GLOBAL EXISTENCE OF WEAK SOLUTIONS TO DISSIPATIVE TRANSPORT EQUATIONS WITH NONLOCAL VELOCITY

HANTAEK BAE, RAFAEL GRANERO-BELINCHÓN, AND OMAR LAZAR

Abstract. We consider 1D dissipative transport equations with nonlocal velocity field:

$$\theta_t + u\theta_x + \delta u_x\theta + \Lambda^\gamma\theta = 0, \quad u = \mathcal{N}(\theta),$$

where $\mathcal{N}$ is a nonlocal operator given by a Fourier multiplier. Especially we consider two types of nonlocal operators:

(1) $\mathcal{N} = \mathcal{H}$, the Hilbert transform,
(2) $\mathcal{N} = (1 - \partial_{xx})^{-\alpha}$.

In this paper, we show several global existence of weak solutions depending on the range of γ, δ and α. When $0 < \gamma < 1$, we take initial data having finite energy, while we take initial data in weighted function spaces (in the real variables or in the Fourier variables), which have infinite energy, when $\gamma \in (0, 2)$.

1. Introduction

In this paper, we consider transport equations with nonlocal velocity. Here, the non-locality means that the velocity field is defined through a nonlocal operator that is represented in terms of a Fourier multiplier. For example, in the two dimensional Euler equation in vorticity form,

$$\omega_t + u \cdot \nabla\omega = 0,$$

the velocity is recovered from the vorticity ω through

$$u = \nabla^\perp(-\Delta)^{-1}\omega \quad \text{or equivalently} \quad \widehat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2}\widehat{\omega}(\xi).$$

Other nonlocal and quadratically nonlinear equations appear in many applications. Prototypical examples are the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magneto-geostrophic equation in multi-dimensions. For more details on nonlocal operators in these equations, see [1].

We here study 1D models of physically important equations. The 1D reduction idea were initiated by Constatin-Lax-Majda [8]: they proposed the following 1D model

$$\theta_t = \theta\mathcal{H}\theta$$

for the 3D Euler equation in the vorticity form and proved that $\mathcal{H}\theta$ blows up in finite time under certain conditions. Motivated by this work, other similar models were proposed and analyzed in the literature [1, 2, 3, 4, 5, 6, 12, 13, 16, 19, 20, 23]. In this paper, we consider the following 1D equation:

$$\theta_t + u\theta_x + \delta u_x\theta + \nu\Lambda^\gamma\theta = 0, \quad u = \mathcal{N}(\theta). \tag{1.1}$$

Depending on a nonlocal operator $\mathcal{N}$, (1.1) has structural similarity of several important fluid equations as described below. The goal of this paper is to show the existence of weak solutions

Date: August 19, 2021.
2010 *Mathematics Subject Classification.* Primary 35A01, 35D30, 35Q35, 35Q86.
Key words and phrases. Fluid equations, 1D models, Global weak solution.

with rough initial data. To this end, we will choose functionals carefully to extract more information from the structure of the nonlinearity to construct weak solutions.

1.1. **The case $\mathcal{N}=\mathcal{H}$.** We first take the case $\mathcal{N}=\mathcal{H}$, the Hilbert transform. Then, (1.1) becomes

$$\theta_t+(\mathcal{H}\theta)\,\theta_x+\delta\theta\Lambda\theta+\nu\Lambda^\gamma\theta=0, \tag{1.2}$$

where the range of γ and δ will be specified below. We note that (1.2) is considered as an 1D model of the dissipative surface quasi-geostrophic equation. The surface quasi-geostrophic equation describes the dynamics of the mixture of cold and hot air and the fronts between them in 2 dimensions [10, 26]. The equation is of the form

$$\theta_t+u\cdot\nabla\theta+\nu\Lambda^\gamma\theta=0,\quad u=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \tag{1.3}$$

where the scalar function θ is the potential temperature and $\mathcal{R}_j$ is the Riesz transform

$$\mathcal{R}_jf(x)=\frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{(x_j-y_j)f(y)}{|x-y|^3}dy,\quad j=1,2.$$

As Constatin-Lax-Majda did for the Euler equation, the equation (1.2) is derived by replacing the Riesz transforms with the Hilbert transform. The case $\delta=0$ and $\delta=1$ correspond to (1.3) in non-divergence and divergence form, respectively. We take a parameter $\delta\in[0,1]$ to cover more general nonlinear terms in (1.2). We note that there are several singularity formation results when $\nu=0$: $0<\delta<\frac{1}{3}$ and $\delta=1$ [23], $0<\delta\leq 1$ [6], and $\delta=0$ [12, 19, 28]. By contrast, we look for weak solutions of (1.2) globally in time (see e.g. [14]). From now on, we set $\nu=1$ for notational simplicity.

We now consider (1.2). In the direction of seeking a weak solution, we assume that θ_0 satisfies the conditions

$$\theta_0(x)>0,\quad \theta_0\in L^1\cap H^{\frac{1}{2}}. \tag{1.4}$$

Since (1.2) satisfies the minimum principle (see Section 2) when $\delta\geq 0$, $\theta(t,x)\geq 0$ for all time. This sign condition combined with the structure of the nonlinearity enables us to use the following function space

$$\mathcal{A}_T=L^\infty\left(0,T;L^p\cap H^{\frac{1}{2}}\right)\cap L^2\left(0,T;H^{\frac{\gamma+1}{2}}\right)\quad \text{for all } p\in(1,\infty).$$

We note that we add the sign condition of θ_0 for the better understanding of the structure of the nonlinearity in the sense that the sign condition and the dissipative term (of any order) deplete the nonlinear effect. If we remove the sign condition of θ_0, we would have to control higher other norms, but then we will obtain strong solutions locally in time which are not the notion of solutions we are going to obtain in this paper. Note that the positivity condition is something that is also assumed in other works (see e.g. [12], [14], [16]).

Definition 1.1. We say θ is a weak solution of (1.2) if $\theta\in\mathcal{A}_T$ and (1.2) holds in the following sense: for any test function $\psi\in\mathcal{C}_c^\infty([0,T)\times\mathbb{R})$,

$$\int_0^T\int_{\mathbb{R}}\left[\theta\psi_t+(\mathcal{H}\theta)\,\theta\psi_x+(1-\delta)\Lambda\theta\theta\psi-\theta\Lambda^\gamma\psi\right]dxdt=\int_{\mathbb{R}}\theta_0(x)\psi(0,x)dx$$

holds for any $0<T<\infty$.

Theorem 1.1. *Let $\gamma\in(0,1)$ and $\delta\geq\frac{1}{2}$. Then, for any θ_0 satisfying (1.4), there exists a weak solution of (1.2) in $\mathcal{A}_T$ for all $T>0$. Moreover, a weak solution is unique when $\gamma=1$.*

For $\gamma\in(0,2)$, we consider infinite energy solutions of (1.2). More precisely, we take a family of weights $w_\beta=\left(1+|x|^2\right)^{-\frac{\beta}{2}}$, $0<\beta<\gamma$ and we shall prove various existence theorems. For the critical case, we take initial data in the following weighted Sobolev space

$$\theta_0\in H^{\frac{1}{2}}(w_\beta dx)\cap L^\infty. \tag{1.5}$$

These weighted spaces are defined in Section 2. We note that θ_0 can decay (slowly) at infinity. For example, as long as $\beta - 2\eta > 1$, $|\theta_0(x)| \simeq |x|^{-\eta}$, $\eta > 0$ is allowed to stay in $L^2(w_\beta dx)$

$$\int_{|x|\geq 1} \frac{|x|^{2\eta}}{(1+|x|^2)^{\frac{\beta}{2}}} dx < \infty.$$

But, we can still use the energy method to obtain a weak solution of (1.2). Let

$$\mathcal{B}_T = L^\infty \left(0, T; H^{\frac{1}{2}}(w_\beta dx)\right) \cap L^2 \left(0, T; H^1(w_\beta dx)\right).$$

Theorem 1.2. *Assume $\gamma = 1$, then, for any θ_0 satisfying (1.5) with $\|\theta_0\|_{L^\infty}$ being sufficiently small, there exists a unique weak solution of (1.2) in $\mathcal{B}_T$ for all $T > 0$.*

Remark 1. It is worth mentioning that this theorem is also true in the unweighted setting, this is an important point because one would need to use the existence of solutions in the unweighted setting to prove Th. 1.2. Indeed, one has nice *a priori* estimates in the unweighted setting, it suffices to observe that the evolution of the $\dot{H}^{1/2}$ (semi-)norm is obtained via the classical Hardy-BMO duality along with formula 2.2. Indeed, one writes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \|\Lambda^{1/2}\theta\|_{L^2}^2 \, dx + \int |\Lambda\theta|^2 \, dx &= -\int \Lambda\theta\mathcal{H}\theta\theta_x \, dx - \int \theta|\Lambda\theta|^2 \, dx \\ &\leq \|\theta_x\mathcal{H}\theta_x\|_{\mathcal{H}^1}\|\mathcal{H}\theta\|_{BMO} + \|\theta_0\|_{L^\infty}\|\Lambda\theta\|_{L^2}^2 \\ &\leq 2\|\theta_x\|_{L^2}^2\|\theta_0\|_{L^\infty} = 2\|\mathcal{H}\theta_x\|_{L^2}^2\|\theta_0\|_{L^\infty}. \end{aligned}$$

Hence, if $\|\theta_0\|_{L^\infty} < 1/2$, one has $\theta \in \mathcal{C}([0,T], \dot{H}^{1/2}) \cap L^2([0,T], \dot{H}^1)$ for all finite $T > 0$. For the construction using compactness we refer to [18].

In the subcritical case i.e. $\gamma \in (1,2)$, we have global existence of weak solutions for any arbitrary initial data in the weighted Sobolev space $H^1(wdx)$, with $w(x) = w_\beta(x) = (1+x^2)^{-\beta/2}$, $\beta \in (0,1)$.

Theorem 1.3. *Let $\gamma \in (1,2)$, for all $\theta_0 \in H^1(wdx) \cap L^\infty$ there exists at least one global weak solution to the equation $\mathcal{T}_\alpha$, which verifies, for all finite $T > 0$*

$$\theta \in \mathcal{C}([0,T], H^1(wdx)) \cap L^2([0,T], \dot{H}^{1+\alpha/2}(wdx)).$$

Moreover, for all $T < \infty$, we have

$$\|\theta(T)\|_{H^1(wdx)}^2 \leq \|\theta_0\|_{H^1(wdx)}^2 e^{CT}$$

The constant $C > 0$ depends only on $\|\theta_0\|_{L^\infty}$, β, k, δ and ν.

In the supercritical case, one can prove the following local existence theorem for data in the weighted $H^2(wdx)$ Sobolev space.

Theorem 1.4. *Assume that $0 < \alpha < 1$ and $\delta \geq 0$, then for all positif initial data $\theta_0 \in H^2(wdx)$ where the weight is given by $w_\beta(x) = (1+|x|^2)^{-\beta/2}$ with $\beta \in (0, \alpha/2)$, there exists a time $T^*(\theta_0) > 0$ such that $(\mathcal{T}_\alpha)$ admits at least one solution that verifies*

$$\theta \in \mathcal{C}([0,T], H^2(wdx)) \cap L^2([0,T], \dot{H}^{2+\frac{\alpha}{2}}(wdx))$$

for all $T \leq T^*$.

We have some restrictions on the sign of initial data and the range of δ in Theorem 1.1, and the smallness condition of $\|\theta_0\|_{L^\infty}$ in Theorem 1.2. We can remove these conditions by looking for a solution of (1.2) in function spaces defined by the Fourier transform. Let

$$A^\alpha = \left\{ f \in L^1_{loc} : \|f\|_{A^\alpha} = \int_{\mathbb{R}} (1+|\xi|^\alpha)|\hat{f}(\xi)|d\xi < \infty \right\}.$$

We also define

$$\mathcal{W}_T = L^\infty\left(0,T;W^{1,\infty}\right) \cap W^{1,\infty}(0,T;L^\infty) \cap L^1(0,T;W^{2,\infty}).$$

Theorem 1.5. *Let $\gamma = 1$ and $\delta \in \mathbb{R}$. Then, for any $\theta_0 \in A^1$ with*

$$\|\theta_0\|_{A^0} < \frac{\sqrt{\pi}}{\sqrt{2}(1+|\delta|)}, \tag{1.6}$$

there exists a unique weak solution of (1.2) verifying the following inequality for all $T > 0$

$$\theta \in \mathcal{W}_T, \quad \sup_{t\in[0,T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}(1+|\delta|)\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right)\int_0^T \|\theta_x(t)\|_{A^1} dt \le \|\theta_0\|_{A^1}.$$

We note that $\theta_0 \in A^1$ can have infinite energy. For example, we take $\widehat{\theta_0}(\xi) = \frac{e^{-|\xi|}}{\sqrt{|\xi|}}$ for $\xi \neq 0$. Then, $\theta_0 \in A^1$ but $\theta_0 \notin L^2$.

We observe that the proof of Theorem 1.5 is due to the perfect balance (in the critical case $\gamma = 1$) between the derivatives in the nonlinearity and the diffusive linear operator. This is due to the fact that, in Wiener spaces, the parabolic gain of regularity for Λ is $L^1_t A^1_x$ (*i.e.* a full derivative in the Wiener space). This is in contrast with the case of L^2, where the parabolic gain of regularity for Λ is $L^2_t H^{1/2}_x$ (*i.e.* just half derivative in L^2). In particular, the proof is based in an inequality of the type

$$\frac{d}{dt}E(t) + E(t)D(t) \le -D(t),$$

where E and D are the appropriate energy and dissipation. From such an inequality the decay of the energy for small enough initial energy can be easily obtained. In the case where $\gamma > 1$, that balance is broken and the previous inequality has to be replaced by

$$\frac{d}{dt}E(t) + E^{2-\alpha}(t)D^\alpha(t) \le -D(t),$$

with $0 < \alpha < 1$. In our setting, the lack of Poincaré inequality that relates E and D, makes this new inequality less suited for our approach.

1.2. The case $\mathcal{N} = (1-\partial_{xx})^{-\alpha}$ and $\delta = 0$.

In this case, (1.1) is changed to the equation

$$\theta_t + u\theta_x + \delta\theta\Lambda\theta + \Lambda^\gamma\theta = 0, \quad u = (1-\partial_{xx})^{-\alpha}\theta. \tag{1.7}$$

This equation is closely related to a generalized Proudman-Johnson equation [25, 27, 31]:

$$f_{txx} + ff_{xxx} + \delta f_x f_{xx} = \nu f_{xxxx}$$

which is derived from the 2D incompressible Navier-Stokes equations via the separation of space variables when $\delta = 1$. By taking $w = f_{xx}$,

$$w_t + fw_x + \delta f_x w = \nu w_{xx}, \quad f = (\partial_{xx})^{-1}w.$$

The inviscid case with $\delta = 2$ is equivalent to the Hunter-Saxton equation arising in the study of nematic liquid crystals [15]. The equation (1.7) is also considered as a model equation of the Lagrangian averaged Navier-Stokes equations [21] which are given by

$$\partial_t\left(1-\sigma^2\Delta\right)u + u\cdot\nabla\left(1-\sigma^2\Delta\right)u + (\nabla u)^T\cdot\left(1-\sigma^2\Delta\right)u = -\nabla p + \nu\Delta\left(1-\sigma^2\Delta\right)u, \quad \nabla\cdot u = 0.$$

But, we here consider (1.7) with $\delta = 0$ to see how the regularizing effect in u overcomes difficulties from the quadratic term $u\theta_x$. Along this direction, we look closely to see if there is a solution when α and γ meet certain conditions.

We first deal with (1.7) with initial data in $L^2 \cap L^\infty$. Let

$$\mathcal{C}_T = L^\infty\left(0,T;L^p\right) \cap L^2\left(0,T;H^{\frac{\gamma}{2}}\right) \quad \text{for all } p \in [2,\infty].$$

Definition 1.2. We say θ is a weak solution of (1.7) if $\theta \in \mathcal{C}_T$ and (1.7) holds in the following sense: for any test function $\psi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t + u_x\theta\psi + u\theta\psi_x - \theta\Lambda^\gamma\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(0,x)dx$$

holds for any $0 < T < \infty$.

Theorem 1.6. *Let $\gamma \in (0,2)$ and $\alpha = \frac{1}{2} - \frac{\gamma}{4}$. Then, for any $\theta_0 \in L^2 \cap L^\infty$, there exists a weak solution of (1.7) in $\mathcal{C}_T$ for all $T > 0$. Moreover, such a weak solution is unique if $\gamma > 1$ and $\alpha > \frac{1}{4}$.*

We note that $\theta_0 \in L^2 \cap L^\infty$ is enough to construct a weak solution in Theorem 1.6, but we need to strengthen γ and α to get the uniqueness of weak solutions.

When $\gamma = 1$, we consider weights $w_\beta(x) = \left(1+|x|^2\right)^{-\frac{\beta}{2}}$ with $0 < \beta < 1$, and take initial data in $H^1(w_\beta dx) \cap L^\infty$. Let $\alpha = \frac{1}{4}$ and

$$\mathcal{D}_T = L^\infty\left(0,T;H^1(w_\beta dx)\right) \cap L^2\left(0,T;H^{\frac{3}{2}}(w_\beta dx)\right).$$

Theorem 1.7. *Let $\gamma = 1$ and $\alpha = \frac{1}{4}$. Then, for any $\theta_0 \in H^1(w_\beta dx) \cap L^\infty$, there exists a unique global weak solution of (1.7) in $\mathcal{D}_T$ for all $T > 0$.*

Compared to Theorem 1.2, we do not assume that $\|\theta_0\|_{L^\infty}$ is small to prove Theorem 1.7.

In Theorem 1.6 and Theorem 1.7, we have restrictions on the range of α. Again, we can remove these conditions by looking for a solution of (1.7) in function spaces defined by the Fourier variables.

Theorem 1.8. *Let $\gamma = 1$ and $\alpha \geq 0$. Then, for any $\theta_0 \in A^1$ satisfying*

$$\|\theta_0\|_{A^0} < \frac{\sqrt{\pi}}{\sqrt{2}}, \tag{1.8}$$

there exists a unique weak solution of (1.7) verifying the following inequality for all $T > 0$

$$\theta \in \mathcal{W}_T, \quad \sup_{t\in[0,T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right) \int_0^T \|\theta_x(t)\|_{A^1} dt \leq \|\theta_0\|_{A^1}.$$

Remark 2. We note that Theorem 1.8 remains valid with straightforward changes in the spirit of Theorem 1.5 when $\delta \neq 0$.

2. Preliminaries

All constants will be denoted by C that is a generic constant. In a series of inequalities, the value of C can vary with each inequality. For $s \in \mathbb{R}$, H^s is a Hilbert space with

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s \left|\hat{f}(\xi)\right|^2 d\xi.$$

2.1. **Hilbert transform and fractional Laplacian.** The Hilbert transform is defined as

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

The differential operator $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ is defined by the action of the following kernels [11]:

$$\Lambda^\gamma f(x) = c_\gamma \text{p.v.} \int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\gamma}} dy, \tag{2.1}$$

where $c_\gamma > 0$ is a normalized constant. When $\gamma = 1$,

$$\Lambda f(x) = \mathcal{H} f_x(x).$$

Moreover, we have the following identity:

$$\mathcal{H}\left(\theta_x\left(\mathcal{H}\theta_x\right)\right) = \frac{1}{2}\left[(\Lambda\theta)^2 - (\theta_x)^2\right]. \tag{2.2}$$

We also recall the following pointwise property of Λ^α.

Lemma 2.1. [11] *Let $0 \le \alpha \le 2$ and $f \in \mathcal{S}$. Then,*

$$f(x)\Lambda^\alpha f(x) \ge \frac{1}{2}\Lambda^\alpha\left(f^2(x)\right),$$
$$f^2(x)\Lambda f(x) \ge \frac{1}{3}\Lambda\left(f^3(x)\right) \quad \textit{when } f \ge 0$$

2.2. **Minimum and Maximum Principles.** In Theorem 1.1, we assume $\theta_0 > 0$. To obtain global-in-time solutions, we need $\theta(t,x) \ge 0$ for all time. We first assume that $\theta(t,x) \in C^1([0,T]\times\mathbb{R})$ and x_t be a point such that $m(t) = \theta(t,x_t)$. If $m(t) > 0$ for all time, nothing is left to prove. So, we check a point (t,x_t) where $m(t) = 0$. Since $m(t)$ is a continuous Lipschitz function, it is differentiable at almost every t by Rademacher's theorem. From the definition of Λ^γ,

$$\begin{aligned}\frac{d}{dt}m(t) &= -\delta\theta(t,x_t)\text{p.v.}\int_{\mathbb{R}} \frac{\theta(t,x_t)-\theta(t,y)}{|x_t-y|^{1+\gamma}}dy - \text{p.v.}\int_{\mathbb{R}} \frac{\theta(t,x_t)-\theta(t,y)}{|x_t-y|^{1+\gamma}}dy \\ &\ge \left[-\delta\text{p.v.}\int_{\mathbb{R}} \frac{\theta(t,x_t)-\theta(t,y)}{|x_t-y|^{1+\gamma}}dy\right] m(t).\end{aligned}$$

Since the quantity in the bracket is nonnegative when $\delta \ge 0$, we have that $m(t)$ is non-decreasing in time if $\theta_0 > 0$ and thus $\theta(t,x) \ge 0$ for all time. Similarly, maximum values of $\theta(t,x)$ are non-increasing in time when $\theta_0 > 0$ with $\theta_0 \in L^\infty$. For general initial data satisfying (1.4) and (1.5), we can use regularization method. For such a regularized problem with smooth solution θ^ϵ, the same argument works. Then, we construct θ as the limit of θ^ϵ. As θ will be also the pointwise limit of θ^ϵ almost everywhere, we conclude that $\theta(t,x) \ge 0$.

Since (1.7) is purely a dissipative transport equation, we immediately have

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}.$$

2.3. **The Wiener spaces A^α.** The Wiener space is defined as

$$A^0 = \left\{ f \in L^1_{loc} : \hat{f}(\xi) \in L^1 \right\},$$

where $\widehat{f}$ denotes the Fourier transform of f

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\cdot\xi} dx.$$

A^0 is a Banach space endowed with the norm

$$\|f\|_{A^0} = \|\widehat{f}\|_{L^1}.$$

Furthermore, using Fubini's Theorem, A^0 is a Banach algebra, *i.e.*

$$\|fg\|_{A^0} \leq \|f\|_{A^0}\|g\|_{A^0}.$$

Once we have defined A^0, we can define the full scale of homogeneous, $\dot{A}^\alpha$, and inhomogeneous, A^α, Wiener spaces as

$$\begin{aligned} \dot{A}^\alpha &= \left\{ f \in L^1_{loc} : \|f\|_{\dot{A}^\alpha} = \int_{\mathbb{R}} |\xi|^\alpha |\hat{f}(\xi)| d\xi < \infty \right\}, \\ A^\alpha &= \left\{ f \in L^1_{loc} : \|f\|_{A^\alpha} = \int_{\mathbb{R}} (1+|\xi|^\alpha) |\hat{f}(\xi)| d\xi \right\}. \end{aligned} \tag{2.3}$$

For these spaces, the following inequalities hold

$$\|f\|_{C(\mathbb{R})} \leq \|f\|_{A^0(\mathbb{R}^d)} \quad \forall\, f \in A^0(\mathbb{R}) \tag{2.4}$$

$$\|f\|_{\dot{A}^\alpha(\mathbb{R})} \leq \|f\|^{1-\theta}_{A^0(\mathbb{R}^d)} \|f\|^{\theta}_{\dot{A}^{\frac{\alpha}{\theta}}(\mathbb{R}^d)} \quad \forall\, 0<\theta<1,\ \alpha \geq 0,\ f \in A^0(\mathbb{R}) \cap A^{\frac{\alpha}{\theta}}(\mathbb{R}). \tag{2.5}$$

As a consequence of (2.4), we obtain that if $u \in A^0$ has infinite energy then

$$\limsup_{|x|\to\infty} |u(x)| + \liminf_{|x|\to\infty} |u(x)| < \infty.$$

2.4. **Commutator estimate.** In the proof of Theorem 1.1, we need to estimate a commutator term involving $\Lambda^{\frac{1}{2}}$. To do this, we first recall Hardy-Littlewood-Sobolev inequality in 1D. Let $K_\alpha(x) = \frac{1}{|x|^\lambda}$ and $T_\lambda f = K_\lambda * f$. Then,

$$\|T_\lambda f\|_{L^q} \leq C\|f\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{p} + \lambda.$$

Lemma 2.2. *For $f \in L^{\frac{3}{2}}$, $g \in L^{\frac{3}{2}}$ and $\psi \in W^{1,\infty}$,*

$$\left\| \left[\Lambda^{\frac{1}{2}}, \psi\right] f - \left[\Lambda^{\frac{1}{2}}, \psi\right] g \right\|_{L^6} \leq C\|\psi\|_{W^{1,\infty}} \|f-g\|_{L^{\frac{3}{2}}}.$$

Proof. By the definition of $\Lambda^{\frac{1}{2}}$, we have

$$\left(\left[\Lambda^{\frac{1}{2}}, \psi\right] f - \left[\Lambda^{\frac{1}{2}}, \psi\right] g\right)(x) = c_1 \text{p.v.} \int \frac{(\psi(y)-\psi(x))(f(y)-g(y))}{|x-y|^{\frac{3}{2}}} dy$$

and thus

$$\left| \left[\Lambda^{\frac{1}{2}}, \psi\right] f - \left[\Lambda^{\frac{1}{2}}, \psi\right] g \right|(x) \leq C\|\nabla\psi\|_{L^\infty} \int \frac{|f(y)-g(y)|}{|x-y|^{\frac{1}{2}}} dy. \tag{2.6}$$

Using Hardy-Littlewood-Sobolev inequality, we obtain that

$$\left\| \left[\Lambda^{\frac{1}{2}}, \psi\right] f - \left[\Lambda^{\frac{1}{2}}, \psi\right] g \right\|_{L^6} \leq C \|\nabla\psi\|_{L^\infty} \|f-g\|_{L^{\frac{3}{2}}} \tag{2.7}$$

which completes the proof. □

2.5. **Muckenhoupt weights.** We briefly introduce weighted spaces. A weight w is a positive and locally integrable function. A measurable function θ on $\mathbb{R}$ belongs to the weighted Lebesgue spaces $L^p(wdx)$ with $1 \leq p < \infty$ if and only if

$$\|\theta\|^p_{L^p(wdx)} = \int_{\mathbb{R}} |\theta(x)|^p w(x) dx < \infty.$$

An important class of weights is the Muckenhoupt class $\mathcal{A}_p$ for $1<p<\infty$ [7, 24]. Let $1<p<\infty$, we say that $w\in\mathcal{A}_p$ if and only if there exists a constant $C_{p,w}>0$ such that

$$\sup_{r>0,x_0\in\mathbb{R}}\left(\frac{1}{2r}\int_{[x_0-r,x_0+r]}wdx\right)\left(\frac{1}{2r}\int_{[x_0-r,x_0+r]}w^{\frac{1}{1-p}}dx\right)^{p-1}\leq C_{p,w}.$$

This class satisfies the following properties.

(1) Calderón-Zygmund type operators are bound on $L^p(wdx)$ when $w\in\mathcal{A}_p$ and $1<p<\infty$ [29].
(2) Let $w\in\mathcal{A}_p$. We define weighted Sobolev spaces as follows

$$\begin{aligned}&f\in H^1(wdx)\iff f\in L^2(wdx)\text{ and }f_x\in L^2(wdx),\\&f\in H^1(wdx)\iff(1-\partial_{xx})^{\frac12}f\in L^2(wdx)\iff f\in L^2(wdx)\text{ and }\Lambda f\in L^2(wdx),\\&f\in H^{\frac12}(wdx)\iff(1-\partial_{xx})^{\frac14}f\in L^2(wdx)\iff f\in L^2(wdx)\text{ and }\Lambda^{\frac12}f\in L^2(wdx).\end{aligned}\tag{2.8}$$

(3) Gagliardo-Nirenberg type inequalities (see e.g [22])

$$\begin{aligned}&\left\|\Lambda^{\frac12}f\right\|_{L^2(wdx)}\leq C\|f\|_{L^2(wdx)}^{\frac12}\|\Lambda f\|_{L^2(wdx)}^{\frac12},\\&\|\theta\|_{L^4(wdx)}\leq C\|\theta\|_{L^2(wdx)}^{\frac12}\left\|\Lambda^{\frac12}\theta\right\|_{L^2(wdx)}^{\frac12}.\end{aligned}\tag{2.9}$$

This latter inequality can be proved for instance by using the weighted Sobolev embedding $H^{\frac14}(wdx)\hookrightarrow L^4(wdx)$, and then by weighted interpolation one recover the second inequality in (2.9).

In this paper, we take weights $w_\beta=(1+|x|^2)^{-\frac{\beta}{2}}$, $0<\beta<1$, which belongs to the $\mathcal{A}_p$ class of Muckenhoupt for all $1<p<\infty$. These weights also satisfy the following properties. For the proofs, see [18] (for the first two points), and [17] (for the last two points).

Lemma 2.3. *Let $w_\beta(x)=(1+|x|^2)^{-\frac{\beta}{2}}$.*

- *Let $p\geq 2$ be such that $\frac32-\beta(1-\frac1p)>1$, then the commutator $\frac{1}{w_\beta}[\Lambda^{1/2},w_\beta]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.*
- *Let $2\leq p<\infty$, then the commutator $\frac{1}{\sqrt{w}}[\Lambda,\sqrt{w}]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.*
- *Let $s\in(0,1)$, then for all $\beta\in(0,s)$, the commutator $\frac1w[\Lambda^{s/2},w]$ is bounded from $L^2(wdx)$ to $L^2(wdx)$*
- *Let $s\in(1,2)$, then for all $\beta\in(0,1)$, the commutator $\frac1w[\Lambda^{s/2},w]$ is continuous from $L^2(wdx)$ to $L^2(wdx)$.*
- *The commutator $\frac1w[\Lambda^{1/2},w]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$*

Remark 3. Actually, the last point of 2.3 is not proved in [18] or [17]. However, a slight modification of the proof given in [18] of the second point of 2.3 gives the last point of 2.3. Indeed, the idea of the proof of all of those commutator estimates is to split the domain of integration into 3 regions ($\Delta_1,\Delta_2, and\Delta_3$ according to the notation in [18]). Since we are dealing with a quite singular kernel, we need to do a sort of second Taylor expansion in the region where the kernel is very singular (the region Δ_1), the same argument works writting w in stead of γ. As well, in the region Δ_2 one can follow [18]. In the region Δ_3, the estimates of the kernel change a little bit, indeed, we have the following estimate

$$|K(x,y)|\leq C\frac{w(x)^{\frac1p-1}+w(y)^{-\frac1p}}{|x-y|^{3/2}}\leq\frac{C'}{|x-y|^{2-\beta(1-\frac1p)}}+\frac{C'}{|x-y|^{2-\frac{\beta}{p}}}\leq\frac{C'}{|x-y|^{2-\beta\max(1-\frac1p,\frac1p)}}$$

where we used that, on Δ_3 we have $1 \leq w(x)^{-1} \leq C|x-y|^{\beta}$ and $1 \leq w(y)^{-1} \leq C|x-y|^{\beta}$. To conclude, it suffices to observe that since $0 < \beta < 1$ and $\max(1-\frac{1}{p}, \frac{1}{p}) \in (0,1]$ for all $p \geq 2$, we have $K \in L^1$.

We shall also need to estimate $|\Lambda^s w|$ for $s \in (0,2)$, the estimate of such a term will be done using the following lemma (see [18] and [17] for the proof).

Lemma 2.4. *Let* $w_\beta(x) = (1+|x|^2)^{-\frac{\beta}{2}}$*, with* $\beta \in (0,1)$.

- $|\partial_x w_\beta(x)| \leq C(\beta) w_\beta(x)$
- *For all* $s \in (0,2)$*, there exists a constant* $C = C(s,\beta) > 0$ *such that the following estimate holds*

$$|\Lambda^s w_\beta(x)| \leq C w_\beta(x)$$

2.6. **Littlewood-Paley decomposition.** Consider a fixed function $\phi_0 \in \mathcal{D}(\mathbb{R})$ that is non-negative and radial and is such that $\phi_0(\xi) = 1$, if $|\xi| \leq 1/2$ and $\phi_0(\xi) = 0$ if $|\xi| \geq 1$. Then, we define a new function $\psi_0 : \psi_0(\xi) = \phi_0(\xi/2) - \phi_0(\xi)$ (which is supported in a corona). Then, for $j \in \mathbb{Z}$, we define the two distributions $S_j f = \mathcal{F}^{-1}(\phi_0(2^{-j}\xi)\hat{f}(\xi))$ and $\Delta_j f = \mathcal{F}^{-1}(\psi_0(2^{-j}\xi)\hat{f}(\xi))$ and we get the so-called inhomogeneous Littlewood-Paley decompositon of $f \in \mathcal{S}'(\mathbb{R})$ that is for all $K \in \mathbb{Z}$ the following inequality holds in $\mathcal{S}'(\mathbb{R})$

$$f = S_K f + \sum_{j \geq K} \Delta_j f. \tag{2.10}$$

Passing to the limit in equality 2.10 as $K \to -\infty$ in the $\mathcal{S}'(\mathbb{R})$ topology one obtains

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f. \tag{2.11}$$

The equality 2.11 is called homogeneous decomposition of f and is defined modulo polynomials. We are now ready to define the homogeneous weighted Sobolev spaces $\dot{H}^s_w$ for $|s| < 1/2$, they are defined as follows

$$f \in \dot{H}^s_w \Longleftrightarrow f = \sum_{j \in \mathbb{Z}} \Delta_j f, \quad \text{in } \mathcal{S}'(\mathbb{R}) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} 2^{2j} \|\Delta_j f\|^2_{L^2_w} < \infty.$$

We shall use the Bernstein's inequality, that is for all $f \in \mathcal{S}'(\mathbb{R})$ and $(j,s) \in \mathbb{Z} \times \mathbb{R}$, and for all $1 \leq p \leq q \leq \infty$ and all weights $w \in A_\infty$, we have

$$\|\Lambda^s \Delta_j f\|_{L^p_w} \lesssim 2^{js} \|\Delta_j f\|_{L^p_w} \quad \text{also} \quad \|\Delta_j f\|_{L^q_w} \lesssim 2^{j(\frac{1}{p}-\frac{1}{q})} \|\Delta_j f\|_{L^p_w} \text{ and } \|\Lambda^s S_j f\|_{L^p_w} \lesssim 2^{js} \|S_j f\|_{L^p_w}$$

Finally, we recall that for any distributions f and g that are in $\mathcal{S}'(\mathbb{R})$, one has the following paraproduct formula

$$fg = \sum_{q \in \mathbb{Z}} S_{q+1} f \Delta_q g + \sum_{j \in \mathbb{Z}} \Delta_j f S_j g. \tag{2.12}$$

2.7. **Compactness.** Since we look for weak solutions, we use compactness arguments when we pass to the limit in weak formulations.

Lemma 2.5. [30] *Let* X_0, X, X_1 *be reflexive Banach spaces such that*

$$X_0 \subset\subset X \subset X_1,$$

where X_0 *is compactly embedded in* X*. Let* $T > 0$ *be a finite number and let* α_0 *and* α_1 *be two finite numbers such that* $\alpha_i > 1$*. Then,* $Y = \{u \in L^{\alpha_0}(0,T;X_0),\ \partial_t u \in L^{\alpha_1}(0,T;X_1)\}$ *is compactly embedded in* $L^{\alpha_0}(0,T;X)$.

Lemma 2.6 ([9]). *Consider a sequence* $(\theta^\epsilon) \in C([0,T] \times B_R(0))$ *that is uniformly bounded in* $L^\infty([0,T], W^{1,\infty}(B_R(0)))$. *Assume further that the weak derivative* $\frac{d\theta^\epsilon}{dt}$ *is in* $L^\infty([0,T], L^\infty(B_R(0)))$ *(not necessarily uniform) and is uniformly bounded in* $L^\infty([0,T], W_*^{-2,\infty}(B_R(0)))$. *Finally suppose that* $\theta^\epsilon_x \in C([0,T] \times B_R(0))$. *Then there exists a subsequence of* (θ^ϵ) *that converges strongly in* $L^\infty([0,T] \times B_R(0))$.

3. Proof of Theorem 1.1

3.1. A priori estimates.

We first obtain a priori bounds of the equation

$$\theta_t + (\mathcal{H}\theta)\,\theta_x + \delta\theta\Lambda\theta + \Lambda^\gamma\theta = 0, \tag{3.1}$$

We note that by the minimum principle applied to (3.1), we have $\theta(t,x) \geq 0$ for all $t \geq 0$.

To obtain $H^{\frac{1}{2}}$ bound of θ, we begin with the L^2 bound. We multiply (3.1) by θ and integrate over $\mathbb{R}$. Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2 = -\int \left[(\mathcal{H}\theta)\,\theta_x\theta\right] dx - \delta\int\left[\theta^2\Lambda\theta\right] dx = \left(\frac{1}{2}-\delta\right)\int\left[\theta^2\Lambda\theta\right] dx.$$

Since $\theta \geq 0$, we have

$$\int\left[\theta^2\Lambda\theta\right] dx = \int\int \frac{(\theta(x)-\theta(y))^2}{|x-y|^2}\cdot\frac{\theta(x)+\theta(y)}{2}dxdy \geq 0$$

and thus

$$\|\theta(t)\|_{L^2}^2 + 2\int_0^t \left\|\Lambda^{\frac{\gamma}{2}}\theta(s)\right\|_{L^2}^2 ds \leq \|\theta_0\|_{L^2}^2. \tag{3.2}$$

We next estimate θ in $\dot{H}^{\frac{1}{2}}$. We multiply (3.1) by $\Lambda\theta$ and integrate over $\mathbb{R}$:

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2}^2 + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|_{L^2}^2 = -\int\left[(\mathcal{H}\theta)\,\theta_x\Lambda\theta\right] dx - \delta\int\left[\theta\left(\Lambda\theta\right)^2\right] dx.$$

By (2.2), we have

$$-\int\left[(\mathcal{H}\theta)\,\theta_x\Lambda\theta\right] dx = \int\left[\theta\mathcal{H}\left(\theta_x\left(\mathcal{H}\theta_x\right)\right)\right] dx = \frac{1}{2}\int\left[\theta\left((\Lambda\theta)^2 - (\theta_x)^2\right)\right] dx,$$

and hence

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2}^2 + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|_{L^2}^2 = \left(\frac{1}{2}-\delta\right)\int\left[\theta\left(\Lambda\theta\right)^2\right] dx - \frac{1}{2}\int\left[\theta\left(\theta_x\right)^2\right] dx \leq 0,$$

where we use the sign conditions $\theta \geq 0$ and $\delta \geq \frac{1}{2}$. This leads to the inequality

$$\left\|\Lambda^{\frac{1}{2}}\theta(t)\right\|_{L^2}^2 + 2\int_0^t\left\|\Lambda^{\frac{1+\gamma}{2}}\theta(s)\right\|_{L^2}^2 ds \leq \left\|\Lambda^{\frac{1}{2}}\theta_0\right\|_{L^2}^2. \tag{3.3}$$

By (3.2) and (3.3), we obtain that

$$\|\theta(t)\|_{H^{\frac{1}{2}}}^2 + 2\int_0^t\left\|\Lambda^{\frac{\gamma}{2}}\theta(s)\right\|_{H^{\frac{1}{2}}}^2 ds \leq \|\theta_0\|_{H^{\frac{1}{2}}}^2. \tag{3.4}$$

We finally estimate θ in L^1. Since $\theta \geq 0$,

$$\frac{d}{dt}\|\theta\|_{L^1} = \frac{d}{dt}\int\theta dx = (1-\delta)\int\theta\Lambda\theta dx \leq C\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2}^2$$

and thus we conclude that

$$\|\theta(t)\|_{L^1} \leq \|\theta_0\|_{L^1} + C\int_0^t\|\theta(s)\|_{\dot{H}^{\frac{1}{2}}}^2 ds \leq \|\theta_0\|_{L^1} + Ct\,\|\theta_0\|_{H^{\frac{1}{2}}}^2. \tag{3.5}$$

3.2. **Approximation and passing to limit.** We first regularize initial data as $\theta_0^\epsilon = \rho_\epsilon * \theta_0$ where ρ_ϵ is a standard mollifier. We then regularize the equation by putting the Laplacian with the coefficient ϵ:

$$\theta_t^\epsilon + (\mathcal{H}\theta^\epsilon)\,\theta_x^\epsilon + \delta\theta^\epsilon\Lambda\theta^\epsilon + \Lambda^\gamma\theta^\epsilon = \epsilon\theta_{xx}^\epsilon. \tag{3.6}$$

For the proof of the existence of a global-in-time smooth solution, see [18] (Section 6). Moreover, (θ^ϵ) satisfies that

$$\|\theta^\epsilon(t)\|_{L^1} + \|\theta^\epsilon(t)\|_{H^{\frac12}}^2 + 2\int_0^t \left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(s)\right\|_{H^{\frac12}}^2 ds + \epsilon\|\nabla\theta^\epsilon\|_{H^{\frac12}}^2 \le \|\theta_0\|_{L^1} + C(1+t)\|\theta_0\|_{H^{\frac12}}^2 .$$

Therefore, (θ_ϵ) is bounded in $\mathcal{A}_T$ uniformly in $\epsilon > 0$. From this, we have uniform bounds

$$\mathcal{H}\theta^\epsilon \in L^4\left(0,T;L^4\right), \quad \theta^\epsilon \in L^2\left(0,T;L^4\right)$$

and hence

$$\left((\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\right)_x \in L^{\frac43}\left(0,T;H^{-1}\right).$$

Moreover,

$$\Lambda^\gamma\theta^\epsilon + \epsilon\theta_{xx}^\epsilon \in L^2\left(0,T;H^{-1}\right).$$

To estimate $\theta^\epsilon\Lambda\theta^\epsilon$, we use the duality argument. For any $\chi \in L^2\left(0,T;H^2\right)$,

$$\begin{aligned}
|\langle\theta^\epsilon\Lambda\theta^\epsilon,\chi\rangle| &\le \int \left|\widehat{\theta^\epsilon\Lambda\theta^\epsilon}(\xi)\widehat{\chi}(\xi)\right| d\xi \le \int\int \left|\widehat{\theta^\epsilon}(\xi-\eta)\right| |\eta| \left|\widehat{\theta^\epsilon}(\eta)\right| |\widehat{\chi}(\xi)|\, d\eta d\xi \\
&\le \int\int \left|\widehat{\theta^\epsilon}(\xi-\eta)\right| \left(|\eta|^{\frac12}\left(|\xi-\eta|^{\frac12} + |\xi|^{\frac12}\right)\right) \left|\widehat{\theta^\epsilon}(\eta)\right| |\widehat{\chi}(\xi)|\, d\eta d\xi \\
&= \int\int |\xi-\eta|^{\frac12}\left|\widehat{\theta^\epsilon}(\xi-\eta)\right| |\eta|^{\frac12}\left|\widehat{\theta^\epsilon}(\eta)\right| |\widehat{\chi}(\xi)|\, d\eta d\xi \\
&+ \int\int \left|\widehat{\theta^\epsilon}(\xi-\eta)\right| |\eta|^{\frac12}\left|\widehat{\theta^\epsilon}(\eta)\right| |\xi|^{\frac12}|\widehat{\chi}(\xi)|\, d\eta d\xi \\
&\le \left\|\Lambda^{\frac12}\theta^\epsilon\right\|_{L^2}^2 \|\widehat{\chi}\|_{L^1} + \left\|\Lambda^{\frac12}\theta^\epsilon\right\|_{L^2}\left\|\widehat{\theta}\right\|_{L^1}\left\|\Lambda^{\frac12}\chi\right\|_{L^2} \le C\left\|(1+\Lambda)^{\frac12+\frac{\gamma}{4}}\theta^\epsilon\right\|_{L^2}^2 \|\chi\|_{H^2}.
\end{aligned}$$

Since $(1+\Lambda)^{\frac12+\frac{\gamma}{4}}\theta^\epsilon \in L^4\left(0,T;L^2\right)$ uniformly in $\epsilon > 0$, we have

$$\int |\langle\theta^\epsilon\Lambda\theta^\epsilon,\chi\rangle|\, dt \le C\left\|(1+\Lambda)^{\frac12+\frac{\gamma}{4}}\theta^\epsilon\right\|_{L_T^4L^2}^2 \|\chi\|_{L_T^2H^2}.$$

This implies that $\theta^\epsilon\Lambda\theta^\epsilon \in L^2\left(0,T;H^{-2}\right)$. So, we conclude that from the equation of θ_t^ϵ

$$\theta_t^\epsilon \in L^{\frac43}\left(0,T;H^{-2}\right).$$

We now extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{A}_T$ such that

$$\begin{aligned}
&\theta^\epsilon \stackrel{\star}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty\left(0,T;L^p\cap H^{\frac12}\right) \quad \text{for all } p\in(1,\infty), \\
&\theta^\epsilon \rightharpoonup \theta \quad \text{in} \quad L^2\left(0,T;H^{\frac{\gamma+1}{2}}\right), \\
&\theta^\epsilon \to \theta \quad \text{in} \quad L^2\left(0,T;H^{\frac12}\right), \\
&\theta^\epsilon \to \theta \quad \text{in} \quad L^2\left(0,T;L_{\text{loc}}^p\right) \quad \text{for all } p\in(1,\infty)
\end{aligned} \tag{3.7}$$

where we use Lemma 2.5 for the strong convergence.

We now multiply (3.6) by a test function $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R})$ and integrate over $\mathbb{R}$. Then,

$$\int_0^T\int \left[\theta^\epsilon\psi_t + (\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\psi_x + (1-\delta)\Lambda\theta^\epsilon\theta^\epsilon\psi - \theta^\epsilon\Lambda^\gamma\psi + \epsilon\theta^\epsilon\psi_{xx}\right] dxdt = \int \theta_0^\epsilon(x)\psi(0,x)dx\,,$$

which can be rewritten as

$$\begin{aligned}&\int_0^T\int\Big[\theta^\epsilon\psi_t+\underbrace{(\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\psi_x}_{\text{I}}-\theta^\epsilon\Lambda^\gamma\psi+\epsilon\theta^\epsilon\psi_{xx}\Big]dxdt-\int\theta_0^\epsilon(x)\psi(0,x)dx\\&=-(1-\delta)\int_0^T\int\underbrace{\Lambda^{\frac12}\theta^\epsilon\left[\Lambda^{\frac12},\psi\right]\theta^\epsilon}_{\text{II}}\,dxdt-(1-\delta)\int_0^T\int\underbrace{\left|\Lambda^{\frac12}\theta^\epsilon\right|^2\psi}_{\text{III}}\,dxdt.\end{aligned}\tag{3.8}$$

By Lemma 2.5 with

$$X_0=L^2\left(0,T;H^{\frac12}\right),\quad X=L^2\left(0,T;L^2_{\text{loc}}\right),\quad X_1=L^2\left(0,T;H^{-2}\right),$$

we can pass to the limit to I. Moreover, since

$$\left[\Lambda^{\frac12},\psi\right]\theta^\epsilon\to\left[\Lambda^{\frac12},\psi\right]\theta$$

strongly in $L^2\left(0,T;L^6\right)$ by Lemma 2.2 and $\Lambda^{\frac12}\theta^\epsilon$ converges weakly in $L^2\left(0,T;L^2\right)$ by (3.7), we can pass to the limit to II. Lastly, Lemma 2.5 with

$$X_0=L^2\left(0,T;H^{\frac{1+\gamma}{2}}\right),\quad X=L^2\left(0,T;H^{\frac12}_{\text{loc}}\right),\quad X_1=L^2\left(0,T;H^{-2}\right),$$

allows to pass to the limit to III. Combining all the limits together, we obtain that

$$\int_0^T\int\left[\theta\psi_t+(\mathcal{H}\theta)\,\theta\psi_x+(1-\delta)\Lambda\theta\theta^\epsilon\psi\right]dxdt=\int\theta_0(x)\psi(0,x)dx.\tag{3.9}$$

3.3. **Uniqueness when $\gamma=1$.** To show the uniqueness of a weak solution, let $\theta=\theta_1-\theta_2$. Then, θ satisfies the following equation:

$$\theta_t+\Lambda\theta=-\left(\mathcal{H}\theta\right)\theta_{1x}-\left(\mathcal{H}\theta_2\right)\theta_x-\delta\theta\Lambda\theta_1-\delta\theta_2\Lambda\theta,\quad\theta(0,x)=0.\tag{3.10}$$

We multiply θ to (3.10) and integrate over $\mathbb{R}$. Then,

$$\frac12\frac{d}{dt}\|\theta\|^2_{L^2}+\left\|\Lambda^{\frac12}\theta\right\|^2_{L^2}=\int\left[-\left(\mathcal{H}\theta\right)\theta_{1x}-\left(\mathcal{H}\theta_2\right)\theta_x-\delta\theta\Lambda\theta_1-\delta\theta_2\Lambda\theta\right]\theta dx.$$

The first three terms in the right-hand side are easily bounded by

$$C\left\|\theta_{1x}\right\|_{L^2}\|\theta\|^2_{L^4}+C\left\|\theta_{2x}\right\|_{L^2}\|\theta\|^2_{L^4}.$$

Moreover, the last term is bounded by using Lemma 2.1

$$-\delta\int\theta_2\theta\Lambda\theta dx\le-\frac{\delta}{2}\int\theta_2\Lambda\theta^2dx=-\frac{\delta}{2}\int\theta^2\Lambda\theta_2dx\le C\left\|\theta_{2x}\right\|_{L^2}\|\theta\|^2_{L^4}.$$

Hence we derive that

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2_{L^2}+\left\|\Lambda^{\frac12}\theta\right\|^2_{L^2}&\le C\left\|\theta_{1x}\right\|_{L^2}\|\theta\|^2_{L^4}+C\left\|\theta_{2x}\right\|_{L^2}\|\theta\|^2_{L^4}\\&\le C\left(\left\|\theta_{1x}\right\|^2_{L^2}+\left\|\theta_{2x}\right\|^2_{L^2}\right)\|\theta\|^2_{L^2}+\frac12\left\|\Lambda^{\frac12}\theta\right\|^2_{L^2}.\end{aligned}$$

Since

$$\theta_{1x}\in L^2\left(0,T:L^2\right),\quad\theta_{2x}\in L^2\left(0,T:L^2\right)$$

when $\gamma=1$, we conclude that $\theta=0$ in L^2 and thus a weak solution is unique. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

4.1. A priori estimate. We consider the equation

$$\theta_t + (\mathcal{H}\theta)\,\theta_x + \delta\theta\Lambda\theta + \Lambda\theta = 0. \tag{4.1}$$

Since (4.1) satisfies the minimum and maximum principles, we have

$$\theta(t,x) \geq 0, \quad \|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

We begin with the $L^2(w_\beta dx)$ bound. For notational simplicity, we suppress the dependence on β. We multiply (4.1) by θw and integrate in x. Then,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{1/2}\theta\right\|^2_{L^2(wdx)} &= -\int (\mathcal{H}\theta)\,\theta_x\theta w dx - \delta\int \theta\,(\Lambda\theta)\,\theta w dx - \int \Lambda^{1/2}\theta\left[\Lambda^{1/2}, w\right]\theta dx \\
&= -\frac{1}{2}(\mathcal{H}\theta)\left(\theta^2\right)_x w dx - \delta\int \theta^2\,(\Lambda\theta)\,w dx - \int \Lambda^{1/2}\theta\left[\Lambda^{1/2}, w\right]\theta dx \\
&= \left(\frac{1}{2}-\delta\right)\int \theta^2\,(\Lambda\theta)\,w dx + \frac{1}{2}\int (\mathcal{H}\theta)\,\theta^2 w_x dx - \int \Lambda^{1/2}\theta\left[\Lambda^{1/2}, w\right]\theta dx \\
&\leq C(\|\theta_0\|_{L^\infty},\delta)\left(\|\sqrt{w}\theta\|_{L^2}\|\sqrt{w}\Lambda\theta\|_{L^2} + \|\sqrt{w}\theta\|^2_{L^2}\right) + \int \sqrt{w}|\theta|\,\frac{1}{\sqrt{w}}\left|\left[\Lambda^{1/2}, w\right]\theta\right|\,dx
\end{aligned}$$

Then, using lemma 2.3 one obtains, for $C(\|\theta_0\|_{L^\infty},\delta) = C_1 = (1+\delta)\|\theta_0\|_{L^\infty}$ for all $\eta_1 > 0$ and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} &\leq \left(\|\theta\|_{L^2(wdx)}\|\Lambda\theta\|_{L^2_w} + \|\theta\|^2_{L^2_w}\right)C_1 + \|\theta\|^2_{L^2(wdx)} \\
&\leq \left(\frac{C_1}{2\eta_1} + C_1 + 1\right)\|\theta\|^2_{L^2(wdx)} + \frac{C_1\eta_1}{2}\|\Lambda\theta\|^2_{L^2_w}
\end{aligned}$$

In particular,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} \leq \left(\frac{C_1}{2\eta_1} + C_1 + 1\right)\|\theta\|^2_{L^2(wdx)} + \frac{C_1\eta_1}{2}\|\Lambda\theta\|^2_{L^2_w} \tag{4.2}$$

We next multiply (4.1) by $\Lambda^{\frac{1}{2}}\left(w\Lambda^{\frac{1}{2}}\theta\right)$ and integrate in x. We first focus one the term $\delta\theta\Lambda\theta$, the resulting term in the computation of the evolution of the $\dot{H}^{1/2}(wdx)$ norm is

$$\begin{aligned}
T_\delta\theta &= \delta\int \Lambda^{1/2}(w\Lambda^{1/2}\theta)\theta\Lambda\theta\ dx \\
&= \delta\int \theta\sqrt{w}\Lambda\theta\,\frac{1}{\sqrt{w}}\left[\Lambda^{1/2}, w\right]\Lambda^{1/2}\theta\ dx + \int \theta\ w|\Lambda\theta|^2\ dx \\
&\leq \delta\|\theta_0\|_{L^\infty}\left(\|\theta\|_{\dot{H}^1(wdx)}\|\theta\|_{\dot{H}^{1/2}(wdx)} + \|\theta\|^2_{\dot{H}^1(wdx)}\right)
\end{aligned}$$

hence

$$|T_\delta\theta| \leq \left(\frac{\eta_2}{2}+1\right)C_2\|\theta\|^2_{\dot{H}^1(wdx)} + \frac{2C_2}{\eta_2}\|\theta\|^2_{\dot{H}^{1/2}(wdx)}, \tag{4.3}$$

where $C_2 = \delta\|\theta_0\|_{L^\infty}$.

For the transport part (which is the case $\delta = 0$), we only need to use inequality 4.4 of [18], indeed, it is shown that, for some constants $C_3 > 0, C_4 > 0$ one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\dot{H}^{1/2}(wdx)}\ dx &\leq (C_3\|\theta_0\|_\infty - 1)\|\theta\|^2_{\dot{H}^1(wdx)} \\
&+ C_4\left(\int \theta^2\ w\ dx + \int |\Lambda^{1/2}\theta|^2\ \ w\ dx\right).
\end{aligned} \tag{4.4}$$

Hence, we get the following control for the full equation 4.1 by collecting the estimates 4.2, 4.3 and 4.5

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{H^{1/2}(wdx)} \leq & \left(C_3\|\theta_0\|_\infty + \left(\frac{\eta_2}{2}+1\right)C_1 + \frac{C_1\eta_1}{2} - 1\right)\|\theta\|^2_{\dot{H}^1(wdx)} \\ & + (C_4 + \frac{2C_2}{\eta_2})\|\theta\|^2_{\dot{H}^{1/2}(wdx)}\end{aligned} \tag{4.5}$$

Hence, choosing η_1, η_2 and $\|\theta_0\|_\infty$ small enough (for instance less than $\frac{1}{1+\delta}\frac{1}{100}$ so that C_1 would be also small), one gets

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{H^{1/2}(wdx)}\ dx + C_6\|\theta\|^2_{\dot{H}^1(w)} \leq C_7\|\theta\|^2_{\dot{H}^{1/2}(wdx)}. \tag{4.6}$$

Hence, integrating in time $s \in [0,T]$ and using Gronwall's inequality one obtains that for all finite $T > 0$

$$\|\theta(T)\|^2_{H^{1/2}(wdx)} + \|\theta\|^2_{L^2([0,T],\dot{H}^1(wdx))} \leq \|\theta_0\|^2_{H^{1/2}(wdx)} \exp\left(C_2\|\theta_0\|_{L^\infty}T\right). \tag{4.7}$$

4.2. **Approximation and passing to limit.** To show the existence of a weak solution in $\mathcal{D}_T$, we first approximate the initial data θ_0. Let χ be a smooth positive function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Let $\chi_R(x) = \chi(x/N)$, $N \in \mathbb{N}$, and consider truncated initial data $\theta_0^N(x) = \theta_0(x)\chi_N(x)$. Then, a direct computation shows that

$$\lim_{N\to\infty} \left\|\theta_0^N - \theta_0\right\|_{H^{\frac{1}{2}}(wdx)} = 0.$$

Moreover, this truncation does not alter the non-negativity and does not increase the L^∞ norm. So, if $\|\theta_0\|_{L^\infty}$ is sufficiently small, there is a global-in-time solution of

$$\partial_t\theta^N + \mathcal{H}\theta^N\partial_x\theta^N + \delta\theta^N\Lambda\theta^N + \Lambda\theta^N = 0, \quad \theta^N(0,x) = \theta_0^N(x). \tag{4.8}$$

From the a priori estimates, the sequence (θ^N) is bounded in

$$L^\infty([0,T], H^{\frac{1}{2}}(wdx)) \cap L^2([0,T], H^1(wdx))$$

uniformly with respect to N. We now take a test function $\psi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R})$. Then, $\psi\theta^N$ is bounded in $L^2([0,T], H^1)$. Moreover, since $\theta^N \in L^\infty([0,T] \times \mathbb{R})$

$$\begin{aligned}&\psi\left(\mathcal{H}\theta^N\partial_x\theta^N + \delta\theta^N\Lambda\theta^N + \Lambda\theta^N\right) \\ &= (\psi\mathcal{H}\theta^N\theta^N)_x - \psi_x\mathcal{H}\theta^N\theta^N + (\delta-1)\psi\theta^N\Lambda\theta^N + \psi\Lambda\theta^N \in L^2([0,T], H^{-1}).\end{aligned}$$

By Lemma 2.5, we can pass to the limit to the weak formulation,

$$\int_0^T\int\left[\theta^N\psi_t + \left(\mathcal{H}\theta^N\right)\theta^N\psi_x + (1-\delta)\Lambda\theta^N\theta^N\psi - \theta^N\Lambda\psi\right]dxdt = \int\theta_0^N(x)\psi(0,x)dx,$$

to obtain a weak solution θ which is also in

$$L^\infty([0,T], H^{\frac{1}{2}}(wdx)) \cap L^2([0,T], H^1(wdx)).$$

4.3. **Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of $\theta = \theta_1 - \theta_2$ given by

$$\theta_t + \Lambda\theta = -\left(\mathcal{H}\theta\right)\theta_{1x} - \left(\mathcal{H}\theta_2\right)\theta_x - \delta\theta\Lambda\theta_1 - \delta\theta_2\Lambda\theta, \quad \theta(0,x) = 0. \tag{4.9}$$

We multiply $w\theta$ to (4.9) and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} = & \int\left[-\left(\mathcal{H}\theta\right)\theta_{1x} - \left(\mathcal{H}\theta_2\right)\theta_x - \delta\theta\Lambda\theta_1 - \delta\theta_2\Lambda\theta\right]\theta wdx \\ & - \int\Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx.\end{aligned}$$

As before, the last term is bounded by

$$\int \Lambda^{\frac{1}{2}}\theta \left[\Lambda^{\frac{1}{2}}, w\right] \theta dx \leq \frac{1}{2}\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}^2 + C\left\|\theta\right\|_{L^2(wdx)}^2.$$

The first three terms in the right-hand side are easily bounded by

$$C\left(\|\theta_{1x}\|_{L^2(wdx)} + \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)}\right)\|\theta\|_{L^4(wdx)}^2.$$

Moreover, since $\delta > 0$, $\theta_2 \geq 0$ and $w \geq 0$, the fourth term is bounded by using Lemma 2.1

$$\begin{aligned}-\delta\int \theta_2\theta\Lambda\theta w dx &\leq -\frac{\delta}{2}\int \theta_2 w\Lambda\theta^2 dx = -\frac{\delta}{2}\int \theta^2\Lambda(\theta_2 w)dx = \frac{\delta}{2}\int \mathcal{H}(\theta^2)(\theta_2 w)_x dx\\ &\leq C\left(\|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)}\right)\|\theta\|_{L^4(wdx)}^2.\end{aligned}$$

Hence we obtain that

$$\begin{aligned}&\frac{d}{dt}\|\theta\|_{L^2(wdx)}^2 + \left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}^2\\ &\leq C\left(\|\theta_{1x}\|_{L^2(wdx)} + \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)}\right)\|\theta\|_{L^4(wdx)}^2 + C\|\theta\|_{L^2(wdx)}^2\\ &\leq C\left(1 + \|\theta_{1x}\|_{L^2(wdx)}^2 + \|\theta_{2x}\|_{L^2(wdx)}^2 + \|\theta_2\|_{L^2(wdx)}^2\right)\|\theta\|_{L^2(wdx)}^2 + \frac{1}{2}\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}^2,\end{aligned}$$

where we use (2.9) to obtain the last inequality. Since

$$\theta_{1x} \in L^2\left(0, T : L^2(wdx)\right), \quad \theta_{2x} \in L^2\left(0, T : H^1(wdx)\right),$$

we conclude that $\theta = 0$ in $L^2(wdx)$ and thus a weak solution is unique. This completes the proof of Theorem 1.2.

In the two next subsections we shall prove a global existence theorem and a local existence theorem, for respectively the subcritical case and the supercritical case), we shall just focus on the *a priori* estimates since the construction by compactness is classical (see [17]).

5. Proof of Theorem 1.3

5.1. **A priori estimate.** Since the case $\delta = 0$ has been already done in [17], one observes that it suffices to estimate the $L^2(wdx)$ part

$$T_1 = \delta\int w_{\beta,k}\theta^2\Lambda\theta \; dx,$$

and the weighted homogeneous Sobolev part, that is

$$T_2 = \delta\int w_{\beta,k}\Lambda^s\theta \; \Lambda^s(\theta\Lambda\theta) \; dx.$$

We shall control T_1 and T_2, for the sake of readibility, we shall just write w in stead of $w_{\beta,k}$. We first control the L^2 part

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2(wdx)}^2 + \left\|\Lambda^{\alpha/2}\theta\right\|_{L^2(wdx)}^2 &= -\int(\mathcal{H}\theta)\,\theta_x\theta w dx - \delta\int\theta\,(\Lambda\theta)\,\theta w dx - \int \Lambda^{\alpha/2}\theta\left[\Lambda^{\alpha/2}, w\right]\theta dx\\ &= -\frac{1}{2}\int(\mathcal{H}\theta)\left(\theta^2\right)_x w dx - \delta\int\theta^2\,(\Lambda\theta)\,w dx - \int\Lambda^{\alpha/2}\theta\left[\Lambda^{\alpha/2}, w\right]\theta dx\\ &= \left(\frac{1}{2} - \delta\right)\int\theta^2\,(\Lambda\theta)\,w dx + \frac{1}{2}\int(\mathcal{H}\theta)\,\theta^2 w_x dx - \int\Lambda^{\alpha/2}\theta\left[\Lambda^{\alpha/2}, w\right]\theta dx\\ &\leq C(\|\theta_0\|_{L^\infty}, \delta)\left(\|\sqrt{w}\theta\|_{L^2}\|\sqrt{w}\Lambda\theta\|_{L^2} + \|\sqrt{w}\theta\|_{L^2}^2\right) + \int\sqrt{w}|\Lambda^{\alpha/2}\theta|\,\frac{1}{\sqrt{w}}\left|\left[\Lambda^{\alpha/2}, w\right]\theta\right| \; dx\end{aligned}$$

Then, using lemma 2.3 one obtains, for $C(\|\theta_0\|_{L^\infty},\delta)=C_1=(1+\delta)\|\theta_0\|_{L^\infty}$ for all $\eta_1>0$ and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)}+\left\|\Lambda^{\frac{\alpha}{2}}\theta\right\|^2_{L^2(wdx)} &\leq \left(\|\theta\|_{L^2(wdx)}\|\Lambda\theta\|_{L^2_w}+\|\theta\|^2_{L^2(wdx)}\right)C_1+\|\theta\|_{L^2(wdx)}\|\theta\|_{\dot H^{\alpha/2}(wdx)}\\ &\leq C_2\|\theta\|^2_{H^1(wdx)}+\frac{1}{100}\|\theta\|^2_{\dot H^{\alpha/2}(wdx)}\end{aligned}$$

Now, we study the contribution coming from the homogeneous Sobolev part $\dot H^1(w)$. The transport part (corresponding to $\delta=0$) has been treated in [17], indeed, it is shown that

$$\frac{1}{2}\partial_t\|\theta\|^2_{H^1(wdx)}\leq C\|\theta\|^2_{H^1(wdx)}.$$

Therefore, it just remains to estimate

$$I_\delta\theta=-\delta\int w\Lambda\theta\Lambda(\theta\Lambda\theta)\ dx$$

In order to take advantage of the dissipation, we start by rewritting I_δ as follows,

$$\begin{aligned}I_\delta\theta &= -\delta\int\Lambda^\mu(w\Lambda\theta)\Lambda^{1-\mu}(\theta\Lambda\theta)\ dx\\ &= -\int\sqrt{w}\Lambda^{1-\mu}(\theta\Lambda\theta)\frac{1}{\sqrt{w}}[\Lambda^\mu,w]\Lambda\theta\ dx-\int\sqrt{w}\Lambda^{1+\mu}\theta\sqrt{w}\Lambda^{1-\mu}(\theta\Lambda\theta)\ dx\\ &= I_1+I_2\end{aligned}$$

In order to estimate these two terms, we shall use the weighted Littlewood-Paley decomposition. We start with I_1, we have, by using the commutator lemma 2.3 that

$$|I_1|\leq\|\theta\Lambda\theta\|_{\dot H^{1-\mu}(wdx)}\|\theta\|_{\dot H^1(wdx)},$$

and then we estimate the right hand side by using the paraproduct formula (2.12) and we get

$$\|\theta\Lambda\theta\|_{\dot H^{1-\mu}(wdx)}\leq\sum_{q\in\mathbb{Z}}2^{q(1-\mu)}\|S_{q+1}\theta\Delta_q\mathcal{H}\theta_x\|_{L^2(wdx)}+\sum_{j\in\mathbb{Z}}2^{j(1-\mu)}\|\Delta_j\theta S_j\mathcal{H}\theta_x\|_{L^2(wdx)}.$$

The first sum is controlled as follows

$$\begin{aligned}\sum_{q\in\mathbb{Z}}2^{q(1-\mu)}\|S_{q+1}\theta_x\ \Delta_q\mathcal{H}\theta_x\|_{L^2(wdx)} &\leq \sum_{q\in\mathbb{Z}}2^{q(1-\mu)}\|S_{q+1}\theta\|_{L^\infty}\|\Delta_q\mathcal{H}\theta_x\|_{L^2(wdx)}\\ &\leq \|\theta\|_{L^\infty}\sum_{q\in\mathbb{Z}}2^{(1-\mu)q}\|\Delta_q\theta_x\|_{L^2(wdx)}\\ &\leq \|\theta\|_{L^\infty}\|\theta_x\|_{\dot H^{1-\mu}(wdx)}\\ &\leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot H^{2-\mu}(wdx)}\end{aligned}$$

where we used Berstein's inequality and the continuity of the Hilbert transform on $L^2(wdx)$.

The other sum has to be treated in a different manner since we do not want to put L^∞ in the term $S_j\mathcal{H}\theta_x$. For instance, we can use Hölder's inequality and then Bernstein's inequality to recover some sufficiently nice Sobolev norm. We obtain,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} 2^{j(1-\mu)}\|\Delta_j\theta S_j\mathcal{H}\theta_x\|_{L^2(w)} &\leq \sum_{j\in\mathbb{Z}} 2^{j(1-\mu)}\|\Delta_j\theta\|_{L^p(w)}\|S_j\mathcal{H}\theta_x\|_{L^q(wdx)} \\
&\leq C\sum_{j\in\mathbb{Z}} 2^{j(1-\mu)}2^{j(\frac{1}{2}-\frac{1}{p})}\|\Delta_j\theta\|_{L^2(w)}\|\mathcal{H}\theta_x\|_{L^q(wdx)} \\
&\leq C\sum_{j\in\mathbb{Z}} 2^{j(1-\mu)}2^{j(\frac{1}{2}-\frac{1}{p})}\|\Delta_j\theta\|_{L^2(w)}2^j\|\mathcal{H}\theta\|_{L^q(wdx)} \\
&\leq C\|\theta\|_{L^q(wdx)}\sum_{j\in\mathbb{Z}} 2^{j(2-\mu+\frac{1}{q})}\|\Delta_j\theta\|_{L^2(wdx)} \\
&\leq C\|\theta_0\|_{L^\infty}^{1-2/q}\|\theta\|_{L^2(w)}^{2/q}\|\theta\|_{\dot{H}^{2-\mu+\frac{1}{q}}(wdx)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|I_1| &\leq \|\theta\Lambda\theta\|_{\dot{H}^{1-\mu}(wdx)}\|\theta\|_{\dot{H}^1(wdx)} \\
&\leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot{H}^{2-\mu}(wdx)}\|\theta\|_{\dot{H}^1(wdx)} + C\|\theta\|_{\dot{H}^1(wdx)}\|\theta_0\|_{L^\infty}^{1-2/q}\|\theta\|_{L^2(w)}^{2/q}\|\theta\|_{\dot{H}^{2-\mu+\frac{1}{q}}(wdx)}
\end{aligned}$$

For I_2, following I_1, one gets

$$|I_2| \leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot{H}^{1+\mu}(wdx)}\|\theta\|_{\dot{H}^{2-\mu}(w)} + \|\theta_0\|_{L^\infty}^{1-2/q}\|\theta\|_{\dot{H}^{1+\mu}(wdx)}\|\theta\|_{L^2(wdx)}^{2/q}\|\theta\|_{\dot{H}^{2-\mu+\frac{1}{q}}(wdx)}.$$

The idea is then to choose q very small. Indeed, the associated Lebesgue space L^q would be then a good substitute for L^∞. It is worth recalling that this latter space is not well suited to estimate $\mathcal{H}\theta$ since it does not map L^∞ to L^∞ but L^∞ to BMO.

Then, one has to choose μ close enough to $1/2$ because we control $H^{1+\frac{\alpha}{2}}(wdx)$ with $\alpha\in(1,2)$. A good choice is to take for instance $\mu=\frac{1}{2}+\frac{1}{q}$, this choice allows us to interpolate and conclude the estimates. Indeed, for all $\delta\in(0,1)$

$$\|\theta\|_{\dot{H}^{2+\frac{1}{q}-\mu}(wdx)} \leq \|\theta\|_{\dot{H}^{2+\frac{1}{q}-\mu-\delta}(wdx)}^{1-\delta}\|\theta\|_{\dot{H}^{3+\frac{1}{q}-\mu-\delta}(wdx)}^{\delta},$$

Then, one observes that by choosing δ very close to 1 (essentially for the worst term, which is the second estimate), we get

$$\|\theta\|_{\dot{H}^{2+\frac{1}{q}-\mu-\delta}(wdx)}^{1-\delta} = \|\theta\|_{\dot{H}^{3/2-\delta}(wdx)}^{1-\delta} \leq \|\theta\|_{H^1(wdx)}^{1-\delta}$$

and if we choose $\delta=1-\epsilon$, with $\epsilon>0$ is chosen small enough so that $\frac{1}{2}+\epsilon\leq\frac{\alpha}{2}$, this is possible since $\alpha\in(1,2)$.

$$\|\theta\|_{\dot{H}^{3+\frac{1}{q}-\mu-\delta}(wdx)}^{\delta} = \|\theta\|_{\dot{H}^{3/2+\epsilon}(wdx)}^{\delta} \leq \|\theta\|_{H^{1+\frac{\alpha}{2}}(wdx)}^{\delta}$$

hence,

$$\|\theta\|_{\dot{H}^{2+\frac{1}{q}-\mu}(wdx)} \leq \|\theta\|_{H^1}^{1-\delta}\|\theta\|_{H^{1+\frac{\alpha}{2}}(wdx)}^{\delta} \tag{5.1}$$

Therefore, since $\frac{1}{2}+\frac{1}{q}\leq\frac{\alpha}{2}$, one obtains

$$|I_1| \lesssim \|\theta\|_{H^{\frac{1}{2}+\frac{\alpha}{2}}(wdx)}\|\theta\|_{\dot{H}^1(wdx)} + \|\theta\|_{L^2(wdx)}^{2/q}\|\theta\|_{H^1(wdx)}^{2-\delta}\|\theta\|_{H^{1+\frac{\alpha}{2}}(wdx)}^{\delta} \tag{5.2}$$

Then, using Young's inequality with $p_1=\frac{2-\nu}{2-\delta}$ and its conjugate $p_2=\frac{2-\nu}{\delta-\nu}$, with $0<\nu<\delta$ where ν will be chosen later. One gets, for all $\epsilon_1>0$

$$|I_1| \lesssim \frac{\epsilon_1}{2}\|\theta\|_{H^{\frac{1}{2}+\frac{\alpha}{2}}(wdx)}^2 + \frac{1}{2\epsilon_1}\|\theta\|_{\dot{H}^1(wdx)}^{2-\nu} + \frac{1}{2\epsilon_1}\|\theta\|_{L^2(w)}^{\frac{2}{q}\frac{2-\nu}{2-\delta}}\|\theta\|_{H^1(wdx)}^{2-\nu} + \frac{\epsilon_1}{2}\|\theta\|_{H^{1+\frac{\alpha}{2}}(wdx)}^{\frac{\delta}{2-\delta}(2-\nu)}$$

And once again, we use Young's inequality with $p=\frac{2}{2-\nu}$ hence $q=\frac{\nu}{2}$

$$|I_1| \lesssim \frac{\epsilon_1}{2}\|\theta\|^2_{H^{\frac{1}{2}+\frac{\alpha}{2}}(wdx)}+\frac{1}{2\epsilon_1}\|\theta\|^2_{\dot{H}^1(wdx)}+\frac{C_1(\delta,\nu)}{2\epsilon_1}\|\theta\|^{\frac{4}{q}\frac{1}{2-\delta}}_{L^2(wdx)}+\frac{C_2(\delta,\nu)}{2\epsilon_1}\|\theta\|^{(2-\nu)\frac{\nu}{2}}_{H^1(wdx)}+\frac{\epsilon_1}{2}\|\theta\|^{\frac{\delta}{2-\delta}(2-\nu)}_{H^{1+\frac{\alpha}{2}}(wdx)}$$

Since q is chosen large, then $\frac{4}{q}\frac{1}{2-\delta}\leq 2$, we also have that $(2-\nu)\frac{\nu}{2}\leq 2$ (this is a consequence of $(\nu-2)^2\geq 0$). We can choose any value of $\nu\in(0,\delta)$ so that $\frac{\delta}{2-\delta}(2-\nu)\leq 2$. For instance, if $\nu=1/2$ then we obviously have $\frac{\delta}{2-\delta}\leq 4/3$ for all $\delta\in(0,1)$ (importantly, there is no restriction on δ, the previous estimates hold for all $\delta\in(0,1)$, we shall use this fact to control the second term, see 5.4 below).

One finally obtains

$$|I_1|\lesssim \epsilon_1\|\theta\|^2_{H^{1+\frac{\alpha}{2}}(wdx)}+\frac{2}{\epsilon_1}\|\theta\|^2_{H^1(wdx)} \tag{5.3}$$

For I_2, we have seen that

$$|I_2|\leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot{H}^{1+\mu}(wdx)}\|\theta\|_{\dot{H}^{2-\mu}(wdx)}+\|\theta_0\|^{1-2/q}_{L^\infty}\|\theta\|_{\dot{H}^{1+\mu}(wdx)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|_{\dot{H}^{2-\mu+\frac{1}{q}}(wdx)}.$$

Since $\mu=\frac{1}{2}+\frac{1}{q}$ and q is such that $\frac{1}{2}+\frac{1}{q}\leq\frac{\alpha}{2}$ then by using 5.1, the latter inequality becomes

$$\begin{aligned}
|I_2| &\leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot{H}^{\frac{3}{2}+\frac{1}{q}}(wdx)}\|\theta\|_{\dot{H}^{\frac{3}{2}-\frac{1}{q}}(wdx)}+\|\theta_0\|^{1-2/q}_{L^\infty}\|\theta\|_{\dot{H}^{\frac{3}{2}+\frac{1}{q}}(w)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|^{1-\delta}_{H^1(wdx)}\|\theta\|^{\delta}_{H^{1+\frac{\alpha}{2}}(wdx)}\\
&\leq \|\theta_0\|_{L^\infty}\|\theta\|_{\dot{H}^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|_{\dot{H}^{\frac{3}{2}-\frac{1}{q}}(w)}+\|\theta_0\|^{1-2/q}_{L^\infty}\|\theta\|_{\dot{H}^{\frac{3}{2}+\frac{1}{q}}(wdx)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|^{1-\delta}_{H^1(wdx)}\|\theta\|^{\delta}_{H^{1+\frac{\alpha}{2}}(wdx)}\\
&\lesssim \|\theta\|^{2-\frac{1}{q}}_{H^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|^{\frac{1}{q}}_{H^1(wdx)}+\|\theta\|^{1+\delta}_{H^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|^{1-\delta}_{H^1(wdx)}
\end{aligned}$$

where, in the last step, we used the following interpolation inequality

$$\|\theta\|_{H^{\frac{3}{2}-\frac{1}{q}}(wdx)}\leq\|\theta\|^{1-\frac{1}{q}}_{H^{\frac{3}{2}}(wdx)}\|\theta\|^{\frac{1}{q}}_{H^{\frac{1}{2}}(wdx)},$$

Then, we use Young's inequality with $p_3=2q$ and its conjugate $p_4=\frac{2q}{2q-1}$ in the first product

$$\|\theta\|^{2-\frac{1}{q}}_{H^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|^{\frac{1}{q}}_{H^1(wdx)} \lesssim \epsilon_1\|\theta\|^2_{H^{1+\frac{\alpha}{2}}(wdx)}+\frac{1}{\epsilon_1}\|\theta\|^2_{H^1(wdx)}$$

it remains to estimate

$$L=\|\theta\|^{1+\delta}_{H^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|^{1-\delta}_{H^1(wdx)}$$

but $\delta=1-\epsilon$, so that

$$L=\|\theta\|^{2-\epsilon}_{H^{1+\frac{\alpha}{2}}(wdx)}\|\theta\|^{2/q}_{L^2(wdx)}\|\theta\|^{\epsilon}_{H^1(wdx)} \tag{5.4}$$

but, we have already controlled the same term with $\delta>0$ in stead of $\epsilon>0$ (see the second term in the right hand side of 5.2), therefore following the same steps, one arrives at the same conclusion at in I_1 that is to say, for $i=1,2$, one has

$$|I_i|\lesssim\epsilon_1\|\theta\|^2_{H^{1+\frac{\alpha}{2}}(wdx)}+\frac{2}{\epsilon_1}\|\theta\|^2_{H^1(wdx)} \tag{5.5}$$

Hence, to conclude it suffices to choose ϵ_1 sufficiently small.

□

6. Proof of Theorem 1.4

The control of the $L^2(w)$ norm is straightforward, indeed, using the lemma 2.3 along with the weighted Sobolev embedding $\dot{H}^{1/6}(w) \hookrightarrow L^3(w)$, one obtains

$$\begin{aligned}
\partial_t \int \frac{\theta^2}{2} w \, dx + \int |\Lambda^{\alpha/2}\theta|^2 w \, dx &= -\int \theta^2 \Lambda\theta w \, dx + \int \theta^2 \mathcal{H}\theta w_x \, dx + \delta \int w\theta^2 \Lambda\theta \, dx \\
&\quad - \int \Lambda^{\alpha/2}\theta [\Lambda^{\alpha/2}, w]\theta \, dx \\
&\lesssim \|\theta\|^2_{\dot{H}^{1/6}(wdx)} \|\theta\|_{\dot{H}^{7/6}(wdx)} + \|\theta\|^3_{\dot{H}^{1/6}(wdx)} + \|\theta\|^2_{H^{\alpha/2}(wdx)} \lesssim \|\theta\|^3_{H^2(wdx)}
\end{aligned}$$

The control of the homogeneous part $\dot{H}^2(wdx)$ is done as follows.

$$\begin{aligned}
\frac{1}{2}\partial_t \|\theta\|^2_{H^2(wdx)} + \int |\Lambda^{2+\frac{\alpha}{2}}\theta|^2 \, w \, dx &= -\frac{1}{2}\int w_x (\theta_{xx})^2 \mathcal{H}\theta - \frac{3}{2}\int w(\theta_{xx})^2 \Lambda\theta + \int w\theta_{xx}\theta_x \Lambda\theta_x \\
&\quad - \int \Lambda^{2+\frac{\alpha}{2}}\theta [\Lambda^{\alpha/2}, w]\theta_{xx} \\
&\quad - \delta \int w(\theta_{xx})^2 \Lambda\theta \, dx - 2\delta \int w\theta_{xx}\theta_x \Lambda\theta_x \, dx - \delta \int w\theta\theta_{xx}\Lambda\theta_{xx} \, dx \\
&= \sum_{j=1}^{7} I_j
\end{aligned}$$

We already know from [17] that

$$\left|\sum_{j=1}^{4} I_j\right| \lesssim \frac{1}{\epsilon_1^4}\|\theta\|^4_{H^2(wdx)} + \epsilon_1 \|\theta\|^{\frac{16}{3}}_{H^2(wdx)} + \epsilon_1 \|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)}$$

where we crucially used the first point of lemma 2.3 to estimate I_4. Futhermore, one observes that I_5 and I_6 are the same as respectively I_2 and I_3, hence

$$\left|\sum_{j=1}^{6} I_j\right| \lesssim \frac{1}{\epsilon_1^4}\|\theta\|^4_{H^2(wdx)} + \epsilon_1 \|\theta\|^{\frac{16}{3}}_{H^2(wdx)} + \epsilon_1 \|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)} \tag{6.1}$$

The most singular term is I_7. Since $\delta \geq 0$ and $\theta \geq 0$ we may use the pointwise inequality 2.1 and then by integrating by parts we get

$$\begin{aligned}
I_7 &= -\delta \int w\theta\theta_{xx}\Lambda\theta_{xx} \, dx \\
&\leq -\delta \int w\theta \Lambda((\theta_{xx})^2) \, dx \\
&= -\delta \int \Lambda(w\theta) \, (\theta_{xx})^2 \, dx
\end{aligned}$$

At this step, it is important to note that one cannot take advantage of the L^∞ maximum principle, indeed, this would imply to estimate the remaning term in $L^1(w)$ and it is well-known that the Hilbert transform is not continuous on that space. The idea here is to write the term $\Lambda(w\theta)$ as a controlled commutator (via lemma 2.3) plus another term which is easy to control since the weight will be outside the differential term. More precisely, we write

$$\begin{aligned} I_7 &= -\delta\int(\theta_{xx})^2[\Lambda,w]\theta\ dx-\delta\int w(\theta_{xx})^2\Lambda\theta\ dx\\ &= I_{7,1}+I_{7,2}\end{aligned}$$

To estimate $I_{7,1}$ we use Hölder's inequality with $\frac{\alpha}{2}+\frac{1-\alpha}{2}=\frac{1}{2}$, and the two following weighted Sobolev embeddings $\dot{H}^{\alpha/2}(wdx)\hookrightarrow L^{\frac{2}{1-\alpha}}(wdx)$ and $\dot{H}^{\frac{1-\alpha}{2}}(wdx)\hookrightarrow L^{\frac{2}{\alpha}}(wdx)$, and the last point of lemma 2.3, one gets

$$\begin{aligned} |I_{7,1}| &\leq \|\theta_{xx}\|_{L^2(wdx)}\|\theta_{xx}\|_{L^{\frac{2}{1-\alpha}}(wdx)}\left\|\frac{1}{w}[\Lambda,w]\theta\right\|_{L^{\frac{2}{\alpha}}(wdx)}\\ &\leq \|\theta\|_{\dot{H}^2(wdx)}\|\theta_{xx}\|_{\dot{H}^{\alpha/2}(wdx)}\|\theta\|_{L^{\frac{2}{\alpha}}(wdx)}\\ &\leq \|\theta\|_{\dot{H}^2(wdx)}\|\theta\|_{\dot{H}^{2+\alpha/2}(wdx)}\|\theta\|_{\dot{H}^{\frac{1-\alpha}{2}}(wdx)}\\ &\leq \|\theta\|_{\dot{H}^2(wdx)}\|\theta\|_{\dot{H}^{2+\alpha/2}(wdx)}\|\theta\|_{H^{\frac{1}{2}}(wdx)}\\ &\leq \frac{1}{\epsilon_1}\|\theta\|^4_{H^2(wdx)}+\epsilon_1\|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)}\end{aligned}$$

For $I_{7,2}$ we have, for all $\epsilon_1>0$,

$$\begin{aligned} |I_{7,2}| &\leq \|w^{\frac{1}{2}}\theta_{xx}\|_{L^2}\|w^{\frac{1-\alpha}{2}}\theta_{xx}\|_{L^{\frac{2}{1-\alpha}}}\left\|w^{\frac{\alpha}{2}}\Lambda\theta\right\|_{L^{2/\alpha}}\\ &\leq \|\theta\|_{\dot{H}^2(wdx)}\|\theta\|_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)}\|\theta\|_{\dot{H}^{\frac{3-\alpha}{2}}(wdx)}\\ &\leq \frac{1}{\epsilon_1}\|\theta\|^4_{H^2(wdx)}+\epsilon_1\|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)}\end{aligned}$$

Therefore,

$$|I_7|\leq\frac{2}{\epsilon_1}\|\theta\|^4_{H^2(wdx)}+2\epsilon_1\|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)} \tag{6.2}$$

Combining 6.3 and 6.2 we conclude by choosing $\epsilon_1>0$

$$\left|\sum_{j=1}^4 I_j\right|\lesssim\frac{1}{\epsilon_1^4}\|\theta\|^4_{H^2(wdx)}+\epsilon_1\|\theta\|^{\frac{16}{3}}_{H^2(wdx)}+\epsilon_1\|\theta\|^2_{\dot{H}^{2+\frac{\alpha}{2}}(wdx)} \tag{6.3}$$

hence, by choosing $\epsilon_1>0$ suficiently small, one finds

$$\frac{1}{2}\partial_t\|\theta\|^2_{H^2wdx}+C\int|\Lambda^{2+\frac{\alpha}{2}}\theta|^2\ w\ dx \quad\lesssim\quad \|\theta\|^4_{H^2(wdx)}+\|\theta\|^{\frac{16}{3}}_{H^2(wdx)}$$

By using Grönwall's inequality, and the classical truncation and compactness arguments (see [17] for instance), one gets the theorem. □

7. Proof of Theorem 1.5

7.1. **A priori estimates.** Taking the Fourier transform of (1.2), we have that

$$\begin{aligned}\partial_t|\hat{\theta}(\xi)| &= \frac{\bar{\hat{\theta}}(\xi)\partial_t\hat{\theta}(\xi)+\hat{\theta}(\xi)\partial_t\bar{\hat{\theta}}(\xi)}{2|\hat{\theta}(\xi)|}=\frac{\text{Re}\left(\bar{\hat{\theta}}(\xi)\partial_t\hat{\theta}(\xi)\right)}{|\hat{\theta}(\xi)|}\\ &=-\text{Re}\left[\int_{\mathbb{R}}\frac{-i\zeta}{|\zeta|}\hat{\theta}(\zeta)i(\xi-\zeta))\hat{\theta}(\xi-\zeta)-\delta\hat{\theta}(\zeta)|\xi-\zeta|\hat{\theta}(\xi-\zeta)d\zeta\frac{\bar{\hat{\theta}}(\xi)}{|\hat{\theta}(\xi)|}\right]\frac{1}{\sqrt{2\pi}}-|\xi||\hat{\theta}|.\end{aligned}$$

Consequently,

$$\frac{d}{dt}\|\theta\|_{A^0} \leq (1+|\delta|)\int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{\theta}(\zeta)||\xi-\zeta||\hat{\theta}(\xi-\zeta)|d\zeta d\xi \frac{1}{\sqrt{2\pi}} - \|\theta\|_{\dot{A}^1}$$
$$\leq (1+|\delta|)\int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{\theta}(\zeta)||\xi-\zeta||\hat{\theta}(\xi-\zeta)|d\xi d\zeta \frac{1}{\sqrt{2\pi}} - \|\theta\|_{\dot{A}^1} \leq \left(\frac{(1+|\delta|)\|\theta\|_{A^0}}{\sqrt{2\pi}} - 1\right)\|\theta\|_{\dot{A}^1}.$$

Thus, if θ_0 satisfies the condition (1.6), we have

$$\|\theta(t)\|_{A^0} + \left(1 - \frac{(1+|\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right)\int_0^\infty \|\theta(s)\|_{\dot{A}^1} ds \leq \|\theta_0\|_{A^0}. \tag{7.1}$$

Similarly,

$$\partial_t|\xi\hat{\theta}| = -\mathrm{Re}\Bigg[\int_{\mathbb{R}} \frac{-i\zeta}{|\zeta|}\hat{\theta}(\zeta)(i(\xi-\zeta))^2\hat{\theta}(\xi-\zeta) + |\zeta|\hat{\theta}(\zeta)i(\xi-\zeta))\hat{\theta}(\xi-\zeta)d\zeta$$
$$-\delta\int_{\mathbb{R}} i\zeta\hat{\theta}(\zeta)|\xi-\zeta|\hat{\theta}(\xi-\zeta) + \hat{\theta}(\zeta)i(\xi-\zeta)|\xi-\zeta|\hat{\theta}(\xi-\zeta)d\zeta\frac{\overline{i\xi\hat{\theta}(\xi)}}{|\xi||\hat{\theta}(\xi)|}\Bigg]\frac{1}{\sqrt{2\pi}} - |\xi|^2|\hat{\theta}|.$$

Thus, using (2.5), we have that

$$\frac{d}{dt}\|\theta\|_{\dot{A}^1} \leq \left(\|\theta\|_{A^0}\|\theta\|_{\dot{A}^2} + \|\theta\|_{\dot{A}^1}^2\right)\frac{1+|\delta|}{\sqrt{2\pi}} - \|\theta\|_{\dot{A}^2} \leq \left(\frac{2(1+|\delta|)\|\theta\|_{A^0}}{\sqrt{2\pi}} - 1\right)\|\theta\|_{\dot{A}^2}.$$

As a consequence, we obtain that, if θ_0 satisfies the condition (1.6), we also have

$$\|\theta(t)\|_{\dot{A}^1} + \left(1 - \frac{2(1+|\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right)\int_0^\infty \|\theta(s)\|_{\dot{A}^2} ds \leq \|\theta_0\|_{\dot{A}^1}. \tag{7.2}$$

By (7.1) and (7.2), we conclude that

$$\|\theta(t)\|_{A^1} + \left(1 - \frac{2(1+|\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right)\int_0^t \|\theta_x(s)\|_{A^1} ds \leq \|\theta_0\|_{A^1} \tag{7.3}$$

for all $t \geq 0$.

7.2. **Approximation and passing to the limit.** Define $e^{\epsilon\partial_x^2}$ the heat semigroup, *i.e.*

$$\widehat{e^{\epsilon\partial_x^2} f(x)} = e^{-\epsilon\xi^2}\hat{f}(\xi),$$

and $g_\epsilon(x) = e^{-\epsilon x^2}$. Note that

$$\hat{g}_\epsilon(\xi) = \frac{1}{\sqrt{2\epsilon}}e^{-\frac{\xi^2}{4\epsilon}}, \quad \|\hat{g}_\epsilon\|_{L^1} = \sqrt{2\pi}.$$

Given $\theta_0(x) \in A^1$, we consider $\theta_0^\epsilon(x) = g_\epsilon(x)e^{\epsilon\partial_x^2}\theta_0(x)$. As $\theta_0(x)$ is a bounded function, we have that θ_0^ϵ is infinitely smooth and has finite total mass:

$$\|\theta_0^\epsilon\|_{L^1} \leq \|g_\epsilon\|_{L^1}\left\|e^{\epsilon\partial_x^2}\theta_0\right\|_{L^\infty} \leq \sqrt{\frac{\pi}{\epsilon}}\|\theta_0\|_{L^\infty}.$$

Furthermore, using Young's inequality and the definition of g_ϵ,

$$\|\theta_0^\epsilon\|_{A^0} = \frac{1}{\sqrt{2\pi}}\left\|\hat{g}_\epsilon * \left(e^{-\epsilon\xi^2}\hat{\theta}_0\right)\right\|_{L^1} \leq \left\|e^{-\epsilon\xi^2}\hat{\theta}_0\right\|_{L^1} \leq \|\theta_0\|_{A^0}.$$

Similarly,

$$\|\theta_0^\epsilon\|_{\dot{A}^1} = \|\partial_x\theta_0^\epsilon\|_{A^0} \leq \|\theta_0\|_{\dot{A}^1} + \left\|\partial_x g_\epsilon e^{\epsilon\partial_x^2}\theta_0\right\|_{A^0} = \|\theta_0\|_{\dot{A}^1} + c\epsilon\|\theta_0\|_{A^0}.$$

Now we define the approximated problems

$$\theta_t^\epsilon + (\mathcal{H}\theta^\epsilon)\,\theta_x^\epsilon + \delta\theta^\epsilon\Lambda\theta^\epsilon + \Lambda\theta^\epsilon = \epsilon\partial_x^2\theta^\epsilon, \tag{7.4}$$

with finite energy approximated initial data θ_0^ϵ. These problems have unique smooth solutions denoted by θ^ϵ. Moreover, (θ^ϵ) satisfies a uniform bound

$$\|\theta^\epsilon(t)\|_{A^1}+\left(1-\frac{2(1+|\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right)\int_0^t\|\theta_x^\epsilon(s)\|_{A^1}ds+\epsilon\int_0^t\|\theta_x^\epsilon(s)\|_{A^2}ds\leq\|\theta_0\|_{A^1}.$$

uniformly in ϵ. Thus, the a priori estimates lead to the following uniform-in-ϵ bounds

$$\begin{aligned}
&\sup_{t\in[0,\infty)}\|\theta^\epsilon(t)\|_{C^0(\mathbb{R})}\leq\sup_{t\in[0,\infty)}\|\theta^\epsilon(t)\|_{A^0}\leq\|\theta_0\|_{A^0}<\frac{\sqrt{2\pi}}{1+|\delta|},\\
&\sup_{t\in[0,\infty)}\|\theta^\epsilon(t)\|_{\dot{C}^1(\mathbb{R})}\leq\sup_{t\in[0,\infty)}\|\theta^\epsilon(t)\|_{\dot{A}^1}<\|\theta_0\|_{\dot{A}^1}+c\epsilon\|\theta_0\|_{A^0},\\
&\sup_{t\in[0,\infty)}\|\Lambda\theta^\epsilon(t)\|_{C^0(\mathbb{R})}<\|\theta_0\|_{\dot{A}^1}+c\epsilon\|\theta_0\|_{A^0}.
\end{aligned}\tag{7.5}$$

Moreover, from the equation (7.4) we also obtain uniform bounds

$$\begin{aligned}
&\sup_{t\in[0,\infty)}\|\partial_t\theta^\epsilon(t)\|_{C^0(\mathbb{R})}+\sup_{t\in[0,\infty)}\|\partial_tH\theta^\epsilon(t)\|_{C^0(\mathbb{R})}\leq2\sup_{t\in[0,\infty)}\|\partial_t\theta^\epsilon(t)\|_{A^0(\mathbb{R})}\leq F_1(\|\theta_0\|_{A^1},\delta),\\
&\|H\theta^\epsilon\|_{C^1([0,\infty)\times\mathbb{R})}+\|\theta^\epsilon\|_{C^1([0,\infty)\times\mathbb{R})}\leq F_2(\|\theta_0\|_{A^1},\delta)
\end{aligned}\tag{7.6}$$

where F_1 and F_2 only depend on the quantities in the right-hand side of (7.5). Due to Banach-Alaoglu Theorem, there exists a subsequence (denoted by ϵ) and a limit function $\theta\in W^{1,\infty}([0,\infty)\times\mathbb{R})$ such that

$$\theta^\epsilon\overset{\star}{\rightharpoonup}\theta\quad\text{in}\quad L^\infty\left(0,T;W^{1,\infty}\right),\quad\Lambda\theta^\epsilon\overset{\star}{\rightharpoonup}\Lambda\theta\quad\text{in}\quad L^\infty\left(0,T;L^\infty\right)$$

for all $T>0$. Using Lemma 2.6, we have the following strong convergence

$$\lim_{\epsilon\to0}\|\theta^\epsilon-\theta\|_{L^\infty(K)}+\|H\theta^\epsilon-H\theta\|_{L^\infty(K)}=0$$

for $K=[0,T]\times[-R,R]$ for any $R>0$. Since $L^\infty(K)\subset L^2(K)$, we can pass to the limit to the weak formulation

$$\int_0^T\int\left[\theta^\epsilon\psi_t+(\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\psi_x+(1-\delta)\Lambda\theta^\epsilon\theta^\epsilon\psi-\theta^\epsilon\Lambda\psi+\epsilon\theta^\epsilon\psi_{xx}\right]dxdt=\int\theta_0^\epsilon(x)\psi(0,x)dx$$

to obtain a weak solution θ satisfying

$$\theta\in\mathcal{W}_T,\quad\sup_{t\in[0,T]}\|\theta(t)\|_{A^1}+\left(1-\frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right)\int_0^T\|\theta_x(t)\|_{A^1}dt\leq\|\theta_0\|_{A^1}$$

for all $T>0$.

7.3. **Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of $\theta=\theta_1-\theta_2$ given by

$$\theta_t+\Lambda\theta=-\left(\mathcal{H}\theta\right)\theta_{1x}-\left(\mathcal{H}\theta_2\right)\theta_x-\delta\theta\Lambda\theta_1-\delta\theta_2\Lambda\theta,\quad\theta(0,x)=0.\tag{7.7}$$

Taking the Fourier transform of (7.7) and multiply by $\frac{\hat{\theta}}{|\hat{\theta}|}$, we have

$$\frac{d}{dt}\|\theta\|_{A^0}+\|\theta\|_{\dot{A}^1}\leq\frac{1+|\delta|}{\sqrt{2\pi}}\|\theta_1\|_{\dot{A}^1}\|\theta\|_{A^0}+\frac{1+|\delta|}{\sqrt{2\pi}}\|\theta_2\|_{A^0}\|\theta\|_{\dot{A}^1}.\tag{7.8}$$

Since

$$\frac{(1+|\delta|)\|\theta_2\|_{A^0}}{\sqrt{2\pi}}<1,\quad\|\theta_1(t)\|_{\dot{A}^1}\in L^1([0,\infty)),$$

we have $\theta(t,x)=0$ in A^0 for all time. This implies that a weak solution is unique.

8. Proof of Theorem 1.6

8.1. A priori estimate. We consider the equation

$$\theta_t + u\theta_x + \Lambda^\gamma \theta = 0, \quad u = (1-\partial_{xx})^{-\alpha}\theta \tag{8.1}$$

Since (8.1) satisfies the maximum principle, we have

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}.$$

We also obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2 &= -\int u\theta_x\theta dx \le \|\theta\|_{L^\infty}\|u_x\|_{L^2}\|\theta\|_{L^2} \\ &\le C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2\right)\|\theta\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2\end{aligned}$$

where we use the condition $\alpha = \frac{1}{2} - \frac{\gamma}{4}$ to bound u_x as

$$\|u_x\|_{L^2} \le C\left(\|\theta\|_{L^2} + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}\right).$$

Therefore, we obtain that

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \left\|\Lambda^{\frac{\gamma}{2}}\theta(s)\right\|_{L^2}^2 ds \le \|\theta_0\|_{L^2}^2 e^{C\left(1+\|\theta_0\|_{L^\infty}^2\right)t}.$$

8.2. Approximation and passing to limit. We consider the following equation with regularized initial data:

$$\theta_t^\epsilon + u^\epsilon\theta_x^\epsilon + \Lambda^\gamma\theta^\epsilon = 0, \quad \theta_0^\epsilon = \rho_\epsilon * \theta_0. \tag{8.2}$$

Then, there exists a global-in-time smooth solution θ^ϵ. Moreover, θ^ϵ satisfies that

$$\|\theta^\epsilon(t)\|_{L^\infty} + \|\theta^\epsilon(t)\|_{L^2}^2 + \int_0^t \left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(s)\right\|_{L^2}^2 ds + \epsilon\|\nabla\theta^\epsilon\|_{L^2}^2 \le \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^2}^2 e^{C\left(1+\|\theta_0\|_{L^\infty}^2\right)t}. \tag{8.3}$$

Therefore, (θ_ϵ) is bounded in $\mathcal{C}_T$ uniformly in $\epsilon > 0$. This implies the uniform bounds

$$u^\epsilon \in L^4\left(0,T;L^4\right), \quad \theta^\epsilon \in L^4\left(0,T;L^4\right), \quad u_x^\epsilon \in L^2\left(0,T;L^2\right).$$

Moreover, these bounds with the equation

$$\theta_t^\epsilon = -u^\epsilon\theta_x^\epsilon - \Lambda^\gamma\theta^\epsilon + \epsilon\theta_{xx}^\epsilon = -\left(u^\epsilon\theta^\epsilon\right)_x + u_x^\epsilon\theta^\epsilon - \Lambda^\gamma\theta^\epsilon + \epsilon\theta_{xx}^\epsilon$$

we also have that

$$\theta_t^\epsilon \in L^{\frac{4}{3}}\left(0,T;H^{-2}\right).$$

We now extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{C}_T$ and $u = (1-\partial_{xx})^{-\alpha}\theta$ such that

$$\begin{aligned} &\theta^\epsilon \overset{\star}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty\left(0,T;L^p\right) \quad \text{for all } p \in (1,\infty), \\ &\theta^\epsilon \rightharpoonup \theta \quad \text{in} \quad L^2\left(0,T;H^{\frac{\gamma}{2}}\right), \\ &\theta^\epsilon \to \theta \quad \text{in} \quad L^2\left(0,T;L^p_{\text{loc}}\right) \quad \text{for all } p \in (1,\infty), \\ &u_x^\epsilon \rightharpoonup u_x \quad \text{in} \quad L^2\left(0,T;L^2\right), \end{aligned} \tag{8.4}$$

where we use Lemma 2.5 to obtain the strong convergence.

We now multiply (8.2) by a test function $\psi \in \mathcal{C}_c^\infty\left([0,T)\times\mathbb{R}\right)$ and integrate over $\mathbb{R}$. Then,

$$\int_0^T\int \left[\theta^\epsilon\psi_t + u^\epsilon\theta^\epsilon\psi_x + u_x^\epsilon\theta^\epsilon\psi - \theta^\epsilon\Lambda^\gamma\psi + \epsilon\theta^\epsilon\psi_{xx}\right] dxdt = \int \theta_0^\epsilon(x)\psi(0,x)dx.$$

By Lemma 2.5 with

$$X_0 = L^2\left(0,T;H^{\frac{\gamma}{2}}\right),\quad X = L^2\left(0,T;L^2_{\text{loc}}\right),\quad X_1 = L^2\left(0,T;H^{-2}\right)$$

and using (8.4), we can pass to the limit to obtain that

$$\int_0^T\int\left[\theta\psi_t + u\theta\psi_x + u_x\theta\psi - \theta\Lambda^\gamma\psi\right]dxdt = \int\theta_0(x)\psi(0,x)dx \tag{8.5}$$

This completes the proof.

8.3. **Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of $\theta = \theta_1 - \theta_2$ given by

$$\theta_t + \Lambda^\gamma\theta = -u_1\theta_x + u\theta_{2x},\quad \theta(0,x) = 0. \tag{8.6}$$

We multiply (8.6) by θ to obtain

$$\begin{aligned}\frac{1}{2}\|\theta\|_{L^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2 &= -\int u_1\theta_x\theta dx + \int u\theta_{2x}\theta dx = \frac{1}{2}\int u_{1x}\theta^2 dx + \int u\theta_{2x}\theta dx\\ &\le C\|u_{1x}\|_{L^2}\|\theta\|_{L^2}\|\theta\|_{L^\infty} + C\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}\|u\|_{L^\infty}\|\theta\|_{\dot H^{\frac{\gamma}{2}}} + C\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}\|u\|_{\dot H^{\frac{\gamma}{2}}}\|\theta\|_{L^\infty}\\ &\le C\|u_{1x}\|_{L^2}\|\theta\|_{L^2}\|\theta\|_{H^{\frac{\gamma}{2}}} + C\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}\|u\|_{L^\infty}\|\theta\|_{\dot H^{\frac{\gamma}{2}}} + C\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}\|u\|_{\dot H^{\frac{\gamma}{2}}}\|\theta\|_{H^{\frac{\gamma}{2}}}\\ &\le C\left(\|u_{1x}\|_{L^2} + \|u_{1x}\|_{L^2}^2\right)\|\theta\|_{L^2}^2 + C\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}^2\|u\|_{L^\infty}^2\\ &+ C\left(\|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}} + \|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}}^2\right)\|u\|_{\dot H^{\frac{\gamma}{2}}}^2 + \frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2,\end{aligned}$$

where we use $\gamma > 1$ to control $\|\theta\|_{L^\infty}$. Since $\gamma > 1$ and $\alpha > \frac{1}{4} \ge \frac{1}{2} - \frac{\gamma}{4}$, we also have

$$\|u_{1x}\|_{L^2} \le C\|\theta_1\|_{H^{\frac{\gamma}{2}}},\quad \|u\|_{L^\infty} \le C\|\theta\|_{L^2},\quad \|\theta_2\|_{\dot H^{1-\frac{\gamma}{2}}} \le C\|\theta_2\|_{H^{\frac{\gamma}{2}}}.$$

Hence

$$\|\theta\|_{L^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2 \le C\left(1 + \|\theta_1\|_{H^{\frac{\gamma}{2}}}^2 + \|\theta_2\|_{H^{\frac{\gamma}{2}}}^2\right)\|\theta\|_{L^2}^2.$$

Since the quantities in the parenthesis are integrable in time and $\theta(0,x) = 0$, we conclude that $\theta = 0$ in L^2 and thus a weak solution is unique. This completes the proof of Theorem 1.6.

9. Proof of Theorem 1.7

9.1. **A priori estimate.** We consider the equation

$$\theta_t + u\theta_x + \Lambda\theta = 0,\quad u = (1-\partial_{xx})^{-\frac{1}{4}}\theta \tag{9.1}$$

We multiply (9.1) by θw and integrate in x. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2(wdx)}^2 + \left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}^2 &= -\int u\theta_x\theta w dx - \int\Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx\\ &= \frac{1}{2}\int u_x\theta^2 w dx + \frac{1}{2}\int u\theta^2 w_x dx - \int\Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx.\end{aligned}$$

As in the proof of Theorem 1.2, we bound the commutator term as

$$\int\Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx \le \frac{1}{4}\left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}^2 + C\|\theta\|_{L^2(wdx)}^2.$$

To estimate terms involving u, we use $\theta = (1-\partial_{xx})^{\frac{1}{4}}u$ and (2.8) to obtain that

$$\begin{aligned}\int u_x\theta^2 w dx + \int u\theta^2 w_x dx &\le C\|\theta_0\|_{L^\infty}\left(\|u\|_{L^2(wdx)} + \|u_x\|_{L^2(wdx)}\right)\|\theta\|_{L^2(wdx)} \\ &\le C\|\theta_0\|_{L^\infty}\left(\|\theta\|_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|_{L^2(wdx)}\right)\|\theta\|_{L^2(wdx)} \\ &\le C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|^2_{L^\infty}\right)\|\theta\|^2_{L^2(wdx)} + \frac{1}{4}\left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)}.\end{aligned}$$

Collecting all terms together, we obtain that

$$\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} \le C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|^2_{L^\infty}\right)\|\theta\|^2_{L^2(wdx)}$$

and hence that

$$\|\theta(t)\|^2_{L^2(wdx)} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta(s)\right\|^2_{L^2(wdx)} ds \le \|\theta_0\|^2_{L^2(wdx)} \exp\left[C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|^2_{L^\infty}\right)t\right]. \tag{9.2}$$

We next multiply (9.1) by $-(\theta_x w)_x$ and integrate in x. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta_x\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)} = \int u\theta_x(\theta_x w)_x dx + \int \Lambda^{\frac{3}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\Lambda\theta dx \\ &= -\frac{1}{2}\int u_x(\theta_x)^2 w dx + \frac{1}{2}\int u(\theta_x)^2 w_x dx - \int \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx + \int \Lambda\theta\theta_x w_x dx.\end{aligned}$$

Following the computation in [18],

$$\begin{aligned}\frac{d}{dt}\|\theta_x\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)} &\le C\|u\|_{H^1(wdx)}\|\theta_x\|^2_{L^4(wdx)} + C\|\theta_x\|^2_{L^2(wdx)} + \frac{1}{4}\left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)} \\ &\le C\|\theta\|_{H^{\frac{1}{2}}(wdx)}\|\theta_x\|^2_{L^4(wdx)} + C\|\theta_x\|^2_{L^2(wdx)} + \frac{1}{4}\left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)},\end{aligned}$$

where we use the relation in (2.8) to bound u in terms of θ. Since

$$\begin{aligned}\|\theta\|_{H^{\frac{1}{2}}(wdx)}\|\theta_x\|^2_{L^4(wdx)} &\le C\|\theta\|_{H^{\frac{1}{2}}(wdx)}\|\theta_x\|_{L^2(wdx)}\left\|\Lambda^{\frac{3}{2}}\theta\right\|_{L^2(wdx)} \\ &\le C\|\theta\|^2_{H^{\frac{1}{2}}(wdx)}\|\theta_x\|^2_{L^2(wdx)} + \frac{1}{4}\left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)}\end{aligned} \tag{9.3}$$

by (2.9), we obtain that

$$\frac{d}{dt}\|\theta_x\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2(wdx)} \le C\left(1 + \|\theta\|^2_{H^{\frac{1}{2}}(wdx)}\right)\|\theta_x\|^2_{L^2(wdx)}. \tag{9.4}$$

Integrating in time (9.4) and using (9.2), we obtain that

$$\begin{aligned}&\|\theta_x(t)\|^2_{L^2(wdx)} + \int_0^t \left\|\Lambda^{\frac{3}{2}}\theta(s)\right\|^2_{L^2(wdx)} \\ &\le \|\theta_{0x}\|^2_{L^2(wdx)} \exp\left[\int_0^t C\left(1 + \|\theta(s)\|^2_{H^{\frac{1}{2}}(wdx)}\right) ds\right] \\ &\le \|\theta_{0x}\|^2_{L^2(wdx)} \exp\left[Ct + \|\theta_0\|^2_{L^2(wdx)}\exp\left[C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|^2_{L^\infty}\right)t\right]\right].\end{aligned} \tag{9.5}$$

By (9.2) and (9.5), we finally obtain that

$$\begin{aligned}&\|\theta(t)\|^2_{H^1(wdx)} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta(s)\right\|^2_{H^1(wdx)} ds \\ &\le C\|\theta_0\|^2_{H^1(wdx)} \exp\left[Ct + \|\theta_0\|^2_{L^2(wdx)}\exp\left[C\left(\|\theta_0\|_{L^\infty} + \|\theta_0\|^2_{L^\infty}\right)t\right]\right].\end{aligned} \tag{9.6}$$

9.2. **Approximation and passing to limit.** Since θ is more regular than a solution in Theorem 1.2, we can follow the procedure in the proof of Theorem 1.2.

9.3. **Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of $\theta = \theta_1 - \theta_2$ given by

$$\theta_t + \Lambda\theta = -u_1\theta_x + u\theta_{2x}, \quad \theta(0,x) = 0. \tag{9.7}$$

We multiply $w\theta$ to (9.7) and integrate over $\mathbb{R}$. Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} = \int [-u_1\theta_x + u\theta_{2x}]\,\theta w dx - \int \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx$$
$$= \frac{1}{2}\int u_{1x}\theta^2 w dx + \frac{1}{2}\int u_1\theta^2 w_x dx + \int u\theta_{2x}\theta w dx - \int \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx.$$

As before, the last term is bounded by

$$\int \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, w\right]\theta dx \leq \frac{1}{2}\left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} + C\|\theta\|^2_{L^2(wdx)}.$$

The first three terms in the right-hand side are easily bounded by

$$C\left(\|\theta_{2x}\|_{L^2(wdx)} + \|\theta_1\|_{H^{\frac{1}{2}}(wdx)}\right)\left(\|\theta\|^2_{L^4(wdx)} + \|\theta\|_{L^4(wdx)}\|u\|_{L^4(wdx)}\right).$$

By (2.9), we obtain that

$$\frac{d}{dt}\|\theta\|^2_{L^2(wdx)} + \left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)} \leq C\left(1 + \|\theta_2\|^2_{H^1(wdx)} + \|\theta_1\|^2_{H^{\frac{1}{2}}(wdx)}\right)\|\theta\|^2_{L^2(wdx)} + \frac{1}{2}\left\|\Lambda^{\frac{1}{2}}\theta\right\|^2_{L^2(wdx)}.$$

Since

$$\theta_2 \in L^2\left(0, T : H^1(wdx)\right), \quad \theta_1 \in L^2\left(0, T : H^{\frac{1}{2}}(wdx)\right),$$

we conclude that $\theta = 0$ in $L^2(wdx)$ and thus a weak solution is unique. This completes the proof of Theorem 1.7.

10. Proof of Theorem 1.8

Taking the Fourier transform of (1.7), we have that

$$\partial_t|\hat{\theta}(\xi)| = -\mathrm{Re}\left[\int_{\mathbb{R}} \frac{1}{(1+|\zeta|^2)^\alpha}\hat{\theta}(\zeta)i(\xi-\zeta))\hat{\theta}(\xi-\zeta)d\zeta\frac{\bar{\hat{\theta}}(\xi)}{|\hat{\theta}(\xi)|}\right]\frac{1}{\sqrt{2\pi}} - |\xi||\hat{\theta}|.$$

Consequently, ignoring the factor $\frac{1}{(1+|\zeta|^2)^\alpha}$, we follow the proof of Theorem 1.5 with $\delta = 0$ and the smallness condition (1.8) to obtain that

$$\|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right)\int_0^t \|\theta_x(s)\|_{A^1} ds \leq \|\theta_0\|_{A^1} \tag{10.1}$$

for all $t \geq 0$. We also follow the proof of Theorem 1.5 to obtain a unique weak solution via the approximation procedure. This completes the proof.

Acknowledgments

H.B. was supported by NRF-2015R1D1A1A01058892. R.G.B. is funded by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program “Investissements d’Avenir” (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). Both O. L. and R.G.B. were partially supported by the Grant MTM2014-59488-P from the former Ministerio de Economía y Competitividad (MINECO, Spain). O.L. was partially supported by the Marie-Curie Grant, acronym: TRANSIC, from the FP7-IEF program.

References

[1] H. Bae, R. Granero-Belinchón, *Global existence for some transport equations with nonlocal velocity*, Adv. Math. **269** (2015), 197–219.

[2] G.R. Baker, X. Li, A.C. Morlet, *Analytic structure of 1D transport equations with nonlocal fluxes*, Physica D. **91** (1996), 349–375.

[3] J. A. Carrillo, L.C.F. Ferreira, J.C. Precioso, *A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity*, Adv. Math. **231** (2012), no.1, 306–327.

[4] A. Castro, D. Córdoba, *Global existence, singularities and ill-posedness for a nonlocal flux*, Adv. Math. **219** (2008), no. 6, 1916–1936.

[5] A. Castro, D. Córdoba, *Self-similar solutions for a transport equation with non-local flux*, Chinese Annals of Mathematics, Series B **30** (2009), no. 5, 505–512.

[6] D. Chae, A. Cordoba, D. Cordoba, M. Fontelos, *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Adv. Math. **194** (2005), no. 1, 203–223.

[7] R. Coifman, Y. Meyer, *Wavelets: Calderón-Zygmund and Multilinear Operators*, Cambridge Studies in Advanced Mathematics, 48 (2000).

[8] P. Constantin, P. Lax, A. Majda, *A simple one-dimensional model for the three dimensional vorticity*, Comm. Pure Appl. Math. **38** (1985), 715–724.

[9] P. Constantin, D. Córdoba, F. Gancedo, R. Strain, *On the global existence for the Muskat problem*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 1, 201–227.

[10] P. Constantin, A. Majda, E. Tabak, *Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar*, Nonlinearity **7** (1994), 1495–1533.

[11] A. Córdoba, D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249** (2004), no. 3, 511–528.

[12] A. Córdoba, D. Córdoba, M. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. (2) **162** (2005), 1–13.

[13] S. De Gregorio, *On a one-dimensional model for the 3D vorticity equation*, J. Statist. Phys. **59** (1990), 1251–1263.

[14] H. Dong, *Well-posedness for a transport equation with nonlocal velocity*, J. Funct. Anal, 255, 3070-3097, (2008).

[15] J.K. Hunter, R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. **51** (1991), 1498–1521.

[16] A. Kiselev, *Regularity and blow up for active scalars*, Math. Model. Math. Phenom. **5** (2010), 225–255.

[17] O. Lazar, *On a 1D nonlocal transport equation with nonlocal velocity and subcritical or supercritical diffusion*, Journal of Differential Equations, doi 10.1016/j.jde.2016.07.009, in press

[18] O. Lazar, P-G. Lemarié-Rieusset, *Infinite energy solutions for a 1D transport equation with nonlocal velocity*, Dynamics of PDEs, Vol. 13, No. 2 (2016), pp. 107-131.

[19] D. Li, J. Rodrigo, *Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation*, Adv. Math. **217** (2008), no. 6, 2563–2568.

[20] D. Li, J. Rodrigo, *On a one-dimensional nonlocal flux with fractional dissipation*, SIAM J. Math. Anal. **43** (2011), no. 1, 507–526.

[21] J.E. Marsden, S. Shkoller, *The anisotropic Lagrangian averaged Euler and Navier-Stokes equations*, Arch. Ration. Mech. Anal. **166** (2003), no. 1, 27–46.

[22] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, Second, revised and augmented edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011.

[23] A. Morlet, *Further properties of a continuum of model equations with globally defined flux*, J. Math. Anal. Appl. **221** (1998), 132–160.

[24] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[25] H. Okamoto, J. Zhu, *Some similarity solutions of the Navier-Stokes equations and related topics. Proceedings of 1999 international conference on nonlinear analysis (Taipei)*, Taiwan. J. Math. **4** (2000), 65–103.

[26] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag (1986).

[27] I. Proudman and K. Johnson, *Boundary-layer growth near a rear stagnation point*, J. Fluid Mech. **12** (1962), 161–168.

[28] L. Silvestre, V. Vicol, *On a transport equation with nonlocal drift.* Transactions of the American Mathematical Society 368 (2016), no. 9, 6159-6188

[29] E. Stein, *Harmonic Analysis: Real variable methods, Orthogonality and Oscillatory Integrals*, Princeton Math. Series 43, Princeton University Press, (1993).

[30] R. Temam, *Navier-Stokes Equations: theory and numerical analysis*, AMS Chelsea publishing, Providence, (2001).

[31] M. Wunsch, *The Generalized Proudman-Johnson Equation Revisited*, J. Math. Fluid Mech. **13** (2011), 147–154.

Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Korea
E-mail address: hantaek@unist.ac.kr

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France.
E-mail address: granero@math.univ-lyon1.fr

Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas, Spain
E-mail address: omar.lazar@icmat.es