|u(t)\|_{L^1} + \|q(t)\|_{L^2}^2/2$. Thus, if we define the rescaled (according to the scaling of the conserved quantity) functions

$$u_\gamma(x,t) = \gamma u\left(\gamma x, \gamma^\alpha t\right), \quad q_\gamma(x,t) = \gamma^{0.5} q\left(\gamma x, \gamma^\alpha t\right).$$

we have that u_γ and q_γ solve

$$\begin{aligned}
\partial_t u_\gamma &= -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5}\partial_x(u_\gamma q_\gamma),\\
\partial_t q_\gamma &= \gamma^{\alpha-1.5} \partial_x u_\gamma.
\end{aligned}$$

A global existence result when α is the range $1.5 \leq \alpha < 2$ for the problem where $f(u) = u$ is left for future research.

2.2. **Results in the one-dimensional case $d = 1$.** One of our main results is

Theorem 1. *Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0\rangle = 0$. Assume that $\alpha \geq 1.5$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying*

$$u \in L^\infty(0,T; H^2(\mathbb{T})) \cap L^2(0,T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0,T; H^2(\mathbb{T})).$$

Furthermore, the solution is uniformly bounded in

$$(u,q) \in C([0,\infty), H^1(\mathbb{T})) \times C([0,\infty), H^1(\mathbb{T})).$$

In the case where the strength of the diffusion, α, is even weaker, we have the following global existence result for small data:

Theorem 2. *Fix T an arbitrary parameter and let $(u_0,q_0)\in H^2(\mathbb{T})\times H^2(\mathbb{T})$ be the initial data such that $0\leq u_0$ and $\langle q_0\rangle=0$. There exists $\mathcal{C}_\alpha$ such that if $1.5>\alpha>1$ and*
$$\|u_0\|^2_{\dot{H}^{\alpha/2}}+\|q_0\|^2_{\dot{H}^{\alpha/2}}\leq\mathcal{C}_\alpha$$
then there exist a unique global solution $(u(t),q(t))$ to problem (1) verifying
$$u\in L^\infty(0,T;H^2(\mathbb{T}))\cap L^2(0,T;H^{2+\alpha/2}(\mathbb{T})),q\in L^\infty(0,T;H^2(\mathbb{T})).$$
Furthermore, the solution verifies
$$\|u(t)\|^2_{\dot{H}^{\alpha/2}}+\|q(t)\|^2_{\dot{H}^{\alpha/2}}\leq\|u_0\|^2_{\dot{H}^{\alpha/2}}+\|q_0\|^2_{\dot{H}^{\alpha/2}}.$$

Corollary 1. *Fix T an arbitrary parameter and let $(u_0,q_0)\in H^2(\mathbb{T})\times H^2(\mathbb{T})$ be the initial data such that $0\leq u_0$ and $\langle q_0\rangle=0$. Assume that $1\geq\alpha\geq0.5$ and*
$$\|u_0\|^2_{\dot{H}^1}+\|q_0\|^2_{\dot{H}^1}<\frac{4}{9C_S^2}$$
where C_S is defined in (7). Then there exist a unique global solution $(u(t),q(t))$ to problem (1) verifying
$$u\in L^\infty(0,T;H^2(\mathbb{T}))\cap L^2(0,T;H^{2+\alpha/2}(\mathbb{T})),q\in L^\infty(0,T;H^2(\mathbb{T})).$$
Furthermore, the solution verifies
$$\|u(t)\|^2_{H^1}+\|q(t)\|^2_{H^1}\leq\|u_0\|^2_{H^1}+\|q_0\|^2_{H^1}.$$

2.3. **Results in the two-dimensional case $d=2$.** In two dimensions the global existence read

Theorem 3. *Fix T an arbitrary parameter and let $(u_0,q_0)\in H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)$ be the initial data such that $0\leq u_0$, $\langle q_0\rangle=0$ and $\mathrm{curl}\, q_0=0$. Assume that $\alpha=2$. Then there exist a unique global solution $(u(t),q(t))$ to problem (1) verifying*
$$u\in L^\infty(0,T;H^2(\mathbb{T}^2))\cap L^2(0,T;H^3(\mathbb{T}^2)),q\in L^\infty(0,T;H^2(\mathbb{T}^2)).$$
Furthermore, the solution is uniformly bounded in
$$(u,q)\in C([0,\infty),H^1(\mathbb{T}^2))\times C([0,\infty),H^1(\mathbb{T}^2)).$$

Corollary 2. *Fix T an arbitrary parameter and let $(u_0,q_0)\in H^2(\mathbb{T}^2)\times H^2(\mathbb{T}^2)$ be the initial data such that $0\leq u_0$, $\langle q_0\rangle=0$ and $\mathrm{curl}\, q_0=0$. Assume that $2>\alpha\geq1$ and*
$$\|u_0\|^2_{\dot{H}^1}+\|q_0\|^2_{\dot{H}^1}<\mathcal{C}$$
where $\mathcal{C}$ is a universal constant. Then there exist a unique global solution $(u(t),q(t))$ to problem (1) verifying
$$u\in L^\infty(0,T;H^2(\mathbb{T}^2))\cap L^2(0,T;H^{2+\alpha/2}(\mathbb{T}^2)),q\in L^\infty(0,T;H^2(\mathbb{T}^2)).$$
Furthermore, the solution verifies
$$\|u(t)\|^2_{H^1}+\|\nabla\cdot q(t)\|^2_{L^2}\leq\|u_0\|^2_{H^1}+\|\nabla\cdot q_0\|^2_{L^2}.$$

Remark 2. *In the case where the domain is the one-dimensional torus, $\mathbb{T}$, local existence of solution for (1)-(2) was proved in [4] for a more general class of kinetic function $f(u)$. The local existence of solution for (1)-(2) the domain is the two-dimensional torus $\mathbb{T}^d$ with $d = 2$ follows from the local existence result in [4] with minor modifications. Consequently, we will focus on obtaining global-in-time* a priori *estimates.*

2.4. **Discussion.** Due to the hyperbolic character of the equation for q, prior available global existence results of classical solution for equation (1) impose several assumptions. Namely,

- either $d = 1$ and $\alpha = 2$ [26, 17],
- or $d = 2, 3$, $\alpha = 2$ and the initial data verifies some smallness condition on *strong* Sobolev spaces H^s, $s \geq 2$ [24, 27].

Our results removed some of the previous conditions. On the one hand, we prove global existence for arbitrary data in the cases $d = 1$ and $\alpha \geq 1.5$ and $d = 2$ and $\alpha = 2$. On the other hand, in the cases where we have to impose size restrictions on the initial data, the Sobolev spaces are bigger than H^2 (thus, the norm is weaker). Finally, let us emphasize that our results can be adapted to the case where the spatial domain is $\mathbb{R}^d$.

A question that remains open is the trend to equilibrium. From (5) is clear that the solution $(u(t), q(t))$ tends to the homogeneous state, namely $(\langle u_0\rangle, 0)$. However, the rate of this convergence is not clear.

3. Proof of Theorem 1

Step 1; H^1 estimate: Testing the first equation in (1) against $\Lambda^2 u$, integrating by parts and using the equation for q, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^1} + \|\partial_x u\|^2_{\dot{H}^{\alpha/2}} &= -\int_{\mathbb{T}} \partial_x(uq)\partial_x^2 u dx\\ &= \frac{1}{2}\int_{\mathbb{T}} \partial_x q(\partial_x u)^2 dx - \int_{\mathbb{T}} \partial_x q u \partial_x^2 u dx\\ &= \frac{1}{2}\int_{\mathbb{T}} \partial_x q(\partial_x u)^2 dx - \int_{\mathbb{T}} \partial_x q(\partial_t\partial_x q - (\partial_x u)^2) dx,\end{aligned}$$

so

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\dot{H}^1} + \|q\|^2_{\dot{H}^1}) + \|u\|^2_{\dot{H}^{1+\alpha/2}} = \frac{3}{2}\int_{\mathbb{T}} \partial_x q(\partial_x u)^2 dx.$$

Denoting

$$I = \frac{3}{2}\int_{\mathbb{T}} \partial_x q(\partial_x u)^2 dx,$$

and using Sobolev embedding and interpolation, we have that

$$I \leq \frac{3}{2}\|q\|_{\dot{H}^1}\|\partial_x u\|^2_{L^4} \leq \frac{3}{2}C_S\|q\|_{\dot{H}^1}\|\partial_x u\|^2_{\dot{H}^{0.25}}, \tag{6}$$

where C_S is the constant appearing in the embedding

$$\|g\|_{L^4} \leq C_S\|g\|_{\dot{H}^{0.25}}. \tag{7}$$

Using the interpolation

$$H^{1+\alpha/2} \subset H^{1.25} \subset H^{\alpha/2},$$

and Poincaré inequality (if $\alpha > 1.5$) we conclude

$$I \leq c\|q\|_{\dot{H}^1}\|\Lambda^{\alpha/2}u\|_{L^2}\|u\|_{\dot{H}^{1+\alpha/2}},$$

Using (4), we have that

$$\frac{d}{dt}E_1 + D_1 \leq c\|u\|^2_{\dot{H}^{\alpha/2}}E_1.$$

Using Gronwall's inequality and the estimate (5), we have that

$$\sup_{0\leq t<\infty} E_1(t) \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}),$$

$$\int_0^T D_1(s)ds \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \; \forall 0 < T < \infty.$$

Step 2; H^2 estimate: Now we prove that the solutions satisfying the previous bounds for E_1 and D_1 also satisfy the corresponding estimate in H^2. We test the equation for u against $\Lambda^4 u$. We have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^2} + \|u\|^2_{\dot{H}^{2+\alpha/2}} &= -\int_{\mathbb{T}} \partial_x^2(uq)\partial_x^3 u dx \\ &= \int_{\mathbb{T}} \partial_x q \frac{5(\partial_x^2 u)^2}{2}dx - \int_{\mathbb{T}} \partial_x^2 q(\partial_t\partial_x^2 q - 5\partial_x u\partial_x^2 u)dx,\end{aligned}$$

so

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\dot{H}^2} + \|q\|^2_{\dot{H}^2}) + \|u\|^2_{\dot{H}^{2+\alpha/2}} = \frac{5}{2}\int_{\mathbb{T}} \partial_x q(\partial_x^2 u)^2 dx + 5\int_{\mathbb{T}} \partial_x^2 q\partial_x^2 u\partial_x u dx.$$

We define

$$J_1 = \frac{5}{2}\int_{\mathbb{T}} \partial_x q(\partial_x^2 u)^2 dx, \; J_2 = 5\int_{\mathbb{T}} \partial_x^2 q\partial_x^2 u\partial_x u dx.$$

Then, we have that

$$J_1 \leq c\|\partial_x q\|_{L^\infty}\|\partial_x^2 u\|^2_{L^2} \leq c\|\partial_x^2 q\|^{0.5}_{L^2}\|u\|^{\alpha}_{\dot{H}^{1+\alpha/2}}\|u\|^{2-\alpha}_{\dot{H}^{2+\alpha/2}},$$

so, using Young's inequality,

$$J_1 \leq c\|\partial_x^2 q\|^{\frac{1}{\alpha}}_{L^2}\|u\|^2_{\dot{H}^{1+\alpha/2}} + \frac{1}{4}\|u\|^2_{\dot{H}^{2+\alpha/2}}.$$

Similarly, using Poincaré inequality and $\alpha \geq 0.5$,

$$J_2 \leq c\|\partial_x u\|_{L^4}\|\partial_x^2 u\|_{L^4}\|\partial_x^2 q\|_{L^2} \leq c\|u\|_{\dot{H}^{1+\alpha/2}}\|u\|_{\dot{H}^{2+\alpha/2}}\|\partial_x^2 q\|_{L^2},$$

and

$$J_2 \leq c\|u\|^2_{\dot{H}^{1+\alpha/2}}\|\partial_x^2 q\|^2_{L^2} + \frac{1}{4}\|u\|^2_{\dot{H}^{2+\alpha/2}}.$$

Finally,

$$\frac{d}{dt}E_2(t) + D_2(t) \leq c\|u\|^2_{\dot{H}^{1+\alpha/2}}(E_2(t) + 1)$$

and we conclude using Gronwall's inequality.

4. Proof of Theorem 2

Step 1; $H^{\alpha/2}$ estimate: Testing the first equation in (1) against $\Lambda^\alpha u$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot H^{\alpha/2}}+\|u\|^2_{\dot H^\alpha}&=\int_{\mathbb{T}}\partial_x(uq)\Lambda^\alpha u dx\\&=-\int_{\mathbb{T}}\Lambda^\alpha(uq)\partial_x u dx\\&=-\int_{\mathbb{T}}\left(\Lambda^\alpha(uq)-u\Lambda^\alpha q\right)\partial_x u dx-\int_{\mathbb{T}}\Lambda^\alpha q u\partial_x u dx,\end{aligned}$$

so

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\dot H^{\alpha/2}}+\|q\|^2_{\dot H^{\alpha/2}}\right)+\|u\|^2_{\dot H^\alpha}\leq-\int_{\mathbb{T}}[\Lambda^\alpha,u]q\partial_x u dx.$$

We define

$$K=-\int_{\mathbb{T}}[\Lambda^\alpha,u]q\partial_x u dx.$$

Using the classical Kenig-Ponce-Vega commutator estimate [10] and Sobolev embedding, we have that

$$\begin{aligned}\|[\Lambda^\alpha,u]q\|_{L^2}&\leq c\left(\|\partial_x u\|_{L^{2+\epsilon}}\|\Lambda^{\alpha-1}q\|_{L^{\frac{4+2\epsilon}{\epsilon}}}+\|\Lambda^\alpha u\|_{L^2}\|q\|_{L^\infty}\right)\\&\leq c\left(\|u\|_{\dot H^{1+\frac{\epsilon}{4+2\epsilon}}}\|\Lambda^{\alpha-1}q\|_{\dot H^{\frac12-\frac{\epsilon}{4+2\epsilon}}}+\|u\|_{\dot H^\alpha}\|q\|_{\dot H^{\frac{\alpha}{2}}}\right).\end{aligned}\tag{8}$$

Thus, taking ϵ such that

$$1+\frac{\epsilon}{4+2\epsilon}=\alpha,\quad i.e.\ \epsilon=\frac{4\alpha-4}{3-2\alpha}$$

Equation (8) reads

$$\|[\Lambda^\alpha,u]q\|_{L^2}\leq c\left(\|u\|_{\dot H^\alpha}\|q\|_{\dot H^{\frac12}}+\|u\|_{\dot H^\alpha}\|q\|_{\dot H^{\frac{\alpha}{2}}}\right)\tag{9}$$

Using (9) and Poincaré inequality, we have that

$$K\leq c\|u\|^2_{\dot H^\alpha}\|q\|_{\dot H^{\frac{\alpha}{2}}}$$

Then, we have that

$$\frac{d}{dt}E_{\frac{\alpha}{2}}+D_{\frac{\alpha}{2}}\leq c\sqrt{E_{\frac{\alpha}{2}}}D_{\frac{\alpha}{2}}.$$

Thus, due to the smallness restriction on the initial data, we obtain

$$E_{\frac{\alpha}{2}}(t)+\delta\int_0^t D_{\frac{\alpha}{2}}(s)ds\leq E_{\frac{\alpha}{2}}(0)$$

for $0<\delta$ small enough.

Step 2; H^1 estimate: Our starting point is (6). Then we use the interpolation

$$\|g\|^2_{\dot H^{0.25}}\leq c\|g\|_{L^2}\|g\|_{\dot H^{0.5}},$$

to obtain

$$I\leq c\|q\|_{\dot H^1}\|u\|_{\dot H^1}\|u\|_{\dot H^{1.5}}\leq cE_1 D_{\frac{\alpha}{2}}+\frac{D_1}{2}.$$

Collecting all the estimates, we have that

$$\frac{d}{dt}E_1+D_1\leq cE_1 D_{\frac{\alpha}{2}},$$

and we conclude using Gronwall's inequality. The H^2 estimates follows as in the proof of Theorem 1.

5. Proof of Corollary 1

Using $\alpha \geq 0.5$ and the estimate (6), we have that

$$I \leq \frac{3}{2} C_S \|q\|_{\dot{H}^1} \|u\|^2_{\dot{H}^{1+\alpha/2}} \leq \frac{3}{2} C_S \sqrt{E_1} D_1.$$

Thus,

$$\frac{1}{2}\frac{d}{dt} E_1 + D_1 \leq \frac{3}{2} C_S \sqrt{E_1} D_1.$$

Thus, due to the smallness restriction on the initial data, we obtain

$$E_1(t) + \delta \int_0^t D_1(s) ds \leq E_1(0)$$

for $0 < \delta$ small enough. Equipped with this estimates, we can repeat the argument as in Step 2 in Theorem 1.

6. Proof of Theorem 3

Recall that the condition

$$\operatorname{curl} q_0 = 0$$

propagates in time, *i.e.*

$$\operatorname{curl} q(t) = \operatorname{curl} q_0 + \frac{1}{2} \int_0^t \operatorname{curl} \nabla u^2 ds = 0.$$

Using Plancherel Theorem, we have that

$$\begin{aligned} \|\nabla q\|^2_{L^2} &= C \sum_{\xi \in \mathbb{Z}^2} |\xi|^2 |\hat{q}(\xi)|^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1^2 + \xi_2^2)(\hat{q}_1^2 + \hat{q}_2^2). \end{aligned}$$

Due to the irrotationality

$$\xi^\perp \cdot \hat{q} = 0.$$

Then, we compute

$$\begin{aligned} \|\nabla \cdot q\|^2_{L^2} &= C \sum_{\xi \in \mathbb{Z}^2} |\xi \cdot \hat{q}(\xi)|^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi) + \xi_2 \hat{q}_2(\xi))^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + 2\xi_1 \hat{q}_1(\xi) \xi_2 \hat{q}_2(\xi) \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + (\xi_2 \hat{q}_1(\xi))^2 + (\xi_1 \hat{q}_2(\xi))^2. \end{aligned}$$

So, the vector field q satisfies

$$\|\nabla q\|_{L^2} \leq \|\nabla \cdot q\|_{L^2}.$$

As a consequence of $\langle \partial_t q_i \rangle = 0$ and $\langle q_0 \rangle = 0$, every coordinate of q satisfy

$$\langle q_i(t) \rangle = 0,$$

and the Poincaré-type inequality

$$\|q\|_{L^2} \le c\|\nabla \cdot q\|_{L^2}. \tag{10}$$

Notice that in two dimensions we also have the energy balance (5). We test equation (1) against $\Lambda^2 u$ and use the equation for q. We obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2\right) = -\|u\|_{\dot{H}^2}^2 - \int_{\mathbb{T}^2} \nabla u \cdot q \Delta u dx + \int_{\mathbb{T}^2} |\nabla u|^2 \nabla \cdot q dx.$$

Using Hölder inequality, Sobolev embedding and interpolation, we have that

$$\begin{aligned}\frac{d}{dt}\left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2\right) + 2\|u\|_{\dot{H}^2}^2 &\le c\left(\|u\|_{\dot{H}^{1.5}}\|q\|_{L^4}\|u\|_{\dot{H}^2} + \|u\|_{\dot{H}^{1.5}}^2\|\nabla \cdot q\|_{L^2}\right)\\ &\le c\|u\|_{\dot{H}^1}^{0.5}\|q\|_{L^2}^{0.5}\|q\|_{H^1}^{0.5}\|u\|_{\dot{H}^2}^{1.5}\\ &\quad + c\|u\|_{\dot{H}^1}\|u\|_{\dot{H}^2}\|\nabla \cdot q\|_{L^2}.\end{aligned}$$

Using the Hödge decomposition estimate together with the irrotationality of q and (10), we have that

$$\|q\|_{H^1} \le c\left(\|q\|_{L^2} + \|\nabla \cdot q\|_{L^2}\right) \le c\|\nabla \cdot q\|_{L^2}. \tag{11}$$

Due to (5), we obtain that

$$\|q\|_{L^\infty(0,\infty,L^2)}^2 + \|u\|_{L^2(0,\infty,\dot{H}^1)}^2 \le C(\|u_0\|_{L^2}, \|q_0\|_{L^2})$$

so,

$$\frac{d}{dt}\left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2\right) + \|u\|_{\dot{H}^2}^2 \le c\|u\|_{\dot{H}^1}^2\|\nabla \cdot q\|_{L^2}^2.$$

Using Gronwall's inequality and the integrability of $\|u\|_{\dot{H}^1}^2$ (see (5)), we obtain

$$E_1 \le C(\|u_0\|_{H^1}, \|q_0\|_{H^1}),$$

$$\int_0^T D_1(s)ds \le C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T),\ \forall 0 < T < \infty.$$

To obtain the H^2 estimates, we test against $\Lambda^4 u$. Then, using the previous H^1 uniform bound and

$$\|q\|_{L^\infty}^2 \le c\|q\|_{L^2}\|q\|_{H^2} \le c\|q\|_{L^2}\|\Delta q\|_{L^2},$$

we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|u\|_{\dot{H}^3}^2 &= -\int_{\mathbb{T}^d} \nabla \Delta u \nabla(\nabla u \cdot q)dx\\ &\quad - \int_{\mathbb{T}^d} u\nabla\Delta u \cdot \nabla(\nabla \cdot q)dx - \int_{\mathbb{T}^d} \nabla u \cdot \nabla \Delta u \nabla \cdot q dx.\end{aligned} \tag{12}$$

Due to the irrotationality of q and the identity

$$\nabla\nabla \cdot q - \Delta q = \mathrm{curl}\,(\mathrm{curl}\, q),$$

we have

$$\partial_t \Delta q = \partial_t \nabla(\nabla \cdot q) = \nabla|\nabla u|^2 + u\nabla\Delta u + \nabla u \Delta u.$$

Applying Sobolev embedding and interpolation, we obtain that (12) can be estimated as

$$\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\Delta q\|_{L^2}^2\right) + \|u\|_{\dot{H}^3}^2 \le c\|\Delta q\|_{L^2}^2,$$

so,

$$\frac{d}{dt}E_2 + D_2 \leq cE_2,$$

and we conclude using Gronwall's inequality.

7. Proof of Corollary 2

We test the equation (1) against $\Lambda^2 u$. We obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\dot{H}^1} + \|\nabla \cdot q\|^2_{L^2}\right) = -\|u\|^2_{\dot{H}^{1+\frac{\alpha}{2}}} + \int_{\mathbb{T}^2} \nabla(\nabla u \cdot q)\nabla u dx + \int_{\mathbb{T}^2} |\nabla u|^2 \nabla \cdot q dx.$$

After a short computation, using Hölder estimates, Sobolev embedding and interpolation, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\dot{H}^1} + \|\nabla \cdot q\|^2_{L^2}\right) + \|u\|^2_{\dot{H}^{1+\frac{\alpha}{2}}} \leq c\|u\|^2_{\dot{H}^{1.5}}\|q\|_{\dot{H}^1}.$$

Using (11) and $\alpha \geq 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\dot{H}^1} + \|\nabla \cdot q\|^2_{L^2}\right) + D_1 \leq cD_1\|\nabla \cdot q\|^2_{L^2}.$$

We conclude the result with the previous ideas.

Acknowledgment

The author is funded by the Labex MILYON and the Grant MTM2014-59488-P from the Ministerio de Economía y Competitividad (MINECO, Spain).

References

[1] N. Bellomo, N. Li, and P. K. Maini. On the foundations of cancer modelling: selected topics, speculations, and perspectives. *Mathematical Models and Methods in Applied Sciences*, 18(04):593–646, 2008.

[2] J. Fan and K. Zhao. Blow up criterion for a hyperbolic–parabolic system arising from chemotaxis. *Journal of Mathematical Analysis and Applications*, 394(2):687–695, 2012.

[3] J. Fleischer and P. Diamond. Burgers' turbulence with self-consistently evolved pressure. *Physical Review E*, 61(4):3912, 2000.

[4] R. Granero-Belinchón. On the fractional fisher information with applications to a hyperbolic-parabolic system of chemotaxis. *Journal of Differential Equations*, http://dx.doi.org/10.1016/j.jde.2016.11.028, 2016.

[5] C. Hao. Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces. *Zeitschrift für angewandte Mathematik und Physik*, 63(5):825–834, 2012.

[6] H.-Y. Jin, J. Li, and Z.-A. Wang. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. *Journal of Differential Equations*, 255(2):193–219, 2013.

[7] G. Jun, X. Jixiong, Z. Huijiang, and Z. Changjiang. Global solutions to a hyperbolic-parabolic coupled system with large initial data. *Acta Mathematica Scientia*, 29(3):629–641, 2009.

[8] S. Kawashima. Large-time behavior of solutions for hyperbolic-parabolic systems of conservation laws. *Proceedings of the Japan Academy. Series A Mathematical sciences*, 62(8):285–287, 1986.

[9] E. Keller and L. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, 26(3):399–415, 1970.

[10] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-De Vries equation via the contraction principle. *Communications on Pure and Applied Mathematics*, 46(4):527–620, 1993.

[11] D. Li, T. Li, and K. Zhao. On a hyperbolic–parabolic system modeling chemotaxis. *Mathematical models and methods in applied sciences*, 21(08):1631–1650, 2011.

[12] D. Li, R. Pan, and K. Zhao. Quantitative decay of a one-dimensional hybrid chemotaxis model with large data. *Nonlinearity*, 28(7):2181, 2015.

[13] H. Li and K. Zhao. Initial–boundary value problems for a system of hyperbolic balance laws arising from chemotaxis. *Journal of Differential Equations*, 258(2):302–338, 2015.

[14] J. Li, T. Li, and Z.-A. Wang. Stability of traveling waves of the Keller–Segel system with logarithmic sensitivity. *Mathematical Models and Methods in Applied Sciences*, 24(14):2819–2849, 2014.

[15] T. Li, R. Pan, and K. Zhao. Global dynamics of a hyperbolic-parabolic model arising from chemotaxis. *SIAM Journal on Applied Mathematics*, 72(1):417–443, 2012.

[16] T. Li and Z.-A. Wang. Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis. *SIAM Journal on Applied Mathematics*, 70(5):1522–1541, 2009.

[17] T. Li and Z.-A. Wang. Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic–parabolic system arising in chemotaxis. *Mathematical models and methods in applied sciences*, 20(11):1967–1998, 2010.

[18] A. J. Majda and J. A. Biello. The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves. *Journal of the atmospheric sciences*, 60(15):1809–1821, 2003.

[19] M. Mei, H. Peng, and Z.-A. Wang. Asymptotic profile of a parabolic–hyperbolic system with boundary effect arising from tumor angiogenesis. *Journal of Differential Equations*, 259(10):5168–5191, 2015.

[20] C. Patlak. Random walk with persistence and external bias. *Bulletin of Mathematical Biology*, 15(3):311–338, 1953.

[21] A. Stevens and H. G. Othmer. Aggregation, blowup, and collapse: the ABC's of taxis in reinforced random walks. *SIAM Journal on Applied Mathematics*, 57(4):1044–1081, 1997.

[22] Z. Wang and T. Hillen. Shock formation in a chemotaxis model. *Mathematical Methods in the Applied Sciences*, 31(1):45–70, 2008.

[23] Z.-A. Wang, Z. Xiang, and P. Yu. Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis. *Journal of Differential Equations*, 260(3):2225–2258, 2016.

[24] W. Xie, Y. Zhang, Y. Xiao, and W. Wei. Global existence and convergence rates for the strong solutions into the 3d chemotaxis model. *Journal of Applied Mathematics*, 2013, 2013.

[25] M. Zhang and C. Zhu. Global existence of solutions to a hyperbolic-parabolic system. *Proceedings of the American Mathematical Society*, 135(4):1017–1027, 2007.

[26] Y. Zhang, Z. Tan, and M.-B. Sun. Global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system. *Nonlinear Analysis: Real World Applications*, 14(1):465–482, 2013.

[27] Y. Zhang and W. Xie. Global existence and exponential stability for the strong solutions in H^2 to the 3-d chemotaxis model. *Boundary Value Problems*, 2015(1):1–13, 2015.

E-mail address: `granero@math.univ-lyon1.fr`

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France.