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Electrostatic potentials in the Euclidean space
(Potenciales electrostáticos en el espacio Euclídeo)

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Abstract

In this work, we study the properties of electric (or electrostatic) potentials and fields generated by a finite number of point charges in the Euclidean space $\mathbb{R}^n$. In particular, we focus our study on the behavior of the equilibrium points of such electric fields. We use tools from Morse theory, real analytic geometry and partial differential equations to investigate, among other things, the location of the equilibrium points, their degeneracy, when there are finitely many of them and whether they form a "big" set when there are infinitely many. Finally, motivated by Maxwell's conjecture and using tools from real algebraic geometry, we obtain some upper bounds for the number of equilibrium points of the electric fields, depending on n and on the number of charges that generate them. We obtain several bounds since certain features of the charges modify the situation, such as the parity of n, whether all the charges have the same sign, or the dimension of the smallest affine set that contains the them.

Key words: electric potential, electric field, equilibrium point, Maxwell's conjecture.

Resumen

En este trabajo, estudiamos las propiedades de potenciales y campos eléctricos (o electrostáticos) generados por un número finito de cargas puntuales en el espacio Euclídeo $\mathbb{R}^n$. En particular, centramos nuestro estudio en el comportamiento de los puntos de equilibrio de tales campos eléctricos. Usamos herramientas de teoría de Morse, geometria analítica real y ecuaciones en derivadas parciales para investigar, entre otras cosas, la ubicación de los puntos de equilibrio, su degeneración, cuándo hay una cantida finita de ellos y si forman un conjunto "grande" cuando hay infinitos. Finalmente, motivados por la conjuetura de Maxwell y usando herramientas de geometría algebraica real, obtenemos algunas cotas superiores para el número de puntos de equilibrio de los campos eléctricos, dependientes de n y del número de cargas que los generan. Obtenemos varias cotas porque algunas características de las cargas modifican la situacion, como la paridad de n, si todas las cargas tienen el mismo signo, o la dimensión del menor conjunto afín que las contiene.

Palabras clave: potencial eléctrico, campo eléctrico, punto de equilibrio, conjetura de Maxwell.

Contents

Introduction

Around the year 1785, Coulomb showed experimentally that the force between two small charged bodies separated by a big distance compared to their size depended directly on the magnitude of each charge, inversely on the square of the distance between them, was directed along the line joining the charges and was attractive if the bodies had opposite charges and repulsive if they had the same type of charge (positive or negative). In particular, if there are two charges with values $q_1 \in \mathbb{R}$ and $q_2 \in \mathbb{R}$ located at $p_1 \in \mathbb{R}^3$ and $p_2 \in \mathbb{R}^3$, in the vacuum, the electric force experimented by the first charge is

$$E_1 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{p}_{12}}{|p_1 - p_2|^2} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{p_1 - p_2}{|p_1 - p_2|^3},$$

where $\hat{p}_{12} = \dfrac{p_1 - p_2}{|p_1 - p_2|}$ is a unit vector pointing from p_2 to p_1 and ε_0 is the permittivity of vacuum. The force experimented by the second charge, according to Newton's third law, is $E_2 = -E_1$. This discovery, together with the similar inverse-square law for gravity due to Newton, represented a significant breakthrough in the Physics of the XVII and XVIII centuries. The law of superposition allows this interaction to be extended to the case where there is any finite number of charges. In this case, we get that the electric force experimented by a test particle located at some point $x \in \mathbb{R}^3$, due to a set of N point charges in the vacuum, is

$$E(x) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i(x - p_i)}{|x - p_i|^3},$$

where $p_1, \ldots, p_N$ are the positions of the charges and $q_1, \ldots, q_N$ are their values. The study of the electric field E has become a classical problem in mathematics and physics.

Given some electric field E, we can define the electric potential at some point x by

$$V_E(x) = -\int_{\mathcal{C}} E \cdot dl,$$

where $\mathcal{C}$ is a path from a fixed point to x. The electric field E is conservative, that is, there exists some C^1 scalar field ϕ such that $E = \nabla\phi$. Therefore, the previous integral does not depend on the chosen path $\mathcal{C}$, only on the endpoints, so V_E is well-defined. By the gradient theorem, we have that

$$E = -\nabla V_E.$$

In particular, the electric potential created at a point $x \in \mathbb{R}^3$ by N charges located at the points $p_1, \ldots, p_N$ with values $q_1, \ldots, q_N$ is

$$V(x) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{q_k}{|x - p_k|}.$$

We can easily see that, indeed,

$$\frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i(x - p_i)}{|x - p_i|^3} = -\nabla V(x).$$

In this text we will work with electric potentials and fields defined in Euclidean spaces of dimension $n \geq 2$, generalizing the previous discussion for $n \neq 3$. We will start by defining the electric potential V generated by a finite set of point charges in $\mathbb{R}^n$ by an expression analogous to the one we just saw, but depending on n. Then, we will define $E = -\nabla V$. The modulus of the electric field generated by a single charge at some point, will depend inversely on the distance between the point and the charge to the power of $n-1$. For more than one charge, we can use again the superposition principle. As a useful fact, we can see, since $E = -\nabla V$, that the zeros of E are precisely the critical points of V. This will allow us to study the equilibrium points of E by dealing with the critical points of V, for which study there exist more tools.

In his great Treatise [10], Maxwell made the following claim about the number of equilibrium points of electric fields generated by a finite set of point charges in dimension 3:

Conjecture ([10], p.136)**.** *The number of points of equilibrium (assumed to be finitely many) of an electric field generated by N point charges in $\mathbb{R}^3$ is at most $(N-1)^2$.*

It is an open problem to know if the number of equilibrium points is finite when all the charges have the same sign. For charges with different signs, as we will see later, there can be infinitely many. It has been done very little work concerning this conjecture. In fact, it was ignored for many years. However, in [4] they did some development on the matter that we will compare with our results. In particular, they obtained the upper bound $4^{N^2}(3N)^{2N}$ for the number of equilibrium points, which is not dependent on the dimension, only on the number of charges.

Our purpose in this work is to investigate the behavior of the equilibrium points of electric fields generated by finitely many charges in $\mathbb{R}^n$. In particular, we will study their location in relation to the charges, we will discuss "how many" electric fields have only non-degenerate equilibrium points and prove that if there are infinitely many equilibrium points, there can not be too many (in terms of an upper bound for the dimension of the set they form), among other things. The last part will be dedicated to obtaining different upper bounds for the number of equilibrium points depending on the dimension and the number of charges, which we will compare with each other and with the one from [4]. In most of the text, we will divide our discussions between the general case and the situation where all the charges have the same sign (positive or negative), since we will be able to get stronger results in the second setting. We now give a brief description of the contents in each chapter, focusing on the main results. Some of them are original and some of them are hard to find in the literature.

In Chapter 1, we introduce some notions from several branches of mathematics. In the first section, about Morse Theory, we introduce the definitions of smooth manifold and map, non-degenerate critical point, index of a critical point and Morse function. Also, we state the Morse Lemma, which shows that the behavior of a function around a non-degenerate critical point depends only on its index and will be useful in our study. In the second Section, we deal with real analytic geometry. We begin defining the notions of power series, real analytic function and analytic submanifold and we state some basic properties about them. Then, we define the concept of analytic set as the zero set of a real analytic function and see how these sets can be decomposed as a finite union of analytic submanifolds. The third and last Section is dedicated to a brief discussion about the some properties of harmonic functions. In particular, we prove they are real analytic functions and that, if a function is harmonic and non-constant, it can not have a relative maximum or minimum. We also talk a little about the Laplace operator and prove that is elliptic, a result that we use to prove Theorem 3.2.4 later.

In Chapter 2, we present the concepts that constitute our object of study. Specifically, in the first section, we define the concept of a configuration of charges $\{(p_k, q_k)\}_{k=1}^N$, where $p_1, \ldots, p_N \in \mathbb{R}^n$ are the different locations of N charges, and $q_1, \ldots, q_N$ are their values, and the electric potential created by it. Then, we introduce the electric field $E = -\nabla V$ associated to an electric potential V, whose equilibrium points (critical points of V) represent the subject of study of this text. In the second section, we explore the simpler case of dimension two and prove the following result:

Theorem. *Let* $\{(p_k, q_k)\}_{k=1}^N$ *be a configuration of* N *point charges in* $\mathbb{R}^2$*. Then, the electric potential* V *has at most* $N-1$ *critical points.*

Lastly, we study the family of configurations in $\mathbb{R}^2$ formed by placing $N \geq 3$ equal charges in the vertices of a regular N-sided polygon.

In Chapter 3 we prove some results about the behavior of the critical points of electric potentials generated by arbitrary configurations of charges. The chapter is divided into three sections. In the first one, we prove some characteristics concerning the location of critical points. First, we obtain some general properties, like the fact that, if the sum of the charges $q_1, \dots, q_N$ of a configuration is nonzero there can not be critical points of V too close to the charges or too far from them. From this, we are able to deduce that Morse electric potentials with nonzero total charge (the sum of the q_i) have finitely many critical points. In the second part of this first section, we restrict our study to the case where all the charges have the same sign and prove that, for any such configuration, all the critical points of V are contained in the interior of the convex hull of the charges (the smallest convex set that contains them).

In the second section, we prove that there can not exist a submanifold of dimension n or $n-1$ of critical points of V, if V is nonzero, with the help of Cauchy-Kovalevskaya Theorem (Theorem A.9). This establishes an upper bound for the dimension of $\mathrm{Cr}(V)$, the set of critical points of V, which tells us that there can not be "too many" critical points:

Theorem. *Let* V *be the electric potential generated by some configuration of charges in* $\mathbb{R}^n$*. Then, the critical set* $\mathrm{Cr}(V)$ *satisfies that* $\dim(\mathrm{Cr}(V)) \leq n-2$*. In particular, if it is nonempty,* $\mathrm{Cr}(V)$ *is a finite union of analytic submanifolds of* $\mathbb{R}^n$ *of dimensions between* 0 *and* $n-2$*.*

In the third section, we show that most configurations generate Morse electric potentials in the following sense:

Theorem. *The set of configurations of* N *charges in* $\mathbb{R}^n$ *with nonzero total charge that are Morse is dense and open in the set of all configurations of* N *charges in* $\mathbb{R}^n$ *(with certain topology).*

This result for $n = 3$ can be found in [14] and we proved it in the general case with a similar method. To finish the section and the chapter, we get a stronger result for configurations with the same sign, which is original:

Theorem. *For* $n \geq m \geq 1$*, the set of* $m-$*dimensional configurations of* N *charges in* $\mathbb{R}^n$ *with the same sign that are Morse is open and dense in the set of all* $m-$*dimensional configurations of* N *charges in* $\mathbb{R}^n$ *with the same sign (with certain topology).*

In this last result, we call a configuration $m-$dimensional if all its charges are contained in an affine set of dimension m and not in one of dimension $m-1$. Therefore, we see that the case $m = n$ corresponds to the previously mentioned result for configurations with the same sign.

In chapter 4 we use tools from real algebraic geometry to obtain upper bounds for the number of critical points of V, when there are finitely many. All the results in this chapter are original. As usual, we divide our study between the general case and the equal-sign case. Also, by the definition of electric field, it is useful to distinguish between even and odd n, since in the first case we can work directly with polynomials while in the second we need more work. We obtain, with help of some bounds for the sum of the Betti numbers of algebraic and semi-algebraic sets, the following results for arbitrary configurations:

Theorem. *Assume that* $\mathrm{Cr}(V)$ *is a finite set (*V *has finitely many critical points). Then,*

- *If* n *is even,* $\#\,\mathrm{Cr}(V) \leq (n(N-1)+1)\,(2n(N-1)+1)^{n-1} - N$*.*

- *If n is odd,*

$$\#\operatorname{Cr}(V) \leq \min \left\{ \begin{array}{c} 2n(N-1)\left(4n(N-1)-1\right)^{n+2N-1} - N, \\ \frac{1}{2}\left[2n(N-1)(2n+3N)+2\right]\left[2n(N-1)(2n+3N)+1\right]^{n+N-1} \end{array} \right\}$$

We see that there are two different bounds for odd n. It is interesting to see, as we prove later, that there are infinitely many pairs (n, N) for which the first is better (smaller) and infinitely many for which the second is better, so they are both relevant.

Finally, for configurations with the same sign, we use a version of Bézout's Theorem for Nash functions [15] to give an upper bound for $\#\operatorname{Cr}(V)$, when V is Morse, depending also on the dimension of the configuration that generates it:

Theorem. *If $\{(p_k, q_k)\}_{k=1}^N$ is an m-dimensional Morse configuration in $\mathbb{R}^n$ whose charges have the same sign, then V has finitely many critical points and*

- $\#\operatorname{Cr}(V) \leq (n(N-1)+1)^m$ *if n is even.*
- $\#\operatorname{Cr}(V) \leq 2^{Nm}(n(N-1)+1)^m$ *if n is odd.*

Chapter 1

Preliminary Material

In this first part of the text we introduce some important concepts and results from different parts of mathematics, which will be useful in order to prove the main results in the rest of the work.

1.1. Morse Theory

As it is said in [9],

> The primary concern of Morse theory is the relation between spaces and functions. The center of interest lies in how the critical points of a function defined on a space affect the topological shape of the space, and conversely, how the shape of a space controls the distribution of the critical points of a function.
>
> Morse theory of finite-dimensional manifolds is a powerful tool for the topology of manifolds, and offers a unified method to "visualize" manifolds with theoretical eyes. On the other hand, Morse theory for infinite-dimensional spaces clarifies the deep relations between variational problems and geometry, and is one of the basic principles of modern mathematics.

In this section, we introduce the basic concepts and results of Morse theory for finite-dimensional manifolds, that are the kind of objects which we will work with (in particular, we are going to deal with open subsets of $\mathbb{R}^n$). The contents are almost entirely extracted from the books [8], [9] and [13].

1.1.1. Smooth Manifolds and Smooth Maps

Definition 1.1.1 (Topological Manifold and Coordinate Chart)**.** Suppose that M is a topological space. We say that M is a topological manifold (without boundary) of dimension n or a topological $n-$manifold if it has the following properties:

(I) M is a Hausdorff space: for every pair of points $p, q \in M$, there are disjoint open neighborhoods $U, V \subseteq M$ of p and q, respectively.

(II) M is second-countable: there exists a countable basis for the topology of M.

(III) M is locally Euclidean of dimension n: for each point $p \in M$ there is an open neighborhood $U \subseteq M$ of p, an open subset $\widehat{U} \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to \widehat{U}$. The pair (U, φ) is called a coordinate chart (or just chart) on M.

Definition 1.1.2 (Transition Map and Smooth Atlas)**.** Let M be a topological $n-$manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the transition map from φ to ψ. It is a homeomorphism. The two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a smooth (C^∞) diffeomorphism. We define an atlas $\mathscr{A}$ for M to be a collection of charts whose domains cover M. An atlas $\mathscr{A}$ is called smooth if any two charts in $\mathscr{A}$ are smoothly compatible with each other.

Definition 1.1.3. A smooth atlas $\mathscr{A}$ on M is said to be maximal if it is not properly contained in any larger smooth atlas.

Proposition 1.1.4 ([8], Proposition 1.17)**.** *Let M be a topological manifold.*

(I) *Every smooth atlas $\mathscr{A}$ of M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathscr{A}$.*

(II) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas too.*

Definition 1.1.5 (Smooth Manifold)**.** A pair $(M, \mathscr{A})$ where M is topological manifold and $\mathscr{A}$ is a maximal smooth atlas of M is called a smooth manifold. Throughout the text we will simply say that M is a smooth manifold, assuming that it is endowed with some maximal atlas.

Definition 1.1.6 (Smooth Map Between Manifolds)**.** Let N and M be smooth manifolds of dimensions n and m respectively, and let $f : M \to N$ a map. We say that f is a smooth (or of class C^∞) map if for every $p \in M$, there exist smooth charts (U, x) containing p an (V, y) containing $f(p)$ such that $f(U) \subseteq V$ and the composition $y \circ f \circ x^{-1}$ is smooth from $x(U) \subseteq \mathbb{R}^n$ to $y(V) \subseteq \mathbb{R}^m$.

1.1.2. Critical Points

Notation 1.1.7. We will denote the tangent space of a manifold M at a point p by T_pM. If $f : M \to N$ is a smooth map with $f(p) = q$, the induced linear map between the tangent spaces will be denoted by $(f_*)_p : T_pM \to T_qN$.

Definition 1.1.8 (Critical Point and Critical Value)**.** Let f be a smooth real valued function on a manifold M. A point $p \in M$ is called a critical point of f if the following two equivalent conditions are satisfied:

(I) The induced map $(f_*)_p : T_pM \to T_{f(p)}\mathbb{R}$ is zero.

(II) If we choose a smooth chart (U, x) containing p, then

$$\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0.$$

Otherwise, p is said to be a regular point. We denote by $\mathrm{Cr}(f)$ the set of critical points of f and call it the critical set of f. A real number c is said to be a critical value of f if there exists some $p \in \mathrm{Cr}(f)$ such that $f(p) = c$. If c is not a critical value, it is called a regular value of f.

Remark 1.1.9. The previous definition makes sense, because p being a critical point does not depend on which chart we choose. This property is a consequence of the following results:

Proposition 1.1.10. *Let f be a smooth real valued function on a manifold M and $p \in M$. If (U, x) and (V, y) are two charts containing p, then*

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p), \quad i = 1, \ldots, n.$$

Corollary 1.1.11. *In the context of the previous proposition, we have that*

$$\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0 \iff \frac{\partial f}{\partial y_1}(p) = \ldots = \frac{\partial f}{\partial y_n}(p) = 0.$$

Definition 1.1.12 (Hessian Matrix)**.** Let M be an n-manifold, p a critical point of a real smooth function $f: M \to \mathbb{R}$ and (U, x) a smooth chart containing p. We define the Hessian of the function f at the critical point p to be the $n \times n$ matrix

$$H_f(p) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(p) & & \cdots & & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p) \\ & \ddots & & & \\ \vdots & & \frac{\partial^2 f}{\partial x_i \partial x_j}(p) & & \vdots \\ & & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p) & & \cdots & & \frac{\partial^2 f}{\partial x_n^2}(p) \end{pmatrix} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{1 \leq i,j \leq n}.$$

As f is smooth, $\frac{\partial^2 f}{\partial x_i \partial x_j}(p) = \frac{\partial^2 f}{\partial x_j \partial x_i}(p)$, and the matrix $H_f(p)$ is symmetric.

We now have the following lemma, that relates the Hessians of f at $p \in M$ in two local coordinate systems of p (smooth charts containing p).

Lemma 1.1.13. *In the same conditions as in the previous definition, let (U, x) and (V, y) be two local coordinate systems of p and $H_f(p)$, $\mathcal{H}_f(p)$ their respective Hessian matrices. Then they are related by*

$$\mathcal{H}_f(p) = J^t(p) H_f(p) J(p),$$

where $J(p)$ is the Jacobian matrix of the function $x : U \to \mathbb{R}^n$ with respect to the coordinate system $(y_1, \ldots, y_n)$ evaluated at the point p:

$$J(p) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1}(p) & & \cdots & & \frac{\partial x_1}{\partial y_n}(p) \\ & \ddots & & & \\ \vdots & & \frac{\partial x_i}{\partial y_j}(p) & & \vdots \\ & & & \ddots & \\ \frac{\partial x_n}{\partial y_1}(p) & & \cdots & & \frac{\partial x_n}{\partial y_n}(p) \end{pmatrix}$$

Definition 1.1.14 (Non-degenerate & Degenerate Critical Point)**.** A critical point p of a function $f: M \to \mathbb{R}$ is called non-degenerate if the matrix

$$H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is non-singular, i.e., has non-zero determinant. Otherwise, the critical point is said to be degenerate.

Proposition 1.1.15 (Non-degeneracy is Independent of Coordinate System)**.** *The property of a critical point of a function $f: M \to \mathbb{R}$ being non-degenerate or degenerate does not depend on the choice of the local coordinate system at p.*

Proof. We deduce from Lemma 1.1.13 that

$$\det \mathcal{H}_f(p) = \det J^t(p) \det H_f(p) \det J(p).$$

Since x and y are diffeomorphisms, the Jacobian $J(p)$ has non-zero determinant. Then, we have that

$$\det \mathcal{H}_f(p) = 0 \iff \det H_f(p) = 0,$$

as we wanted. □

Definition 1.1.16 (Index of a Non-Degenerate Critical Point)**.** Let p be a non-degenerate critical point of a function $f: M \to \mathbb{R}$. The matrix $H_f(p)$ defines a symmetric and bilinear form $f_{**} : T_pM \times T_pM \to \mathbb{R}$ with respect to the basis $\left\{ \frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p \right\}$. The index of a bilinear form H on a vector space V is defined to be the maximal dimension of a subspace of V on which H is negative definite. The index of f_{**} will be referred to as the index of f at p. This does not depend on the choice of local coordinate system (smooth chart).

Definition 1.1.17 (Morse Function)**.** We say that a function $f : M \to \mathbb{R}$ is a Morse function if every critical point of f is non-degenerate.

Next, we state one of the fundamental results in Morse theory, the Morse Lemma, which shows that the behavior of a function f around a non-degenerate critical point can be completely described by its index.

Theorem 1.1.18 (Morse Lemma)**.** *Let p be a non-degenerate critical point of a function $f : M \to \mathbb{R}$. Then, there is a local coordinate system (U, y) of p with $y_i(p) = 0$ for all i and such that*

$$f = f(p) - (y_1)^2 - \ldots - (y_\lambda)^2 + (y_{\lambda+1})^2 + \ldots + (y_n)^2$$

in U, where λ is the index of f at p.

Corollary 1.1.19. *A non-degenerate critical point of a function $f : M \to \mathbb{R}$ is isolated (from other critical points of f).*

Corollary 1.1.20. *Let $f : M \to \mathbb{R}$ be a Morse function on a manifold M and let K be a compact subset (compact space with the subspace topology) of M. Then, f has only finitely many critical points in K.*

Proof. If f had infinitely many critical points in K, we could form a sequence $\{p_i\}_{i=1}^{\infty} \subseteq \mathrm{Cr}(f) \cap K$ of different points. Since K is compact, there must be a subsequence $\{p_{k_i}\}_{i=1}^{\infty}$ that converges to some $p_0 \in K$. Now, as the function f is smooth in M, all its partial derivatives are continuous, so

$$\frac{\partial f}{\partial x_j}(p_0) = \frac{\partial f}{\partial x_j}\left(\lim_{i\to\infty} p_{k_i}\right) = \lim_{i\to\infty} \frac{\partial f}{\partial x_j}(p_{k_i}) = \lim_{i\to\infty} 0 = 0$$

for all j. Thus, p_0 is a critical point of f so, as f is a Morse function, by Corollary 1.1.19, p_0 must be isolated from other ones. However, we have constructed a sequence of critical points (different from p_0) that converges to p_0, which is a contradiction. Therefore, there cannot be infinitely many critical points of f contained in K. □

Remark 1.1.21 (Types of Non-degenerate Critical Points)**.** Let p be a non-degenerate critical point of a function $f : M \to \mathbb{R}$ and let λ be the index of f at p. Then,

(I) If $\lambda = 0$, f has a local minimum at p.

(II) If $\lambda = n$, f has a local maximum at p.

(III) If $0 < \lambda < n$, p is said to be a saddle critical point of f.

1.2. Real Analytic Geometry

In this section, we briefly introduce the theory of real analytic functions defined in open subsets of $\mathbb{R}^n$, with a small background on power series, and the definition and some properties of analytic sets (or analytic varieties) in $\mathbb{R}^n$. The properties that we describe here can be generalized by considering abstract real analytic manifolds instead of just open subsets of $\mathbb{R}^n$, but we will not need such a general setting. The part about power series and real analytic functions has been extracted from [6], and the one about analytic sets, from [17]. We will only include the proofs of a few results, because the study of real analytic functions is not the main topic of the text and a lot of them are not elementary. Still, the proofs of the results can be found in the previously mentioned books.

1.2.1. Power Series

Definition 1.2.1 (Multiindices). Let $\mathbb{Z}^+$ be the set $\{0, 1, 2, \ldots\}$. A multiindex μ is an element of $(\mathbb{Z}^+)^n$, with $n \geq 1$. We will write $\Lambda(n) = (\mathbb{Z}^+)^n$. It is the set of all multiindices of size n.

Definition 1.2.2. If

$$\mu = (\mu_1, \ldots, \mu_n) \in \Lambda(n) \quad \text{and} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

we define

$$\mu! = \mu_1! \ldots \mu_n! \quad , \quad |\mu| = \mu_1 + \cdots + \mu_n,$$

$$x^\mu = x_1^{\mu_1} \ldots x_n^{\mu_n} \quad , \quad |x|^\mu = |x_1|^{\mu_1} \ldots |x_n|^{\mu_n},$$

$$\partial^\mu = \frac{\partial^\mu}{\partial x^\mu} = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_n}}{\partial x_n^{\mu_n}},$$

$$(x)_\mu = \prod_{i=1}^n (x_i)_{\mu_i} = \prod_{i=1}^n \left[x_i(x_i - 1) \ldots (x_i - \mu_i + 1)\right].$$

Definition 1.2.3 (Power Series). A formal expression

$$\sum_{\mu \in \Lambda(n)} a_\mu (x - \alpha)^\mu = \sum_{(\mu_1, \ldots, \mu_n) \in \Lambda(n)} a_{(\mu_1, \ldots, \mu_n)} (x_1 - \alpha_1)^{\mu_1} \ldots (x_n - \alpha_n)^{\mu_n}$$

with $\alpha \in \mathbb{R}^n$ and $a_\mu \in \mathbb{R}$ for each μ, is called a power series in n variables. It is said to converge at $x \in \mathbb{R}^n$ if some rearrangement of the series converges. More precisely, the series converges if there is a bijective map $\phi : \mathbb{Z}^+ \to \Lambda(n)$ such that the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (x - \alpha)^{\phi(j)}$$

converges.

Proposition 1.2.4. *Let $\sum_\mu a_\mu (x - \alpha)^\mu$ be a power series and define the set $\mathcal{B} = \{x \in \mathbb{R}^n : |a_\mu||x - \alpha|^\mu \text{ is bounded}\}$. If the power series converges at x, then $x \in \mathcal{B}$.*

The series does not necessarily converge for every point of $\mathcal{B}$, but we will determine a subset of $\mathcal{B}$ for which it does.

Definition 1.2.5 (Domain of Convergence). For a fixed power series $\sum_\mu a_\mu (x - \alpha)^\mu$, we set

$$\mathcal{C} = \bigcup_{r > 0} \left\{ x \in \mathbb{R}^n : \sum_{\mu \in \Lambda(n)} |a_\mu (y - \alpha)^\mu| < +\infty \text{ for all } |y - x| < r \right\}.$$

The set $\mathcal{C}$ is called the domain of convergence of the power series.

This set is open. In fact, we have the following result:

Proposition 1.2.6. *A power series converges at every point of its domain of convergence and $\mathcal{C} = \operatorname{int} \mathcal{B}$.*

Then, the set of points for which the series converges is "between" $\operatorname{int} \mathcal{B}$ and $\mathcal{B}$. For $n = 1$, we know that the domain of convergence of a power series is an open interval centered at α (or just α). In several variables, the structure of $\mathcal{C}$ is not that simple, but we can can say something about its the shape:

Definition 1.2.7 (Logarithmically Convex). For a set $S \subseteq \mathbb{R}^n$, we define $\log ||S||$ by

$$\log ||S|| = \{(\log |s_1|, \ldots, \log |s_n|) : (s_1, \ldots, s_n) \in S\}.$$

The set S is said to be logarithmically convex if $\log ||S||$ is a convex subset of $\mathbb{R}^n$.

Proposition 1.2.8. *The domain of convergence $\mathcal{C}$ of a power series is logarithmically convex.*

1.2.2. Analytic Functions

Now, we can proceed to define the concept of real analytic function. These functions are, so to speak, locally equal to convergent power series.

Definition 1.2.9 (Real Analytic Function)**.** A real-valued function f, with domain an open subset $U \subset \mathbb{R}^n$, is called real analytic on U, written $f \in C^\omega(U)$, if for each $\alpha \in U$ the function f may be represented by a convergent power series in some neighborhood of α. A map $f : U \to \mathbb{R}^k$ given by $f = (f_1, \dots, f_k)$ is said to be real analytic on U if each f_i is real analytic on U. As we will always work here with $\mathbb{R}$, when referring to real analytic functions, we can omit the word "real" and just say that they are analytic.

Proposition 1.2.10 (Derivatives of Analytic Functions)**.** *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be an analytic function. Then, f is continuous and has continuous, real analytic partial derivatives of all orders. In fact, if f can be represented by the power series*

$$\sum_{\mu\in\Lambda(n)} a_\mu (x-\alpha)^\mu$$

around $\alpha \in U$, then, for each $\nu \in \Lambda(n)$, the partial derivative $\partial^\nu f$ of f is given by the following convergent power series around $\alpha \in U$:

$$\sum_{\mu\in\Lambda(n)} (\mu+\nu)_\nu a_{\mu+\nu} (x-\alpha)^\mu.$$

It is, in fact, the expression that is obtained by differentiating each term of the power series. We deduce that $C^\omega(U) \subseteq C^\infty(U)$.

Remark 1.2.11. We can relate the coefficients of the power series representing f around α to the partial derivatives of the function at α. Concretely,

$$a_\mu = \frac{1}{\mu!} \frac{\partial^\mu f}{\partial x^\mu}(\alpha).$$

It is interesting to verify that power series define analytic functions in their domain of convergence. Concretely, we have the following result:

Proposition 1.2.12. *Let*

$$\sum_{\mu\in\Lambda(n)} a_\mu (x-\alpha)^\mu$$

be a power series and $\mathcal{C}$ its (nonempty) domain of convergence. If $f : \mathcal{C} \to \mathbb{R}$ is defined by

$$f(x) = \sum_{\mu\in\Lambda(n)} a_\mu (x-\alpha)^\mu,$$

then f is real analytic.

Example 1.2.13. Some examples of real analytic functions are:

(I) Every polynomial in $\mathbb{R}[X_1, \dots, X_n]$, as they are power series around $(0, \dots, 0) \in \mathbb{R}^n$ with $a_\mu = 0$ for all μ except a finite number of them.

(II) For $n = 1$, the functions

$$x \mapsto e^x \quad , \quad x \mapsto \sin(x) \quad , \quad x \mapsto cos(x).$$

In fact, they can be represented by a power series centered at the origin, that converges in all $\mathbb{R}^n$, in the following way:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^3}{24} + \dots$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

It is easy to see that if we differentiate termwise the series for $\sin(x)$, we get the one for $\cos(x)$, as we expected from Proposition 1.2.10.

Example 1.2.14. We can find examples of functions of class C^∞ that are not analytic, which means that the inclusion in 1.2.10 can be proper. For example, take the function

$$\begin{array}{cccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \end{array}$$

It can be verified that f has continuous derivatives of all orders, so $f \in C^\infty(\mathbb{R})$. However, it satisfies that

$$f(0) = 0 \quad \text{and} \quad f^{(k)}(0) = 0 \text{ for all } k \geq 1,$$

where $f^{(k)}(0)$ denotes the k-th derivative of f at 0. Therefore, if f were analytic at 0, we would have that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0$$

in a neighborhood of 0. But $f(x) >$ for $x > 0$, so this is not possible. We deduce that $f \notin C^\omega(\mathbb{R})$.

Definition 1.2.15 (Analytic Submanifold)**.** A nonempty subset $S \subset \mathbb{R}^n$ is called an m-dimensional real analytic submanifold (or submanifold of class C^ω) of $\mathbb{R}^n$ if, for each $p \in S$, there exists an open subset $U \subset \mathbb{R}^m$ and a real analytic function $f : U \to \mathbb{R}^n$ which maps open subsets of U onto relatively open subsets of S and which is such that

$$p \in f(U) \quad \text{and} \quad \operatorname{rank}[Df(u)] = m, \ \forall u \in U,$$

where $Df(u)$ is the Jacobian matrix of f at u. Submanifolds of dimension n of $\mathbb{R}^n$ coincide with non-empty open subsets of $\mathbb{R}^n$. Replacing "real analytic" by C^k ($1 \leq k \leq \infty$), we get the definition of m-dimensional submanifold of class C^k.

Remark 1.2.16. By the explanation given in Remark A.2, we see that $(n-1)$-dimensional submanifolds of class C^k are the same as hypersurfaces of class C^k (cf. Definition A.1), for $1 \leq k \leq \infty$ or $k = \omega$.

We can extend the definition of real analytic function to analytic submanifolds. We will not need that here, as was said in the beginning of the section, but it is included anyway.

Definition 1.2.17. Let S be a real analytic submanifold of $\mathbb{R}^n$, and let $h : S \to \mathbb{R}$. We say that h is real analytic at $p \in S$ if, for f as in Definition 1.2.15 and for a point u_0 such that $f(u_0) = p$, the function $h \circ f$ is real analytic at u_0.

1.2.3. Analytic Sets

Definition 1.2.18 (Analytic Subset)**.** Let U be an open subset of $\mathbb{R}^n$. A (globally) analytic subset (also called analytic space or analytic variety) of U is a set of the form

$$\mathcal{Z}_U(f_1, \dots, f_k) = \{(x_1, \dots, x_n) \in U : f_1(x_1, \dots, x_n) = \dots = f_k(x_1, \dots, x_n) = 0\},$$

where $f_1, \dots, f_k : U \to \mathbb{R}$ are real analytic functions. When the set U is clear by the context, we will simply write $\mathcal{Z}(f_1, \dots, f_k)$. We will say that some $E \subseteq \mathbb{R}^n$ is an analytic set if it is an analytic subset of some U.

Remark 1.2.19. By the properties of the field $\mathbb{R}$, we can see that every analytic set can be described by a single analytic function. In fact, if we define

$$f = f_1^2 + \ldots + f_k^2,$$

then f is analytic too and we have that

$$\mathcal{Z}_U(f_1, \ldots, f_k) = \mathcal{Z}_U(f).$$

Proposition 1.2.20. *Let U be a connected open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a real analytic function. If $\operatorname{int} Z_U(f) \neq \emptyset$, then $f \equiv 0$.*

Proof. Call $E = Z_U(f)$. Now, let (α_k) be a sequence in $\operatorname{int} E$ that converges to some $\alpha \in U$. As f is analytic, we know that there is some neighborhood $N \subset U$ of α such that

$$f(x) = \sum_{\mu \in \Lambda(m)} \frac{1}{\mu!} \frac{\partial^\mu f}{\partial x^\mu}(\alpha)(x - \alpha)^\mu$$

in N. Also, again for being analytic, all the partial derivatives of f are continuous in U (by Proposition 1.2.10). Then,

$$\frac{\partial^\mu f}{\partial x^\mu}(\alpha) = \frac{\partial^\mu f}{\partial x^\mu}\left(\lim_{k \to \infty} \alpha_k\right) = \lim_{k \to \infty} \frac{\partial^\mu f}{\partial x^\mu}(\alpha_k) = \lim_{k \to \infty} 0 = 0,$$

because $f = 0$ in some neighborhood of α_k for each k. Consequently, $f = 0$ in N. This means that $\alpha \in \operatorname{int} E$, concluding that $\operatorname{int} E$ is closed. As it is nonempty by hypothesis and clearly open, and U is connected, $\operatorname{int} E$ has to be equal to U, so

$$U = \operatorname{int} E \subseteq E \subseteq U.$$

The result follows. □

Corollary 1.2.21. *A nonzero analytic function defined on a connected set cannot have an open set of zeros.*

Definition 1.2.22 (Dimension of a Set)**.** Let E be a nonempty subset of $\mathbb{R}^n$. Then, the dimension of E is defined by

$$\dim E = \sup\{\dim \Gamma : \Gamma \text{ is a real analytic submanifold contained in } E\}.$$

For the empty set, we assume that $\dim \emptyset = -\infty$.

Proposition 1.2.23. *The dimension of any countable non-empty set is equal to* 0*. If $E \subseteq F$, then $\dim E \leq \dim F$. Also, if $\{E_i\}_{i \in I}$ is a family of sets such that each E_i is open in $\bigcup_{i \in I} E_i$, then*

$$\dim \left(\bigcup_{i \in I} E_i \right) = \max_{i \in I} \dim E_i.$$

Definition 1.2.24 (F-sigma Set)**.** A subset of a topological space is said to be an F_σ set if it is a countable union of closed sets.

Proposition 1.2.25. *Let (X, d) be a metric space and let U be a nonempty open subset of X. Then, U is an F_σ set in the topology induced by the metric d.*

Proof. We define the set

$$F_i = \{x \in U : d(x, X \setminus U) \geq 2^{-n}\}$$

for each $i \in \mathbb{N}$, where $d(x, X \setminus U)$ is the distance from the point x to the set $X \setminus U$, that is,

$$d(x, X \setminus U) := \inf \{d(x, y) : y \in X \setminus U\}.$$

Since the mapping $x \mapsto d(x, X \setminus U)$ is continuous, every F_i is closed. We have that $U = \bigcup_{i=1}^{\infty} F_i$, so U is an F_σ set. □

Proposition 1.2.26. *If $\{E_i\}_{i=1}^{\infty}$ is a countable family of F_σ subsets of $\mathbb{R}^n$, then*

$$\dim\left(\bigcup_{i=1}^{\infty} E_i\right) = \max_{i\in\mathbb{N}} \dim E_i.$$

Definition 1.2.27 (Nowhere Dense Set)**.** Let U be a subset of a topological space. Then, U is said to be nowhere dense if

$$\operatorname{int}(\overline{U}) = \emptyset.$$

Proposition 1.2.28. *Let U be a connected open subset of $\mathbb{R}^n$ and let E be a proper analytic subset of U. Then, E is nowhere dense in U.*

Proof. As E is the zero set of some analytic function in U, it is closed in U. If $\operatorname{int}_U E \neq \emptyset$, by Proposition 1.2.20, we know that $E = U$. But $\operatorname{int}_U E = \operatorname{int}_U(\overline{E})$, as E is closed. This means that if E is not nowhere dense, then $E = U$, and the result follows. □

Proposition 1.2.29. *For a nonempty subset $E \subseteq \mathbb{R}^n$,*

$$0 \leq \dim E \leq n.$$

Also,

$$\dim E = n \iff \operatorname{int} E \neq \emptyset.$$

Consequently, if E is nowhere dense, then $\dim E \leq n-1$.

Analytic sets have an interesting structure. We describe it below:

Theorem 1.2.30. *Let E be a nonempty analytic set. The points of E at which E is an analytic submanifold (of dimension m), i.e. the points $p \in E$ that satisfy the properties mentioned in Definition 1.2.15, are called the regular points (of dimension m) of E. The remaining points of E are said to be its singular points. We denote by $E^*, E^0, E^{(m)}$ the set of singular points (or singular locus), the set of regular points, and the set of regular points of dimension m, respectively, of E. Thus we have the decompositions*

$$E = E^* \cup E^0 \quad \text{and} \quad E^0 = E^{(0)} \cup \cdots \cup E^{(n)}.$$

The set $E^{(m)}$ is an $m-$dimensional analytic submanifold and is open in E. Therefore, E^0 is open in E, while E^* is closed. Moreover, the submanifolds $E^{(m)}$ are open and closed subsets of E^0.

Theorem 1.2.31. *In the context of the previous theorem,*

(I) The set E^0 of regular points is open and dense in E, whereas the set E^* of singular points is closed and nowhere dense.

(II) E^* is itself an analytic subset of U and $\dim E^* \leq \dim E - 1$.

As a direct consequence of Theorems 1.2.30 and 1.2.31(ii), we have the following:

Corollary 1.2.32. *Every analytic set is a finite union of analytic submanifolds.*

We can figure out the dimension of an analytic set if we know the dimensions of the analytic submanifolds that form its set of regular points.

Proposition 1.2.33. *Let $E = E^* \cup E^0 = E^* \cup E^{(0)} \cup \cdots \cup E^{(n)}$ be a nonempty analytic set. Then,*

$$\dim E = \max\left\{\dim E^{(0)}, \dim E^{(1)}, \ldots, \dim E^{(n)}\right\},$$

Proof. We know that E is a metric space with the Euclidean distance in $\mathbb{R}^n$ restricted to E. Also, E^* is closed in E and E^0 is open (cf. Theorem 1.2.30). Then, by Proposition 1.2.25, both E^* and E^0 are F_σ sets. We deduce from Proposition 1.2.26 that

$$\dim E = \max\left\{\dim E^*, \dim E^0\right\}.$$

But we know that $\dim E^* < \dim E$ (cf. Theorem 1.2.31(*ii*)), so $\dim E = \dim E^0$. Now, as each $E^{(i)}$ is open in E^0, we have, by Proposition 1.2.23, that

$$\dim E = \dim E^0 = \max\left\{\dim E^{(0)}, \dim E^{(1)}, \dots, \dim E^{(n)}\right\},$$

as we wanted. □

Remark 1.2.34. We can see that $\dim E^{(i)} = i$ if and only if $E^{(i)} \neq \emptyset$, as $E^{(i)}$ is an $i-$dimensional analytic submanifold (if it is nonempty). Otherwise, by definition, $\dim E^{(i)} = -\infty$. Thus, we have that

$$\dim E = \max\left\{i \in \{0, 1, \dots, n\} : E^{(i)} \neq \emptyset\right\},$$

because E is nonempty.

1.3. Laplace Operator and Harmonic Functions

In this section we explore briefly some features of what is, perhaps, one of the most important partial differential operators, the Laplace operator or Laplacian Δ (or ∇^2), and of harmonic functions, which are defined by that operator. These properties will help us investigate the behavior of electric potentials and their critical points in the following chapters because potentials are harmonic, as we will prove later.

Definition 1.3.1 (Laplace Operator)**.** The Laplace operator on $\mathbb{R}^n$ is defined by

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Proposition 1.3.2 (Laplace Operator is Elliptic)**.** *The operator Δ is elliptic in all $\mathbb{R}^n$ (see Appendix A). In particular, every analytic hypersurface in $\mathbb{R}^n$ is characteristic for the operator Δ.*

Proof. Take any point $x \in \mathbb{R}^n$ and some nonzero vector $\xi \in \mathbb{R}^n$. Then,

$$\chi_\Delta(x, \xi) = \sum_{i=1}^{n} \xi_i^2 \neq 0.$$

We deduce that $\mathrm{char}_x(\Delta) = \emptyset$ for all $x \in \mathbb{R}^n$, and the result follows. □

Definition 1.3.3 (Harmonic Function)**.** Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a function of class C^2. Then, f is said to be a harmonic function if it satisfies Laplace equation in U, that is, if

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \dots + \frac{\partial^2 f}{\partial x_n^2}(x) = 0$$

for all $x \in U$.

The following proposition will allow us to apply the results from the previous section to harmonic functions:

Proposition 1.3.4 (Harmonic Functions are Real Analytic ([2], Theorem 10 on p. 31))**.** *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a harmonic function. Then, f is real analytic in U.*

Proposition 1.3.5 (Maximum Principle ([3], Theorem 2.13)). *Let U be a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a harmonic function. If $\sup_{x\in U} f(x) = A < \infty$, then either $f(x) < A$ for all $x \in U$ or $f(x) = A$ for all $x \in U$. That is, if f is non-constant, it cannot have a (global) maximum in U.*

Corollary 1.3.6 ([3], Corollary 2.14). *If $\overline{U}$ is compact and f is continuous on $\overline{U}$, then the maximum of f on $\overline{U}$ is achieved in $\partial\overline{U}$.*

Proposition 1.3.7 (Liouville's Theorem([3], Theorem 2.16)). *If f is bounded and harmonic in all $\mathbb{R}^n$, then f is constant.*

Propositions 1.2.20, 1.3.4, and 1.3.5 allows us to restrict the types of non-degenerate critical points harmonic functions can have.

Proposition 1.3.8 (Critical Points in Harmonic Functions). *If f is harmonic and non-constant, it cannot have a relative maximum or a relative minimum in U. Thus, all non-degenerate critical points of f are saddle points (cf. Remark 1.1.21).*

Proof. Suppose $x_0 \in U$ is a relative maximum of f and let $M = f(x_0)$. As f is analytic by Proposition 1.3.4, so is $g = f - M$. Since x_0 is a relative maximum, there exists some $\varepsilon > 0$ such that

$$f(x) \leq M \quad \text{for every} \quad x \in B_\varepsilon(x_0).$$

In that case, we can apply the Maximum Principle (Proposition 1.3.5) to the restriction

$$f\big|_{B_\varepsilon(x_0)} : B_\varepsilon(x_0) \to \mathbb{R},$$

which is clearly harmonic, and get that

$$f(x) = M \quad \text{for every} \quad x \in B_\varepsilon(x_0).$$

Therefore, the set

$$S = f^{-1}(\{M\}) = \{x \in U : f(x) = M\} = V_U(g)$$

has nonempty interior. Thus, from Proposition 1.2.20, we deduce that $g \equiv 0$, what means that

$$\{x \in U : f(x) = M\} = V_U(g) = U,$$

so f is constantly equal to M, a contradiction. Consequently, f cannot have a relative maximum in U. To prove the result for relative minimums, we notice that $-f$ is harmonic and non-constant, and that

$$x_0 \text{ is a relative minimum of } f \quad \Longleftrightarrow \quad x_0 \text{ is a relative maximum of } -f,$$

so we can follow the previous reasoning with $-f$. □

Chapter 2

Electrostatics

In this chapter, we introduce the concepts that make up the core of this work. Specifically, we start defining the electric potential created when a finite number of point charges are located in different points of the Euclidean space $\mathbb{R}^n$, with $n \geq 2$. Then, we talk about the electric field $E = -\nabla V$ associated to each potential, whose zeros (critical points of V) represent the subject of study of this text, and next we explore the simpler case of dimension two.

2.1. Basic Definitions

Definition 2.1.1 (Configuration of Charges). Let $n, N \in \mathbb{N}$ be natural numbers such that $n \geq 2$ and $N \geq 1$. A finite set $\{(p_k, q_k)\}_{k=1}^N$ with $p_k \in \mathbb{R}^n$ ($p_i \neq p_j$ if $i \neq j$), $q_k \in \mathbb{R}$ ($q_k \neq 0$) is called a configuration of N point charges in the Euclidean space $\mathbb{R}^n$. The points $p_k \in \mathbb{R}^n$ are called electric charges and $q_k \in \mathbb{R}$ are the respective values of the charges.

Definition 2.1.2 (Electric Potential). Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of point charges as in the previous definition. We define the electric potential generated by the configuration as

$$\begin{aligned} V_{\{(p_k,q_k)\}_{k=1}^N} : \mathbb{R}^n \setminus \{p_1, \ldots, p_N\} &\longrightarrow \mathbb{R} \\ p &\longmapsto \begin{cases} -\dfrac{1}{4\pi\epsilon_0} \displaystyle\sum_{k=1}^N q_k \ln |p - p_k|, & \text{if} \quad n = 2, \\ \dfrac{1}{4\pi\epsilon_0} \displaystyle\sum_{k=1}^N \dfrac{q_k}{|p - p_k|^{n-2}}, & \text{if} \quad n \geq 3, \end{cases} \end{aligned}$$

where the constant ϵ_0 is the permittivity of vacuum, $\frac{1}{4\pi\epsilon_0}$ is called the Coulomb constant and $|p - p_k|$ is the Euclidean distance between the points p and p_k in $\mathbb{R}^n$. As this text focuses on the study of critical points of the electric potential, multiplication by a constant is irrelevant. Therefore, we will omit it and define the potential by

$$p \longmapsto \begin{cases} -\displaystyle\sum_{k=1}^N q_k \ln |p - p_k|, & \text{if} \quad n = 2, \\ \displaystyle\sum_{k=1}^N \dfrac{q_k}{|p - p_k|^{n-2}}, & \text{if} \quad n \geq 3, \end{cases}$$

When the configuration is clear by the context or is not important, we will simply write V when talking about the function $V_{\{(p_k,q_k)\}_{k=1}^N}$.

Definition 2.1.3 (Total Charge). Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of charges. Its total charge is just the sum of its charges, i.e., $\sum_{k=1}^N q_k$.

Remark 2.1.4. The definition of electric potential for dimension 2 can look strange compared the rest, which have a similar description. The reason for this difference will be clear in a moment, when we see the definition of electric field E, which is what we care about, as $-\nabla V$. In fact, we will see that the gradient of V has a very similar structure for each n.

Definition 2.1.5 (Electric Field)**.** Given an electric potential V for some configuration of charges, its associated electric field is defined by

$$E = -\nabla V = \left(-\frac{\partial V}{\partial x_1}, \dots, -\frac{\partial V}{\partial x_n}\right)$$

Concretely, for each point $p \in \mathbb{R}^n \setminus \{p_1, \dots, p_N\}$,

$$E(p) = \begin{cases} \displaystyle\sum_{k=1}^{N} q_k \frac{p - p_k}{|p - p_k|^2}, & \text{if} \quad n = 2, \\ \displaystyle\sum_{k=1}^{N} (n-2) q_k \frac{p - p_k}{|p - p_k|^n}, & \text{if} \quad n \geq 3. \end{cases}$$

As we have previously said, some $p \in \mathbb{R}^n \setminus \{p_1, \dots, p_N\}$ is a critical point of V ($\nabla V(p) = 0$) if and only if $E(p) = 0$. Then, all the results about critical points of V throughout the text will talk about equilibrium points of the electric field E as well.

Remark 2.1.6. From now on, we will consider the constants

$$C_2 = 1, \quad C_n = (n-2) \quad \text{for} \quad n \geq 3,$$

so we can write

$$\nabla V(p) = -C_n \sum_{k=1}^{N} q_k \frac{p - p_k}{|p - p_k|^n}$$

instead of distinguishing by cases.

Definition 2.1.7 (Morse configuration)**.** Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of charges in $\mathbb{R}^n$ for some n. We say that it is a Morse configuration if the electric potential V they generate is a Morse function.

Remark 2.1.8. We are always going to consider configurations with more than one point, i.e., $N \geq 2$. The reason for this is the fact that the electric potential created by a single charge configuration has no critical points (and therefore it is of no interest to us). Actually, the electric field generated by $(p_1, q_1) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$ in a point $p \in \mathbb{R}^n \setminus \{p_1\}$ is given by

$$E(p) = C_n q_1 \frac{p - p_1}{|p - p_1|^n}.$$

As $p \neq p_1$ and $q_1 \neq 0$, we have two possibilities:

(I) If $q_1 > 0$, then $E(p) = \frac{C_n q_1}{|p - p_1|^n}(p - p_1)$ with $\frac{C_n q_1}{|p - p_1|^n} > 0$, so it is a nonzero vector pointing outwards from the charge at p_1.

(II) If $q_1 < 0$, then $E(p) = \frac{C_n q_1}{|p - p_1|^n}(p - p_1)$ with $\frac{C_n q_1}{|p - p_1|^n} < 0$, so it is a nonzero vector pointing inwards to the charge at p_1.

In both cases, V has no critical points and the integral curves of the field E are radial centered at p_1.

Remark 2.1.9. In some parts of this text, we will need to use the expressions of the second order partial derivatives of V. For this reason, it is convenient to compute them all now. First of all, let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of point charges in $\mathbb{R}^n$, and V its corresponding potential. Also, denote $p_k = (x_{k,1}, \dots, x_{k,n})$ for each k and let $p = (x_1, \dots, x_n)$ be some point in $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$. Then,

$$\frac{\partial^2 V}{\partial x_i^2}(p) = C_n \sum_{k=1}^{N} q_k \frac{n(x_i - x_{k,i})^2 - |p - p_k|^2}{|p - p_k|^{n+2}}$$

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(p) = \frac{\partial^2 V}{\partial x_j \partial x_i}(p) = C_n \sum_{k=1}^{N} q_k \frac{n(x_i - x_{k,i})(x_j - x_{k,j})}{|p - p_k|^{n+2}} \quad \text{if} \quad i \neq j$$

2.2. Simple Case: Electric Potentials in Dimension 2

In this section, we begin investigating the behavior of critical points of electric potentials by restricting our view to the case of dimension 2. The expression of the electric field for N charges is similar in all cases (depending on n), as we saw in Definition 2.1.5. Consequently, it is reasonable to think that electric potentials in all dimensions share some properties concerning their critical points. This general properties will be explored in Chapter 3. However, the case of dimension 2 has some interesting features, like the possibility of identifying $\mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$ in a convenient way. This allows us to establish an upper bound for the number of critical points, depending on the number of charges, by relating the critical points of V to the roots of a complex polynomial and using the Fundamental Theorem of Algebra. In fact, we will see later that this bound is sharp, i.e., we can find examples at which it is reached.

Throughout this part of the text, we will view points of $\mathbb{R}^2$ as complex numbers in $\mathbb{C}$ by means of the correspondence $(x, y) \mapsto x + iy$. Also, given a configuration of charges $\{(p_k, q_k)\}_{k=1}^N$ in $\mathbb{R}^2$, we are going to consider the electric field generated by that configuration at some $z \in \mathbb{C} \setminus \{z_1, \ldots, z_N\}$ to be

$$\widetilde{E}_{\{(z_k,q_k)\}_{k=1}^N}(z) = \sum_{k=1}^N q_k \frac{z - z_k}{|z - z_k|^2},$$

where $p_k \mapsto z_k$. We are able to work with $\mathbb{C}$ to investigate the critical points of electric potentials in dimension 2 because some $(x, y) \in \mathbb{R}^2$ is a critical point of the potential $V_{\{(p_k,q_k)\}_{k=1}^N}$ if and only if

$$\widetilde{E}_{\{(z_k,q_k)\}_{k=1}^N}(x + iy) = 0.$$

2.2.1. Upper Bound for the Number of Critical Points

Having made this clear, we can proceed to prove the upper bound for the number of critical points of V that we mentioned. This result can be considered "folk wisdom", but it is not easy to find a reference where it is proven. We start by stating a well known and important theorem about roots of complex polynomials in one variable:

Lemma 2.2.1 (Fundamental Theorem of Algebra). *Let $f = a_n X^n +, \ldots +, a_1 X + a_0$ be a polynomial in $\mathbb{C}[X]$ of degree $n \geq 1$. Then, f has n roots in $\mathbb{C}$, counting their multiplicity.*

A direct consequence of this theorem is the fact that a polynomial in one variable, with complex coefficients, of degree $n \geq 1$, has at most n different roots in $\mathbb{C}$. We can now prove the following:

Theorem 2.2.2 (Upper Bound for the Number of Critical Points). *Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of N point charges in $\mathbb{R}^2$. Then, the electric potential has at most $N - 1$ critical points.*

Proof. As was said before, a point $p = (x, y) \in \mathbb{R}^2 \setminus \{p_1, \ldots, p_N\}$ is a critical point of V if and only if $\widetilde{E}(z) = 0$, being $z = x + iy$. That is,

$$p \in \mathrm{Cr}(V) \iff \sum_{k=1}^N q_k \frac{z - z_k}{|z - z_k|^2} = 0 \iff \sum_{k=1}^N \overline{\left(q_k \frac{1}{z - z_k}\right)} = 0 \iff \sum_{k=1}^N q_k \frac{1}{z - z_k} = 0$$

Now, as $z \neq z_k$ for all k,

$$\sum_{k=1}^N q_k \frac{1}{z - z_k} = 0 \iff \prod_{k=1}^N (z - z_k) \sum_{k=1}^N q_k \frac{1}{z - z_k} = 0 \iff \sum_{j=1}^N q_k \prod_{k \neq j} (z - z_k) = 0,$$

because

$$\sum_{j=1}^N q_k \prod_{k \neq j} (z_i - z_k) \neq 0 \quad , \quad \forall i \in \{1, \ldots, N\} \ .$$

Finally, as the expression

$$f = \sum_{j=1}^{N} q_k \prod_{k \neq j} (X - z_k)$$

is a polynomial with complex coefficients of degree $N-1$, by Lemma 2.2.1, it has at most $N-1$ different roots. And, as the roots of f are the zeros of $\widetilde{E}$, and these correspond to the critical points of V, the result follows. □

It is interesting to note, as we mentioned at the beginning of this section, that this bound is reached for some configurations of charges, which means that it is the lowest possible. A simple case is the potential generated by a configuration of N equal charges located in a line, which has a critical point between each pair of adjacent charges:

Example 2.2.3. Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of charges located in a line with all q_k equal to some $q \neq 0$. Also, we are going to consider the case were $p_k \in \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ for all k. That is, $z_k \in \mathbb{R} \subset \mathbb{C}$. In the next chapter, we will see that studying this case is enough to know the properties of all configurations of N charges contained in a line.

We know, form the proof of Theorem 2.2.2, that the critical points of the potential V correspond to the roots of the polynomial

$$f = \sum_{j=1}^{N} q_k \prod_{k \neq j} (X - z_k) = q \sum_{j=1}^{N} \prod_{k \neq j} (X - z_k) \in \mathbb{R}[X].$$

We can consider f as a continuous function $f : \mathbb{R} \to \mathbb{R}$. Besides, if we define

$$g = q \prod_{k=1}^{N} (X - z_k) \in \mathbb{R}[X],$$

whose roots are $z_1, \ldots, z_N$, we have that $f = g'$. By the Mean Value Theorem, we know that there is a root of f between each pair of adjacent points of the set $\{z_1, \ldots, z_N\} \subset \mathbb{R}$. Thus, as they have to be different, f has at least $N-1$ roots, so it has exactly $N-1$, by Theorem 2.2.2. We deduce that V has $N-1$ critical points, reaching the bound.

2.2.2. Example: Configurations of Equal Charges Forming Regular Polygons

In this section, we are going to consider the example of a family of configurations of charges that generate electric potentials with a single critical point, which is degenerate. Let $N \geq 3$ and let $\{(z_k, q_k)\}_{k=1}^N$ be a configuration of equal charges in $\mathbb{R}^2$ located at the vertices of an N-sided regular polygon. As before, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and consider $z_k = x_k + iy_k$. Without loss of generality, we may assume that $z_k = e^{\frac{2\pi k}{N} i}$ and that $q_k = 1$. To avoid being repetitive, we will say that some $z = x + iy \in \mathbb{C}$ is a critical point of V when referring to the fact that $p = (x, y)$ is a critical point of V.

We have that

$$\widetilde{E}(0) = \sum_{k=1}^{N} q_k \frac{z_k}{|z_k|^2} = \sum_{k=1}^{N} e^{\frac{2\pi k}{N} i} = 0,$$

because $-\sum_{k=1}^{N} e^{\frac{2\pi k}{N} i}$ is the coefficient of X^{N-1} in the polynomial

$$\prod_{k=1}^{N} (X - e^{\frac{2\pi k}{N} i}) = X^N - 1.$$

Then, $z = 0$ is a critical point of V. If there were some other critical point $z' \neq 0$, all the N different points in the set

$$\left\{ z' e^{\frac{2\pi}{N} i}, z' e^{\frac{2\pi 2}{N} i}, \dots, z' e^{\frac{2\pi (N-1)}{N} i} \right\}$$

should be too. As this would contradict Theorem 2.2.2, $z = 0$ is the only critical point of V. We may now check if it is degenerate or not. We will use the following lemma.

Lemma 2.2.4. *Let N be a natural number such that $N \geq 2$. Then,*

$$\sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) = \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right) = 0.$$

Proof. By what we previously proved,

$$0 = \sum_{k=1}^{N} e^{\frac{2\pi k}{N} i} = \sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) + i \sin\left(\frac{2\pi k}{N}\right) = \sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) + i \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right).$$

The result follows. □

Now, by Remark 2.1.9 and Lemma 2.2.4, we can compute the following:

$$\frac{\partial^2 V}{\partial x^2}(0) = \sum_{k=1}^{N} \frac{|z_k|^2 - 2(x_k)^2}{|z_k|^4} = \sum_{k=1}^{N} 1 - 2(x_k)^2 = \sum_{k=1}^{N} 1 - 2\left(\cos\left(\frac{2\pi k}{N}\right)\right)^2 = -\sum_{k=1}^{N} \cos\left(\frac{4\pi k}{N}\right) =$$

$$= \begin{cases} -2\sum_{k=1}^{N/2} \cos\left(\frac{2\pi k}{N/2}\right) = 0 & \text{if } N \text{ is even} \\ -\sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) = 0 & \text{if } N \text{ is odd} \end{cases}$$

$$\frac{\partial^2 V}{\partial y^2}(0) = \sum_{k=1}^{N} \frac{|z_k|^2 - 2(y_k)^2}{|z_k|^4} = \sum_{k=1}^{N} 1 - 2(y_k)^2 = \sum_{k=1}^{N} 1 - 2\left(\sin\left(\frac{2\pi k}{N}\right)\right)^2 = \sum_{k=1}^{N} \sin\left(\frac{4\pi k}{N}\right) =$$

$$= \begin{cases} 2\sum_{k=1}^{N/2} \sin\left(\frac{2\pi k}{N/2}\right) = 0 & \text{if } N \text{ is even} \\ \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right) = 0 & \text{if } N \text{ is odd} \end{cases}$$

$$\frac{\partial^2 V}{\partial x \partial y}(0) = \frac{\partial^2 V}{\partial y \partial x}(0) = -\sum_{k=1}^{N} 2 x_k y_k = -\sum_{k=1}^{N} 2\cos\left(\frac{2\pi k}{N}\right) \sin\left(\frac{2\pi k}{N}\right) = -\sum_{k=1}^{N} \sin\left(\frac{4\pi k}{N}\right) = 0.$$

Then, the Hessian of V at the point 0 corresponds to the zero matrix, so it is singular, that is, the critical point is degenerate.

Chapter 3

General Results about the Critical Points of Electric Potentials

In this chapter we proceed to prove some results about the behavior of the critical points of electric potentials generated by arbitrary configurations of charges. These properties will help us understand the nature of electric potentials. The chapter is divided into three sections. In the first one we prove some properties concerning the location of critical points, first some general features and then some that are satisfied for configurations of charges with the same sign. Then, in the second section, we give an upper bound for the dimension of the set of critical points of any nonzero electric potential (in the sense of Definition 1.2.22). Lastly, we show that most configurations generate Morse electric potentials, in the sense that they form an open and dense subset of the set of all configurations with certain topology.

We start with some particularly important properties about electric potentials that will allow us to prove others later.

Proposition 3.0.1 (Electric Potentials are Harmonic)**.** *If V is the electric potential generated by some configuration of charges $\{(p_k, q_k)\}_{k=1}^N$ in $\mathbb{R}^n$, then V is a harmonic function in $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$.*

Proof. The potential V is clearly of class C^2 by Remark 2.1.9. Now, take any $p \in \mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$. Then,

$$\Delta V(p) = \frac{\partial^2 V}{\partial x_1^2}(p) + \ldots + \frac{\partial^2 V}{\partial x_n^2}(p) = C_n \sum_{k=1}^{N} \sum_{i=1}^{n} q_k \frac{n(x_i - x_{k_i})^2 - |p - p_k|^2}{|p - p_k|^{n+2}} =$$

$$= C_n \sum_{k=1}^{N} q_k \frac{n \sum_{i=1}^{n} (x_i - x_{k_i})^2 - n|p - p_k|^2}{|p - p_k|^{n+2}} = C_n \sum_{k=1}^{N} q_k \frac{n|p - p_k|^2 - n|p - p_k|^2}{|p - p_k|^{n+2}} = 0.$$

The result follows. □

From this and Proposition 1.3.4, we deduce the following:

Corollary 3.0.2 (Electric Potentials are Real Analytic)**.** *If V is the electric potential generated by some configuration of charges $\{(p_k, q_k)\}_{k=1}^N$ in $\mathbb{R}^n$, then V is a real analytic function in $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$.*

Remark 3.0.3. Since the electric potential

$$V : \mathbb{R}^n \setminus \{p_1, \ldots, p_N\} \to \mathbb{R}$$

is real analytic, it is in particular, by Proposition 1.2.10, a function of class C^∞ in $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$, which is a smooth manifold (without boundary). Therefore, we can use the tools introduced in Chapter 1 to investigate its properties.

3.1. Location of the Critical Points

As was said at the beginning of the present chapter, we will devote this first section to the study of the location of the critical points of electric potentials. To begin with, we will prove some results for arbitrary configurations. Later, we will restrict ourselves to the situation of equal-sign configurations in order to obtain a considerably stronger result.

3.1.1. Arbitrary Configurations

The following result for the case of $n = 3$ corresponds to Lemma 32.1 of [14]. We can adapt the proof in that book to show it for the general case:

Proposition 3.1.1. *Let V be the electric potential generated by a configuration of charges in $\mathbb{R}^n$ with nonzero total charge. Then, $\mathrm{Cr}(V) \subset \mathbb{R}^n$ is a bounded set.*

Proof. Take some $M_0 > 0$ such that

$$|p_k| < B_0 \quad \text{for all} \quad k \in \{1, \dots, N\}.$$

Then, $\nabla V(p)$ is defined for $|p| \geq M_0$. For every such p we have that

$$p \cdot \nabla V(p) = C_n \sum_{k=1}^{N} q_k \frac{p \cdot (p - p_k)}{|p - p_k|^n} = C_n \sum_{k=1}^{N} \sum_{i=1}^{n} q_k \frac{x_i (x_i - x_{k,i})}{|p - p_k|^n},$$

where $C_2 = 1$ and $C_n = n - 2$ for $n \geq 3$. Therefore,

$$|p|^{n-2} \left(p \cdot \nabla V(p)\right) = C_n \sum_{k=1}^{N} \sum_{i=1}^{n} q_k |p|^{n-2} \frac{x_i (x_i - x_{k,i})}{|p - p_k|^n} = C_n \sum_{k=1}^{N} q_k \frac{|p|^{n-2} \sum_{i=1}^{n} x_i (x_i - x_{k,i})}{|p - p_k|^n}.$$

Now, we see that

$$\lim_{|p| \to \infty} \frac{|p|}{|p - p_k|} \leq \lim_{|p| \to \infty} \frac{|p| + |p_k|}{|p - p_k|} \leq \lim_{|p| \to \infty} \frac{|p| + |p_k|}{|p| - |p_k|} = 1,$$

$$\lim_{|p| \to \infty} \frac{|p|}{|p - p_k|} \geq \lim_{|p| \to \infty} \frac{|p| - |p_k|}{|p - p_k|} \geq \lim_{|p| \to \infty} \frac{|p| - |p_k|}{|p| + |p_k|} = 1,$$

so $\lim_{|p| \to \infty} \frac{|p|}{|p - p_k|} = 1$. Also,

$$\lim_{|p| \to \infty} \frac{\sum_{i=1}^{n} x_i (x_i - x_{k,i})}{|p - p_k|^2} = \lim_{|p| \to \infty} \frac{\sum_{i=1}^{n} x_i^2 - x_i x_{k,i}}{|p - p_k|^2} = \lim_{|p| \to \infty} \frac{|p|^2 - p \cdot p_k}{|p - p_k|^2} = 1 - \lim_{|p| \to \infty} \frac{p \cdot p_k}{|p - p_k|^2},$$

$$0 \leq \lim_{|p| \to \infty} \frac{|p \cdot p_k|}{|p - p_k|^2} \leq \lim_{|p| \to \infty} \frac{|p||p_k|}{|p - p_k|^2} = \lim_{|p| \to \infty} \frac{|p_k|}{|p - p_k|} \leq \lim_{|p| \to \infty} \frac{|p_k|}{|p| - |p_k|} = 0,$$

so $\lim_{|p| \to \infty} \frac{\sum_{i=1}^{n} x_i (x_i - x_{k,i})}{|p - p_k|^2} = 1$. Then, we can see that

$$\lim_{|p| \to \infty} |p|^{n-2} \left(p \cdot \nabla V(p)\right) = \lim_{|p| \to \infty} C_n \sum_{k=1}^{N} q_k \frac{|p|^{n-2} \sum_{i=1}^{n} x_i (x_i - x_{k,i})}{|p - p_k|^n} =$$

$$= \lim_{|p| \to \infty} C_n \sum_{k=1}^{N} q_k \frac{\sum_{i=1}^{n} x_i (x_i - x_{k,i})}{|p - p_k|^2} = \lim_{|p| \to \infty} C_n \sum_{k=1}^{N} q_k \frac{\sum_{i=1}^{n} x_i^2}{|p|^2} = C_n \sum_{k=1}^{N} q_k \neq 0.$$

We deduce from this that there exists some $M > M_0$ such that if $|p| > M$, then $|p|^{n-2}(p \cdot \nabla V(p)) \neq 0$, which in turn implies that $\nabla V(p) \neq 0$. Thus, $\mathrm{Cr}(V)$ is contained in the open ball $B_M(0)$ of radius M centered at the origin, so it is bounded.

□

Remark 3.1.2. The result of the previous proposition relies on the property that, at great distance, the electric potential generated by a finite set of charges behaves like if it were just one, with charge equal to the total charge of the configuration. And, as seen in Remark 2.1.8, this generates a potential with no critical points if it is nonzero.

Proposition 3.1.3. *Let V be the electric potential generated by a configuration $\{(p_k, q_k)\}_{i=1}^N$. Then,*

$$\lim_{p \to p_i} |\nabla V(p)| = \infty$$

for every i.

Proof. As the charges are different and isolated, there is some $\varepsilon > 0$ small enough such that the function

$$\left| \sum_{k \neq i} q_k \frac{p - p_k}{|p - p_k|^n} \right|$$

is defined and continuous in the closed ball $\overline{B_\varepsilon(p_i)}$, so $|\nabla V(p)|$ has some maximum M in $\overline{B_\varepsilon(p_i)}$. Therefore, we have that

$$|\nabla V(p)| = \left| \sum_{k=1}^N q_k \frac{p - p_k}{|p - p_k|^n} \right| \geq \frac{|q_i|}{|p - p_i|^{n-1}} - \left| \sum_{k \neq i} q_k \frac{p - p_k}{|p - p_k|^n} \right| \geq \frac{|q_i|}{|p - p_i|^{n-1}} - M$$

in $\overline{B_\varepsilon(p_i)}$. But

$$\frac{|q_i|}{|p - p_i|^{n-1}} \to \infty \quad \text{if} \quad |p - p_i| \to 0,$$

because we work with $n \geq 2$. The result follows.

□

Corollary 3.1.4. *Given any configuration, there exist neighborhoods around the charges that do not contain any critical points of the potential it generates. In other words, there are no critical points near the charges.*

This last results allow us to prove that certain configurations of charges generate electric potentials with only finitely many critical points. We will later see that these configurations constitute the "majority" of them.

Proposition 3.1.5. *Let $\{(p_k, q_k)\}_{k=1}^N$ be a Morse configuration with nonzero total charge. Then, V has finitely many critical points.*

Proof. By Proposition 3.1.1, there is some $M > 0$ such that $\mathrm{Cr}(V) \subseteq \overline{B_M(0)}$. Besides, by Proposition 3.1.3, there exist open balls $B_1, \dots, B_N$ centered at $p_1, \dots, p_N$, respectively, such that $\mathrm{Cr}(V) \cap (B_1 \cup \cdots \cup B_N)$ is empty. Therefore, we have that

$$\mathrm{Cr}(V) \subseteq \overline{B_M(0)} \cap (\mathbb{R}^n \setminus (B_1 \cup \cdots \cup B_N)) = \overline{B_M(0)} \setminus (B_1 \cup \cdots \cup B_N),$$

which is a compact set, call it K. On the other hand, by Corollary 1.1.2, we know that there are finitely many critical points of V contained in K. Then, as $\mathrm{Cr}(V) \subseteq K$, we deduce that V has finitely many critical points. □

3.1.2. Configurations with the Same Sign

Now, we restrict ourselves to configurations whose charges have the same sign. First, we introduce some concepts.

Definition 3.1.6 (Convex Set)**.** A subset $C \subseteq \mathbb{R}^n$ is said to be convex if, for all $x, y \in C$, the set

$$[x, y] := \{tx + (1-t)y : t \in [0,1]\}$$

is contained in S.

Proposition 3.1.7 (Convex Hull)**.** *Let $\{C_i : i \in I\}$ be a family of convex subsets of $\mathbb{R}^n$. Then, their intersection $\cap_{i\in I} C_i$ is convex too. We define the convex hull of a subset $X \subseteq \mathbb{R}^n$ by*

$$\operatorname{conv}(X) := \bigcap \{C \subseteq \mathbb{R}^n : X \subseteq C \textit{ and } C \textit{ is convex}\}.$$

This set is the same as

$$\left\{ \sum_{i=1}^{k} \alpha_i x_i : k \geq 1,\ x_i \in X,\ \alpha_i \geq 0,\ \alpha_1 + \ldots \alpha_k = 1 \right\}$$

We say that the convex hull of X is the "smallest" convex set that contains X.

Proposition 3.1.8 ([5], Theorem 1.4.3 of Chapter III)**.** *If $X \subseteq \mathbb{R}^n$ is bounded (resp. compact), then* $\operatorname{conv}(X)$ *is also bounded (resp. compact).*

Definition 3.1.9 (Affine Set)**.** A subset $A \subseteq \mathbb{R}^n$ is said to be affine if, for all $x, y \in A$, the line though x and y, that is, the set

$$\{tx + (1-t)y : t \in \mathbb{R}\},$$

is contained in A.

Proposition 3.1.10 (Dimension of Affine Set)**.** *Consider $\mathbb{R}^n$ as an $n-$dimensional real vector space and let $A \subseteq \mathbb{R}^n$ be a nonempty affine set. There exists a unique linear subspace U of $\mathbb{R}^n$ such that A is the translate of U. That is, there exists a unique subspace U such that there is some $w \in \mathbb{R}^n$ for which*

$$A = w + U := \{w + u : u \in U\}.$$

In fact, any point in A can be chosen as w. We define the dimension of A as the dimension of that unique subspace U that satisfies the condition.

Proposition 3.1.11 (Affine Hull)**.** *Let $\{A_i : i \in I\}$ be a family of affine subsets of $\mathbb{R}^n$. Then, their intersection $\cap_{i\in I} A_i$ is affine too. We define the affine hull of a subset $X \subseteq \mathbb{R}^n$ by*

$$\operatorname{aff}(X) = \bigcap \{A \subseteq \mathbb{R}^n : X \subseteq A \textit{ and } C \textit{ is affine}\}.$$

This set is the same as

$$\left\{ \sum_{i=1}^{k} \alpha_i x_i : k \geq 1,\ x_i \in X,\ \alpha_i \in \mathbb{R},\ \alpha_1 + \ldots \alpha_k = 1 \right\}.$$

We say that the affine hull of X is the "smallest" affine set that contains X.

Definition 3.1.12 (Relative Interior)**.** Consider $\mathbb{R}^n$ with the euclidean topology and let X be a subset of $\mathbb{R}^n$. The relative interior of X, denoted $\operatorname{relint}(X)$, is defined as the interior of X in $\operatorname{aff}(X)$ with the subspace topology. It can be expressed as the set

$$\{x \in X : \text{ there exists some } \epsilon > 0 \text{ such that } B_\epsilon(x) \cap \operatorname{aff}(X) \subset X\}.$$

If X is not contained in some hyperplane of $\mathbb{R}^n$, then $\operatorname{relint}(X) = \operatorname{int}(X)$, the interior of X in $\mathbb{R}^n$ with the euclidean topology.

Lemma 3.1.13 (Separating Hyperplane Theorem, Corollary 4.1.3 of [5]). *Let A and B be two closed convex subsets of $\mathbb{R}^n$, with at least one of them bounded and such that $A \cap B = \emptyset$. Then, there exists some nonzero $a \in \mathbb{R}^n$ and some $b \in \mathbb{R}$ such that*

$$a \cdot x > b \quad \forall x \in A \quad \textit{and} \quad a \cdot x < b \quad \forall x \in B,$$

where $\cdot$ is the dot product in $\mathbb{R}^n$. In this case, it is said that the hyperplane $\{x \in \mathbb{R}^n : a \cdot x = b\}$ strictly separates the sets A and B.

Lemma 3.1.14 (Supporting Hyperplane Theorem, Lemma 4.2.1 of [5]). *Let X be a convex subset of $\mathbb{R}^n$ and x_0 a point in $\partial X := \overline{X} \setminus \operatorname{int}(X)$. Then, there exists some $a \in \mathbb{R}^n, a \neq 0$, such that*

$$a \cdot x \leq a \cdot x_0 \quad \forall x \in X.$$

The hyperplane $\{a \cdot x = a \cdot x_0\}$ is called a supporting hyperplane to X at the point x_0.

Proposition 3.1.15. *Let $\{(p_k, q_k)\}_{k=1}^N$ be a subset of $\mathbb{R}^m \times \mathbb{R}$ ($n \geq 1$), such that $p_i \neq p_j$ if $i \neq j$ and $q_k \neq 0$, and such that the p_k do not all lie in the same hyperplane, i.e., $\dim(\operatorname{aff}(\{p_1, \ldots, p_N\})) = m$. Let E_n be the function*

$$\begin{aligned} E_n : \quad \mathbb{R}^m \setminus \{p_1, \ldots, p_N\} &\longrightarrow \qquad \mathbb{R}^m \\ p \qquad &\longmapsto \sum_{k=1}^N q_k \frac{p - p_k}{|p - p_k|^n} \end{aligned}$$

for $n \geq 1$. Then, if $q_k > 0$ for all k, the zeros of E_n are in $\operatorname{int}(\operatorname{conv}(\{p_1, \ldots, p_N\}))$.

Proof. Take $p \notin \operatorname{int}(\operatorname{conv}(\{p_1, \ldots, p_N\}))$. There are two possibilities:

(I) If $p \notin \operatorname{conv}(\{p_1, \ldots, p_N\})$, by Lemma 3.1.13, as that set is closed (Proposition 3.1.8), there is some $a \in \mathbb{R}^m$, $a \neq 0$, and some $b \in \mathbb{R}$ such that

$$a \cdot p > b \quad \textit{and} \quad a \cdot x < b \quad \forall x \in \operatorname{conv}(\{p_1, \ldots, p_N\}).$$

We deduce that

$$E_n(p) \cdot a = \sum_{k=1}^N q_k \frac{(p - p_k) \cdot a}{|p - p_k|^n} = \sum_{k=1}^N q_k \frac{p \cdot a - p_k \cdot a}{|p - p_k|^n} > \sum_{k=1}^N q_k \frac{b - b}{|p - p_k|^n} = 0,$$

because $\frac{q_k}{|p - p_k|^n} > 0$. Then, $E_n(p) \neq 0$.

(II) If $p \in \operatorname{conv}(\{p_1, \ldots, p_N\}) \setminus \operatorname{int}(\operatorname{conv}(\{p_1, \ldots, p_N\}))$, by Lemma 3.1.14, there is some $a \in \mathbb{R}^m$, $a \neq 0$, such that

$$a \cdot x \leq a \cdot p \quad \forall x \in \operatorname{conv}(\{p_1, \ldots, p_N\}).$$

Now, by hypothesis, there is some p_k such that $a \cdot p_k < a \cdot p$ because, otherwise, they all would lie in the hyperplane $\{x \in \mathbb{R}^m : a \cdot p = a \cdot x\}$. Again,

$$a \cdot E_n(p) = \sum_{k=1}^N q_k \frac{a \cdot (p - p_k)}{|p - p_k|^n} > 0.$$

Then, $E_n(p) \neq 0$ in this case too.

Combining the two cases, we deduce that there are no zeros of E_n outside $\operatorname{int}(\operatorname{conv}(\{p_1, \ldots, p_N\}))$, as we wanted. □

Now we can prove the following result about the location of the critical points:

Theorem 3.1.16. *Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of N charges in $\mathbb{R}^n$, $n \geq 2$. If all the charges have the same sign, then*

$$\mathrm{Cr}(V) \subseteq \mathrm{relint}(\mathrm{conv}(\{p_1, \ldots, p_N\})).$$

Proof. Without loss of generality, we assume that all the charges are positive. If $\dim(\mathrm{aff}(\{p_1, \ldots, p_N\})) = n$, the result follows from Proposition 3.1.15. Therefore, we shall consider that $\dim(\mathrm{aff}(\{p_1, \ldots, p_N\})) = m$ for some $1 \leq m \leq n-1$. In this situation there is no problem in assuming that the set $\mathrm{aff}(\{p_1, \ldots, p_N\})$ is just $\Pi_m^n = \{x_{m+1} = \ldots = x_n = 0\} \subset \mathbb{R}^n$, as we can construct an isometry of $\mathbb{R}^n$ onto itself that sends $\mathrm{aff}(\{p_1, \ldots, p_N\})$ to Π_m^n, while preserving the properties of the electric potential. There are two different possibilities:

(I) If $p \notin \{x_{m+1} = \ldots = x_n = 0\}$ there is some $i \in \{m+1, \ldots, n\}$ such that $\langle e_i, p\rangle \neq 0$, where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$. Moreover, $e_i \cdot p_k = 0$ for each k. Then, we have that

$$e_i \cdot E(p) = \begin{cases} \displaystyle\sum_{k=1}^N q_k \frac{e_i \cdot (p - p_k)}{|p - p_k|^2} = \sum_{k=1}^N q_k \frac{e_i \cdot p}{|p - p_k|^2} \neq 0, & \text{if} \quad n = 2, \\ \displaystyle\sum_{k=1}^N (n-2) q_k \frac{e_i \cdot (p - p_k)}{|p - p_k|^n} = \sum_{k=1}^N (n-2) q_k \frac{e_i \cdot p}{|p - p_k|^n} \neq 0, & \text{if} \quad n \geq 3, \end{cases}$$

so p is not a critical point of V.

(II) If $p \in \{x_{m+1} = \ldots = x_n = 0\}$ and π is the map

$$\begin{array}{cccc} \pi : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & (x_1, \ldots, x_m, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_m), \end{array}$$

then

$$p \in \mathrm{Cr}(V) \iff \widetilde{E}(\pi(p)) = 0,$$

where the function $\widetilde{E} : \mathbb{R}^m \setminus \{\pi(p_1), \ldots, \pi(p_N)\} \longrightarrow \mathbb{R}^m$ is defined by

$$\widetilde{E}(p) = \sum_{k=1}^N q_k \frac{p - \pi(p_k)}{|p - \pi(p_k)|^n}.$$

By Proposition 3.1.15, the zeros of $\widetilde{E}$ are in $I = \mathrm{int}(\mathrm{conv}(\{\pi(p_1), \ldots, \pi(p_N)\})) \subset \mathbb{R}^m$. Additionally,

$$\pi(p) \in I \iff p \in \pi^{-1}(I),$$

but $\pi^{-1}(I) \cap \{x_{m+1} = \ldots = x_n = 0\}$ is just $\mathrm{relint}(\mathrm{conv}(\{p_1, \ldots, p_N\}))$, and the theorem follows.

□

The condition in Theorem 3.1.16 that all the charges have equal sign is necessary for the statement to be true. In fact, there exist configurations, without all signs equal, that do not satisfy Theorem 3.1.16. We have the following example:

Example 3.1.17. Let V be the electric potential generated by the configuration formed by the following charges in $\mathbb{R}^3$:

$$\begin{array}{ll} p_1 = (1, 0, 0), & q_1 = 1 \\ p_2 = (-1, 0, 0), & q_2 = 1 \\ p_3 = (0, 1, 0), & q_3 = -1 \\ p_4 = (0, -1, 0), & q_4 = -1. \end{array}$$

Then, if $p = (x, y, z)$,

$$p \in \mathrm{Cr}(V) \iff q_1 \frac{p-p_1}{|p-p_1|^3} + q_2 \frac{p-p_2}{|p-p_2|^3} + q_3 \frac{p-p_3}{|p-p_3|^3} + q_4 \frac{p-p_4}{|p-p_4|^3} = 0$$

$$\iff \begin{cases} \dfrac{x-1}{|p-p_1|^3} + \dfrac{x+1}{|p-p_2|^3} - \dfrac{x}{|p-p_3|^3} - \dfrac{x}{|p-p_4|^3} = 0 \\[2ex] \dfrac{y}{|p-p_1|^3} + \dfrac{y}{|p-p_2|^3} - \dfrac{y-1}{|p-p_3|^3} - \dfrac{y+1}{|p-p_4|^3} = 0 \\[2ex] \dfrac{z}{|p-p_1|^3} + \dfrac{z}{|p-p_2|^3} - \dfrac{z}{|p-p_3|^3} - \dfrac{z}{|p-p_4|^3} = 0. \end{cases}$$

It is easy to see that $\{(x, y, x) \in \mathbb{R}^3 : x = y = 0\} \subset \mathrm{Cr}(V)$. Then, $\mathrm{Cr}(V)$ cannot be contained in $\mathrm{conv}(\{p_1, p_2, p_3, p_4\})$, because $\mathrm{conv}(\{p_1, p_2, p_3, p_4\})$ is a subset of $\{z = 0\}$. In particular, this is an example of an electric potential with infinitely many critical points.

3.2. Upper Bound for the Dimension of the Critical Set

In this section, we establish an upper bound for the dimension (in the sense of Definition 1.2.22) of the critical set of the electric potential V generated by any configuration of charges in $\mathbb{R}^n$. In fact, we are going to prove, with the help of Cauchy-Kovalevskaya Theorem (Theorem A.9), that $\mathrm{Cr}(V)$ cannot contain any $n-$dimensional or $(n-1)-$dimensional real analytic submanifold of $\mathbb{R}^n$, deducing that $\dim(\mathrm{Cr}(V)) \leq n-2$. We begin by stating the following well known theorem by Sard, whose proof can be found, for example, in [12] pp. 16-19:

Lemma 3.2.1 (Sard's Theorem). *Let M be a manifold and $f : M \to \mathbb{R}$ a smooth map. Then, the set of critical values of f has (Lebesgue) measure zero in $\mathbb{R}$.*

Proposition 3.2.2. *If $f : X \to Y$ is a continuous function and E is a connected subset of X, then $f(E)$ is a connected subset of Y.*

As a direct consequence of the previous results, we can prove the following:

Proposition 3.2.3. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a smooth map. Then, f is constant in every connected subset of $\mathrm{Cr}(f)$.*

Proof. Let $E \subseteq \mathrm{Cr}(f)$ be connected. If f is not constant in E, then, by Proposition 3.2.2, $f(E)$ is some nontrivial interval $I \subseteq \mathbb{R}$. But, as $I = f(E) \subseteq f(\mathrm{Cr}(f))$, this means that $0 < m(I) \leq m(f(\mathrm{Cr}(f)))$, which contradicts Lemma 3.2.1 because $f(\mathrm{Cr}(f))$ is precisely the set of critical values of f. Then f must be constant in the set E. □

Theorem 3.2.4. *Let V be the electric potential generated by some configuration of charges in $\mathbb{R}^n$. Then, the critical set $\mathrm{Cr}(V)$ satisfies that $\dim(\mathrm{Cr}(V)) \leq n-2$. In particular, if it is nonempty, $\mathrm{Cr}(V)$ is a finite union of analytic submanifolds of $\mathbb{R}^n$ of dimensions between 0 and $n-2$.*

Proof. We may assume that $\mathrm{Cr}(V)$ is nonempty, because otherwise $\dim(\mathrm{Cr}(V)) = -\infty$ which is obviously smaller than $n-2$. In Corollary 3.0.2 we saw that V is a real analytic function away from the charges, that is, in the open subset $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$ of $\mathbb{R}^n$. Then, by Proposition 1.2.10, the functions

$$\frac{\partial V}{\partial x_i} : \mathbb{R}^n \setminus \{p_1, \ldots, p_N\} \to \mathbb{R} \quad , \quad i = 1, \ldots, n$$

are real analytic too. Consequently, $\mathrm{Cr}(V)$ is an analytic subset of $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$ as we have that

$$\mathrm{Cr}(V) = \mathcal{Z}_{\mathbb{R}^n \setminus \{p_1,\dots,p_N\}} \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right).$$

We have assumed that $\mathrm{Cr}(V)$ is nonempty, so $0 \leq \dim(\mathrm{Cr}(V)) \leq n$. By Corollary 3.1.4, we have that $\mathrm{Cr}(V) \neq \mathbb{R}^n \setminus \{p_1, \dots, p_N\}$, because there are no critical points close enough to the charges. Therefore, we deduce from Proposition 1.2.28, and from the fact that $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$ is connected, that $\mathrm{Cr}(V)$ is nowhere dense and closed in $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$. Then, $\mathrm{int}(\mathrm{Cr(V)}) = \emptyset$. By Proposition 1.2.29, this means that $\dim(\mathrm{Cr}(V)) \neq n$.

By what we discussed in Section 1.2.3, if $\dim(\mathrm{Cr}(V)) = n-1$, the set $\mathrm{Cr}(V)^{(n-1)}$ of points where $\mathrm{Cr}(V)$ is an $(n-1)$-dimensional analytic submanifold is nonempty. Then, there is some $p \in \mathrm{Cr}(V)$, a subset $W \subseteq \mathbb{R}^{n-1}$ and a real analytic function $f : W \to \mathbb{R}^n$, which maps open subsets of W onto relatively open subsets of $\mathrm{Cr}(V)$, such that

$$p \in f(W) \quad \text{and} \quad \mathrm{rank}\,[Df(w)] = n-1, \ \forall w \in W.$$

We can take an open ball $B \subseteq W$ such that $p \in f(B) = S$. We have that S is a connected $(n-1)$-dimensional analytic submanifold. So, by Remark 1.2.16, S is a connected analytic hypersurface. If we now take any open neighborhood $U \subseteq \mathbb{R}^n \setminus \{p_1, \dots, p_N\}$ of p, the following statements are true:

(I) $\Delta V = 0$ in U, because V is harmonic away from the charges.

(II) $\nabla V = 0$ in S, because $S \subset \mathrm{Cr}(V)$.

(III) $V = c_0$ in S for some constant $c_0 \in \mathbb{R}$, by Proposition 3.2.3, as S is a connected subset of $\mathrm{Cr}(V)$.

But this is also true for the constant function equal to c_0. Now, define the Cauchy problem

$$\Delta u = 0,$$

$$\frac{\partial u}{\partial \nu} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \nu_i = 0, \quad u = c_0 \quad \text{in S.}$$

From Propositions A.8 and 1.3.2, every hypersurface is non-characteristic for the Laplace operator Δ. Also, constant functions are trivially analytic everywhere. It is easy to check that both V and the constant function equal to c_0 are two different analytic solutions for the problem in every open neighborhood of the point p. This contradicts Cauchy-Kovalevskaya Theorem (Theorem A.9). Then, $\mathrm{Cr}(V)$ must have dimension less than or equal to $n-2$. □

3.3. Density of Morse Configurations

In this section, we will prove that most configurations, in a certain sense that we will describe later, are Morse. First, we will do it for all types of configurations. This result for the case of dimension 3 is proven in [14]. With a little more work, using a similar procedure, we will prove it for every dimension. Later, as we have seen it is useful, we look at the case of configurations with the same sign and prove a stronger result. This last result is new and requires additional considerations, but the argumentation we use is similar to that used in the general case.

3.3.1. Arbitrary Configurations

Definition 3.3.1 (Partial Mappings)**.** Let W be a nonempty open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and

$$\begin{array}{rccc} U: & W & \longrightarrow & \mathbb{R} \\ & (x,a) & \longmapsto & U(x,a) \end{array}$$

a mapping. Let $\pi_1(W)$ and $\pi_2(W)$ be the orthogonal projections of W into $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and define for each $a \in \pi_2(W)$ the open subset

$$W(a) = \{x \in \mathbb{R}^n : (x,a) \in W\}$$

of $\pi_1(W)$. Finally, for each $a \in \pi_2(W)$, we define the partial mapping

$$\begin{array}{rccc} U^a: & W(a) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & U(x,a). \end{array}$$

We can see U as defining a family of partial mappings U^a.

Lemma 3.3.2 ([14], Theorem 6.3)**.** *In the situation of Definition 3.3.1, let*

$$\Omega = \left\{(x,a) \in W : \frac{\partial U}{\partial x_1}(x,a) = \ldots = \frac{\partial U}{\partial x_n}(x,a) = 0\right\}.$$

If the matrix

$$\mathrm{H}(U)(x,a) = \left(\left(\frac{\partial^2 U}{\partial x_i \partial x_j}(x,a)\right), \left(\frac{\partial^2 U}{\partial x_i \partial a_k}(x,a)\right)\right) \in \mathcal{M}_{n,n+m}(\mathbb{R})$$

is such that $\operatorname{rank} \mathrm{H}(U) = n$ *for every* $(x,a) \in \Omega$*, then for almost all* $a \in \pi_2(W)$ *the partial mapping* U^a *is a Morse function.*

Lemma 3.3.3 (Weinstein–Aronszajn-Sylvester Determinant Identity)**.** *If* $A \in \mathcal{M}_{m,n}$ *and* $B \in \mathcal{M}_{n,m}$ *are two matrices, then*

$$\det(I_m + AB) = \det(I_n + BA).$$

Proof. We know that for square matrices M_1 and M_2 of the same dimension,

$$\det(M_1 M_2) = \det(M_1)\det(M_2).$$

We can apply this to the matrices

$$\begin{pmatrix} I_m + AB & 0 \\ B & I_n \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix},$$

$$\begin{pmatrix} I_m & 0 \\ B & I_n + BA \end{pmatrix} = \begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix}\begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$$

to get the what we want. In fact,

$$\det\begin{pmatrix} I_m + AB & 0 \\ B & I_n \end{pmatrix} = \det(I_n)\det(I_m + AB) = \det(I_m + AB),$$

$$\det\begin{pmatrix} I_m & 0 \\ B & I_n + BA \end{pmatrix} = \det(I_m)\det(I_n + BA) = \det(I_n + BA),$$

and the result follows.

□

Proposition 3.3.4. *Let* $q_1, \ldots, q_N \in \mathbb{R}\setminus\{0\}$ *such that* $\sum_{k=1}^N q_k \neq 0$ *and* $p_2, \ldots, p_N \in \mathbb{R}^n$ *be fixed, with all* p_i *different. Then, for almost all* $p_1 \in \mathbb{R}^n \setminus \{p_2, \ldots, p_N\}$*, the electric potential* $V_{\{(p_k,q_k)\}_{k=1}^N}$ *is a Morse function.*

Proof. First of all, we define the open subset

$$W = \{(p,a) \in \mathbb{R}^n \times \mathbb{R}^n : p \neq a, p \neq p_i, a \neq p_i \text{ for } i = 2, \ldots, N\}$$

and the C^2-mapping

$$\begin{array}{rccl} U: & W & \longrightarrow & \mathbb{R} \\ & (p,a) & \longmapsto & \begin{cases} -q_1 \ln|p-a| - \displaystyle\sum_{k=2}^{N} q_k \ln|p-p_k| & \text{if} \quad n = 2, \\ \\ \dfrac{q_1}{|p-a|^{n-2}} + \displaystyle\sum_{k=2}^{N} \dfrac{q_k}{|p-p_k|^{n-2}} & \text{if} \quad n \geq 3. \end{cases} \end{array}$$

Then, for each $(p,a) = (x_1, \ldots, x_n, a_1, \ldots, a_n) \in W$, the function U satisfies the following:

$$\frac{\partial^2 U}{\partial x_i \partial a_i}(p,a) = C_n q_1 \frac{|p-a|^2 - n(x_i - a_i)^2}{|p-a|^{n+2}}$$

$$\frac{\partial^2 U}{\partial x_i \partial a_j}(p,a) = C_n q_1 \frac{-n(x_i - a_i)(x_j - a_j)}{|p-a|^{n+2}} \quad , \quad i \neq j.$$

Besides, using the concepts of Definition 3.3.1, we have that

$$\pi_2(W) = \{a \in \mathbb{R}^n : a \neq p_i \text{ for } i = 2, \ldots, N\} = \mathbb{R}^n \setminus \{p_2, \ldots, p_N\},$$

$$W(a) = \mathbb{R}^n \setminus \{a, p_2, \ldots, p_N\},$$

and the partial mapping U^a is just the electric potential generated by the configuration

$$\{(a, q_1), (p_2, q_2), \ldots (p_N, q_N)\}.$$

Next, we define the matrix

$$\mathrm{H}(p,a) = \left(\frac{\partial^2 U}{\partial x_i \partial a_j}(p,a) \right)_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R}).$$

Its determinant is

$$\det \mathrm{H}(p,a) = \left(\frac{C_n q_1}{|p-a|^n} \right)^n \begin{vmatrix} \frac{|p-a|^2 - n(x_1-a_1)^2}{|p-a|^2} & \cdots & \frac{-n(x_1-a_1)(x_n-a_n)}{|p-a|^2} \\ \vdots & \ddots \; \frac{|p-a|^2 - n(x_i-a_i)^2}{|p-a|^2} \; \ddots & \vdots \\ \frac{-n(x_n-a_n)(x_1-a_1)}{|p-a|^2} & \cdots & \frac{|p-a|^2 - n(x_n-a_n)^2}{|p-a|^2} \end{vmatrix} =$$

$$= \left(\frac{C_n q_1}{|p-a|^n} \right)^n \begin{vmatrix} 1 + \frac{-n(x_1-a_1)^2}{|p-a|^2} & \cdots & \frac{-n(x_1-a_1)(x_n-a_n)}{|p-a|^2} \\ \vdots & \ddots \; 1 + \frac{-n(x_i-a_i)^2}{|p-a|^2} \; \ddots & \vdots \\ \frac{-n(x_n-a_n)(x_1-a_1)}{|p-a|^2} & \cdots & 1 + \frac{-n(x_n-a_n)^2}{|p-a|^2} \end{vmatrix}.$$

Now, let A be the matrix

$$A = \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_i - a_i \\ \vdots \\ x_n - a_n \end{pmatrix} \in \mathcal{M}_{n,1}.$$

We see that

$$\det \mathrm{H}(p,a) = \left(\frac{C_n q_1}{|p-a|^n}\right)^n \left| I_n + \frac{-n}{|p-a|^2} A A^T \right|.$$

Then, by Lemma 3.3.3,

$$\det \mathrm{H}(p,a) = \left(\frac{C_n q_1}{|p-a|^n}\right)^n \left| 1 + \frac{-n}{|p-a|^2} A^T A \right| = \left(\frac{C_n q_1}{|p-a|^n}\right)^n \left| 1 - \frac{n}{|p-a|^2} |p-a|^2 \right| =$$

$$= \left(\frac{C_n q_1}{|p-a|^n}\right)^n (1-n) \neq 0, \quad \forall (p,a) \in W.$$

The theorem follows from Lemma 3.3.2. □

Proposition 3.3.5. *Let A be a subset of $\mathbb{R}^n$ such that $m(A) = 0$, where m is the Lebesgue measure on $\mathbb{R}^n$. Then, $A^c = \mathbb{R}^n \setminus A$ is dense in $\mathbb{R}^n$.*

Proof. If A^c is not dense in $\mathbb{R}^n$, there exists some nonempty open set U such that $U \cap A^c = \emptyset$. This means that $U \subset A$. Then,

$$0 < m(U) \leq m(A),$$

and the result follows. □

As a direct consequence of Proposition 3.3.4, using Proposition 3.3.5, we have the following:

Corollary 3.3.6. *In the conditions of Proposition 3.3.4, the set of $p_1 \in \mathbb{R}^n \setminus \{p_2, \ldots, p_N\}$ for which the electric potential $V_{\{(p_k,q_k)\}_{k=1}^N}$ is a Morse function is dense in $\mathbb{R}^n$.*

Remark 3.3.7. A configuration of N charges in $\mathbb{R}^n$ can be represented by a point $(p_1, \ldots, p_N, q_1, \ldots, q_N) \in (\mathbb{R}^n)^N \times \mathbb{R}^N$. In fact, each one by $N!$ different points. But not all points represent a valid configuration. Concretely, the set of them is

$$\mathcal{C}_{n,N} = \{(p_1, \ldots, p_N, q_1, \ldots, q_N) \in (\mathbb{R}^n)^N \times \mathbb{R}^N : p_i \neq p_j \text{ for } i \neq j, q_k \neq 0\}.$$

We do this in order to give the set of configurations a topological structure to prove some results about them. In fact, we consider $\mathcal{C}_{n,N}$ with the subspace topology induced by $(\mathbb{R}^n)^N \times \mathbb{R}^N$ with the Euclidean topology, in which $\mathcal{C}_{n,N}$ is an open and dense subset. We will call elements of $\mathcal{C}_{n,N}$ configurations of charges too, and denote the potential generated by a configuration $c = (p_1, \ldots, p_N, q_1, \ldots, q_N) \in \mathcal{C}_{n,N}$ by V_c. In what follows, we will work with configurations whose total charge is nonzero, because it simplifies the arguments and still represent the majority of configurations. Let us call this set

$$\mathcal{T}_{n,N} = \{(p_1, \ldots, p_N, q_1, \ldots, q_N) \in \mathcal{C}_{n,N} : \textstyle\sum_{k=1}^N q_k \neq 0\} \subset \mathcal{C}_{n,N},$$

that is also open and dense.

Proposition 3.3.8. *Let X be a subset of $\mathbb{R}^m$, $0 < k < n$ and Y a dense subset of $\mathbb{R}^{m-k}$. If*

$$\{(x_{k+1}, \ldots, x_m) = y\} \cap X$$

is dense in $\{(x_{k+1}, \ldots, x_m) = y\} \subset \mathbb{R}^m$ for each $y \in Y$, then X is dense in $\mathbb{R}^m$.

Proof. Take some fixed $x \in \mathbb{R}^m$ and some arbitrary $\varepsilon > 0$. If we find some $x_\varepsilon \in X$ such that $|x - x_\varepsilon| < \varepsilon$, we are done. First, as Y is dense in $\mathbb{R}^{m-k}$, there exists some $y \in Y$ such that

$$|x - (x_1, \dots, x_k, y_1, \dots, y_{m-k})| = |(x_{k+1}, \dots, x_m) - y| < \frac{\varepsilon}{2}.$$

Also, as $\{(x_{k+1}, \dots, x_m) = y\} \cap X \subset \{(x_{k+1}, \dots, x_m) = y\}$ is dense , there exists some x_ε in the set $\{(x_{k+1}, \dots, x_m) = y\} \cap X$ such that

$$|x_\varepsilon - (x_1, \dots, x_k, y_1, \dots, y_{m-k})| < \frac{\varepsilon}{2}.$$

We conclude that

$$|x - x_\varepsilon| \leq |x_\varepsilon - (x_1, \dots, x_k, y_1, \dots, y_{m-k})| + |x - (x_1, \dots, x_k, y_1, \dots, y_{m-k})| < \varepsilon,$$

as we wanted. □

Proposition 3.3.9. *The set of configurations in $\mathcal{T}_{n,N}$ that are Morse is dense in $(\mathbb{R}^n)^N \times \mathbb{R}^N$.*

Proof. The result follows from Corollary 3.3.6 and Proposition 3.3.8 taking X as the set of Morse configurations in $\mathcal{T}_{n,N}$, $m = nN + N$, $k = n$ and

$$Y = \{(p_2, \dots, p_N, q_1, \dots, q_N) \in (\mathbb{R}^n)^{N-1} \times \mathbb{R}^N : \sum_{k=1}^{N} q_k \neq 0, p_i \neq p_j \text{ for } i \neq j, q_k \neq 0\},$$

which is dense in $(\mathbb{R}^n)^{N-1} \times \mathbb{R}^N$ as it equals $(\mathbb{R}^n)^{N-1} \times \mathbb{R}^N$ minus a finite number of (proper) linear subspaces. □

In particular, as $\mathcal{T}_{n,N} \subset \mathcal{C}_{n,N} \subset (\mathbb{R}^n)^N \times \mathbb{R}^N$, this last result implies the following:

Corollary 3.3.10. *The set of Morse configurations in $\mathcal{T}_{n,N}$ is dense in $\mathcal{C}_{n,N}$.*

We may now prove that the set of Morse configurations in $\mathcal{T}_{n,N}$ is also open.

Lemma 3.3.11 (Implicit Function Theorem)**.** *Let V be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $f \in C^k(V, \mathbb{R}^n)$ with $1 \leq k \leq \infty$. Assume that*

$$(x_0, y_0) \in V, \quad f(x_0, y_0) = 0.$$

If the matrix

$$D_x f(x_0, y_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial f_n}{\partial x_n}(x_0, y_0) \end{pmatrix}$$

is invertible, then there exist open neighborhoods U of x_0 in $\mathbb{R}^n$ and W of y_0 in $\mathbb{R}^m$ such that there is a unique function $g : W \to U$ satisfying

$$g(y_0) = x_0, \quad f(g(y), y) = 0 \text{ for all } y \in W.$$

The function g is of class C^k. Furthermore, the derivative $Dg(y) \in \mathcal{M}_{m \times n}(\mathbb{R})$ of g at $y \in W$ is given by

$$Dg(y) = -D_x f(g(y), y)^{-1} D_y f(g(y), y).$$

We recall some definitions of types of continuity for real functions.

Definition 3.3.12 (Continuous Function)**.** Let X be a subset of $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to be continuous at $x \in X$ if for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $y \in X$ with $|x - y| < \varepsilon$, we have that $|f(x) - f(y)| < \delta$.

Definition 3.3.13 (Uniformly Continuous Function). Let X be a subset of $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to be uniformly continuous if for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $x, y \in X$ with $|x - y| < \varepsilon$, we have that $|f(x) - f(y)| < \delta$.

Lemma 3.3.14. *Let K be a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ a continuous function. Then f is also uniformly continuous on K.*

Proof. Let $\varepsilon > 0$. As f is continuous, there is some δ_x for each $x \in K$ such that $f(B_{\delta_x}(x)) \subset B_{\varepsilon/2}(f(x))$. The family of balls $\left\{B_{\delta_x/2}(x)\right\}_{x\in K}$ forms an open cover of K. Since K is compact, there is some finite subcover $\left\{B_{\delta_{x_i}/2}(x_i)\right\}_{i=1}^{n}$ of K. Next, we define

$$\delta := \min_{1\le i\le n} \frac{\delta_{x_i}}{2}.$$

Now, we consider two points $x, y \in K$ such that $|x - y| < \delta$. As $\left\{B_{\delta_{x_i}/2}(x_i)\right\}_{i=1}^{k}$ covers K, x lies in a ball $B_{\delta_{x_k}/2}(x_k)$ for some k. Also, we have that

$$|y - x_k| \le |y - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \le \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}.$$

Then, $y \in B_{\delta_{x_k}/2}(x_k)$, so

$$|f(x) - f(y)| \le |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The result follows. □

Theorem 3.3.15. *The set of configurations in $\mathcal{T}_{n,N}$ that are Morse is open (in $\mathcal{T}_{n,N}$).*

Proof. Let $c_0 = (p_1, \dots, p_N, q_1, \dots, q_N) \in \mathcal{T}_{n,N}$ be a Morse configuration. Now, take some $\varepsilon_0 > 0$ small enough that $B_{\varepsilon_0}(c_0) \subset \mathcal{T}_{n,N}$, where $B_{\varepsilon_0}(c_0)$ is the open ball centered at c of radius ε_0. This is possible because $\mathcal{T}_{n,N}$ is open. For every point $c \in B_{\varepsilon_0}(c_0)$, the set $\mathrm{Cr}(V_c)$ is bounded. In particular, as V_{c_0} is Morse, $\mathrm{Cr}(V_{c_0})$ is a finite set (Proposition 3.1.5). Let $m_1, \dots, m_k$ be its critical points. We can apply the Implicit Function Theorem (Lemma 3.3.11) around each m_i to the mapping

$$(p, c) \longmapsto \nabla V_c(p).$$

We get, for each $i \in \{1, \dots, k\}$, neighborhoods W_i of c_0 and U_i of m_i, and C^∞ mappings $g_i : W_i \to U_i$ satisfying

$$g_i(c_0) = m_i, \quad \nabla V_c(g_i(c)) = 0 \text{ for each } c \in W_i.$$

We can find some $\varepsilon_1 < \varepsilon_0$ small enough and some $R > 0$ big enough that if $|c - c_0| < \varepsilon_1$, then $\mathrm{Cr}(V_c) \subset B_R(0)$. This is because all those sets are bounded and ∇V_c is differentiable with respect to c, so a small perturbation in the configuration has little effect on the critical points.

By Proposition 3.1.3, the norm of the gradient approaches infinity near the charges. Then, again because ∇V_c is differentiable with respect to c, there is some $\varepsilon_2 < \varepsilon_1$ such that, if $D_1, \dots, D_N$ are the open balls of radius ε_2 around $p_1, \dots, p_N$, the electric potential V_c has no critical point in $D_1 \cup \dots \cup D_N$ if $|c - c_0| < \varepsilon_2$. In this case, $\mathrm{Cr}(V_c) \subset \overline{B_R(0)} \setminus (D_1 \cup \dots \cup D_N)$ if $|c - c_0| < \varepsilon_2$.

Now, if $\mathrm{H}_f(p)$ is the Hessian matrix of a function f at a point p, we define

$$H(p, c) = \det \mathrm{H}_{V_c}(p).$$

This function is continuous as the determinant and the second partial derivatives of V_c are. Also, we have that $H(m_1, c_0), \dots, H(m_N, c_0) \neq 0$, as the m_i are non-degenerate. Then, there is some $\varepsilon_3 < \varepsilon_2$ such that if

$|(p,c) - (m_i, c_0)| < \varepsilon_3$ for any i, then $|H(p,c)| > 0$. We recall that each U_i is a neighborhood of m_i. Then, there is some $\varepsilon_4' < \varepsilon_3/2$ such that $B_{\varepsilon_4'}(m_i) \subseteq U_i$ for every i. Also, as the g_i are continuous, we can find some $\varepsilon_4 < \varepsilon_3/2$ such that

$$g_i(B_{\varepsilon_4}(c_0)) \subseteq B_{\varepsilon_4'}(m_i) \text{ for every } i.$$

Therefore, we have the following chain of implications for every i:

$$|c - c_0| < \varepsilon_4 \implies |g_i(c) - m_i| < \varepsilon_4' \implies |(g_i(c), c) - (m_i, c_0)| < \varepsilon_4 + \varepsilon_4' < \varepsilon_3 \implies H(g_i(c), c) \neq 0.$$

Then, the critical points $g_1(c), \ldots, g_k(c)$ are all non-degenerate and, by the Implicit Function Theorem, each $g_i(c)$ is the only critical point of V_c in $B_{\varepsilon_4'}(m_i)$. We know that in our situation, i.e., when $|c - c_0| < \varepsilon_4$, $\mathrm{Cr}(V_c) \subset \overline{B_R(0)} \setminus (D_1 \cup \ldots \cup D_N)$. To finish the proof, we will show that if we take a configuration close enough to c_0, there are no more critical points than the perturbed of the original ones, i.e., the set

$$\overline{B_R(0)} \setminus \left(D_1 \cup \ldots \cup D_N \cup B_{\varepsilon_4'}(m_1) \cup \ldots \cup B_{\varepsilon_4'}(m_k)\right)$$

does not contain any critical point. This set, let us call it K to simplify things, is clearly compact. We know that $|\nabla V_{c_0}|$ is continuous and nonzero in K, so it has a nonzero minimum, call it l. Recall that we took the ε_0 such that $B_{\varepsilon_0}(c_0) \subset \mathcal{T}_{n,N}$. Then, we can define the function

$$(p,c) \longmapsto |\nabla V_c(p)|$$

in the compact set $K \times \overline{B_{\varepsilon_4}(c_0)}$. It is continuous, so it is uniformly continuous (Lemma 3.3.14). Thus, there is some $\varepsilon < \varepsilon_4$ such that

$$|(p,c) - (\overline{p}, \overline{c})| < \varepsilon \implies ||\nabla V_c(p)| - |\nabla V_{\overline{c}}(\overline{p})|| < l.$$

In particular, this means that if $|c - c_0| < \varepsilon$, then

$$|V_c(p)| = |V_c(p)| - |V_{c_0}(p)| + |V_{c_0}(p)| > |V_{c_0}(p)| - ||V_c(p)| - |V_{c_0}(p)|| > l - l = 0$$

for every $p \in K$. We have proven that if $|c - c_0| < \varepsilon$, the potential V_c is Morse, and the result follows.

□

From the previous result, as $\mathcal{T}_{n,N}$ is open in $\mathcal{C}_{n,N}$, we deduce the following:

Corollary 3.3.16. *The set of Morse configurations with nonzero total charge is open in $\mathcal{C}_{n,N}$.*

Definition 3.3.17 (Generic Set and Property)**.** Let X be a topological space. A subset $U \subseteq X$ is said to be generic if it is open and dense. We say that a property of points in X is generic if it is satisfied by a generic subset.

Putting the previous results together, we have the following:

Theorem 3.3.18. *The set of configurations of N charges in $\mathbb{R}^n$ with nonzero total charge that are Morse is generic in $\mathcal{C}_{n,N}$.*

Remark 3.3.19. This result means that the property of being Morse is generic among the electric configurations of N charges in $\mathbb{R}^n$. We will refer to this fact saying that generic configurations in $\mathbb{R}^n$ are Morse.

3.3.2. Configurations with the Same Sign

Throughout this section, we assume that the configurations we work with are composed of charges with the same sign. Without loss of generality, we will consider that they are all positive, since the arguments do not vary. We are going to use Lemma 3.3.2 again to prove, in this scenario of equal-sign configurations, a stronger result about "how many" configurations are Morse. In particular, we will prove that, for each affine subset A of $\mathbb{R}^n$, the set of configurations whose charges lie in A contains a generic subset (in certain topology) of Morse configurations (notice that the previous section corresponds to the case $A = \mathbb{R}^n$). Thus, we obtain a stronger result in this situation, as could be expected. One of the keys to do so will be Theorem 3.1.16, which restricts considerably the location of the critical points of V for this kind of configurations. Proving this result is not idle. In fact, it will allow us to show in Section 4.2 that, for each affine set A, most configurations such that their charges lie in A satisfy certain upper bound, depending on $\dim A$, for the number of critical points of V.

Definition 3.3.20 (m-dimensional Configuration)**.** Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of charges in $\mathbb{R}^n$. If $m = \dim(\operatorname{aff}(\{p_1, \ldots, p_N\}))$, we say that $\{(p_k, q_k)\}_{k=1}^N$ is an $m-$dimensional configuration.

Remark 3.3.21. Since we are working with $N \geq 2$, we have that $1 \leq \dim(\operatorname{aff}(\{p_1, \ldots, p_N\})) \leq n$ for any configuration in $\mathbb{R}^n$.

Remark 3.3.22. As we discussed in the proof of Theorem 3.1.16, to study the behavior of $m-$dimensional configurations, it is enough to restrict ourselves to the case where

$$\operatorname{aff}(\{p_1, \ldots, p_N\}) = \Pi_m^n = \{x \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0\}.$$

This simplifies considerably the arguments while preserving generality. For this reason, throughout the rest of this section, when we say that some configuration $\{(p_k, q_k)\}_{k=1}^N$ is an $m-$dimensional configuration, we will be referring to the case in which $\operatorname{aff}(\{p_1, \ldots, p_N\}) = \Pi_m$.

Let $1 \leq m \leq n$. It will be helpful in some cases to consider elements of Π_m^n as points in $\mathbb{R}^m$ or the other way around. In fact, we can define the inclusion

$$\begin{array}{cccc} \iota_m^n : & \mathbb{R}^m & \longrightarrow & \Pi_m^n \\ & (x_1, \ldots, x_m) & \longmapsto & (x_1, \ldots, x_m, 0, \ldots, 0) \end{array}$$

and the projection

$$\begin{array}{cccc} \pi_m^n : & \Pi_m & \longrightarrow & \mathbb{R}^m \\ & (x_1, \ldots, x_m, 0, \ldots, 0) & \longmapsto & (x_1, \ldots, x_m) \end{array},$$

which are both homeomorphisms, inverse of each other.

Remark 3.3.23. Similarly to the general case explained in Remark 3.3.7, it will be useful to use points from an open subset of $(\mathbb{R}^m)^N \times \mathbb{R}^N$ to represent $m-$dimensional configurations of N charges. Concretely, we use the set

$$\mathcal{C}_{n,N}^m = \left\{(p_1, \ldots, p_N, q_1, \ldots, q_N) \in (\mathbb{R}^m)^N \times \mathbb{R}^N : p_i \neq p_j \text{ for } i \neq j, \text{ and } q_k > 0 \text{ for all } k\right\},$$

where each $(p_1, \ldots, p_N, q_1, \ldots, q_N)$ represents the configuration $\{(\iota_m^n(p_k), q_k)\}_{k=1}^N \subset \mathbb{R}^n \times \mathbb{R}$.

We can now see, with a method similar to the one we used in Section 3.3.1, "how many" $m-$dimensional configurations are Morse. The idea is to use Lemma 3.3.2 restricted to the m-dimensional plane were the charges are:

Proposition 3.3.24. *For $1 \leq m \leq n$, let $p_2, \ldots, p_N \in \mathbb{R}^m$ and $q_1, \ldots, q_N > 0$ be fixed with the p_k different. Then, for almost all $p_1 \in \mathbb{R}^m \setminus \{p_2, \ldots, p_N\}$, the electric potential generated by the configuration $\{(\iota_m^n(p_k), q_k)\}_{k=1}^N$ is a Morse function.*

Proof. First of all, we define the open set

$$W = \{(p,a) \in \mathbb{R}^n \times \mathbb{R}^m : p \neq \iota_m^n(a),\, p \neq \iota_m^n(p_i),\, a \neq p_i \text{ for } i = 2,\dots,N)\},$$

and the C^2-mapping

$$\begin{array}{rccl} U: & W & \longrightarrow & \mathbb{R} \\ & (p,a) & \longmapsto & \begin{cases} -q_1 \ln|p - \iota_m^n(a)| - \displaystyle\sum_{k=2}^{N} q_k \ln|p - \iota_m^n(p_k)| & \text{if } n = 2 \\ \\ \dfrac{q_1}{|p-\iota_m^n(a)|^{n-2}} + \displaystyle\sum_{k=2}^{N} \dfrac{q_k}{|p-\iota_m^n(p_k)|^{n-2}} & \text{if } n \geq 3 \end{cases} \end{array}.$$

Then, for each $(p,a) = (x_1,\dots,x_n,a_1,\dots,a_m) \in W$, the function U satisfies the following:

$$\frac{\partial^2 U}{\partial x_i \partial a_i}(p,a) = C_n q_1 \frac{|p-a|^2 - n(x_i - a_i)^2}{|p - \iota_m^n(a)|^{n+2}}$$

$$\frac{\partial^2 U}{\partial x_i \partial a_j}(p,a) = C_n q_1 \frac{-n(x_i - a_i)(x_j - a_j)}{|p - \iota_m^n(a)|^{n+2}} \quad \text{if} \quad i \neq j$$

$$\frac{\partial^2 U}{\partial x_i^2}(p,a) = C_n q_1 \frac{n(x_i - a_i)^2 - |p - \iota_m^n(a)|^2}{|p - \iota_m^n(a)|^{n+2}} + C_n \sum_{k=2}^{N} q_k \frac{n(x_i - x_{k,i})^2 - |p - \iota_m^n(p_k)|^2}{|p - \iota_m^n(p_k)|^{n+2}}$$

$$\frac{\partial^2 U}{\partial x_i \partial x_j}(p,a) = C_n q_1 \frac{n(x_i - a_i)(x_i - a_j)}{|p - \iota_m^n(a)|^{n+2}} + C_n \sum_{k=2}^{N} q_k \frac{n(x_i - x_{k,i})(x_j - x_{k,j})}{|p - \iota_m^n(p_k)|^{n+2}} \quad \text{if} \quad i \neq j$$

where $(x_{k,1},\dots,x_{k,n}) = \iota_m^n(p_k)$, so $x_{k,m+1} = \cdots = x_{k,n} = 0$ for all k. Besides, using the concepts of Definition 3.3.1, we have that

$$\pi_2(W) = \{a \in \mathbb{R}^m : a \neq p_i \text{ for } i = 2,\dots,N\} = \mathbb{R}^m \setminus \{p_2,\dots,p_N\},$$

$$W(a) = \mathbb{R}^n \setminus \{\iota_m^n(a), \iota_m^n(p_2), \dots, \iota_m^n(p_N)\},$$

and the partial mapping U^a is just the electric potential generated by the configuration

$$\{(\iota_m^n(a), q_1), (\iota_m^n(p_2), q_2), \dots (\iota_m^n(p_N), q_N)\}.$$

Finally, we have the set

$$\Omega = \left\{(p,a) \in W : \frac{\partial U}{\partial x_1}(p,a) = \cdots = \frac{\partial U}{\partial x_n}(p,a) = 0\right\}$$

of pairs $(p,a) \in W$ such that p is a critical point of the function U^a.

Now, we define the matrix

$$\mathrm{H}(p,a) = \left(\frac{\partial^2 U}{\partial x_i \partial a_1}(p,a), \dots, \frac{\partial^2 U}{\partial x_i \partial a_m}(p,a), \frac{\partial^2 U}{\partial x_i \partial x_{m+1}}(p,a) \dots, \frac{\partial^2 U}{\partial x_n \partial x_3}(p,a)\right)$$

and take some $(p,p_1) \in \Omega$. By Theorem 3.1.16, we know that

$$p \in \operatorname{conv}(\{\iota_m^n(p_1), \iota_m^n(p_2), \dots, \iota_m^n(p_N)\}) \subset \{x \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0\} = \Pi_m^n.$$

Therefore, $H(p,p_1)$ equals

$$C_n \begin{pmatrix} q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_1-x_{1,1})^2}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{-n(x_1-x_{1,1})(x_m-x_{1,m})}{|p-\iota^n_m(p_1)|^{n+2}} & \sum_{k=1}^N q_k \frac{n(x_1-x_{k,1})(x_{m+1})}{|p-\iota^n_m(p_k)|^{n+2}} & \cdots & \sum_{k=1}^N q_k \frac{n(x_1-x_{k,1})(x_n)}{|p-\iota^n_m(p_k)|^{n+2}} \\ \vdots & \ddots & & & & \vdots \\ q_1 \frac{-n(x_m-x_{1,m})(x_1-x_{1,1})}{|p-\iota^n_m(p_1)|^{n+2}} & & q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_m-x_{1,m})^2}{|p-\iota^n_m(p_1)|^{n+2}} & \sum_{k=1}^N q_k \frac{n(x_m-x_{k,m})(x_{m+1})}{|p-\iota^n_m(p_k)|^{n+2}} & & \sum_{k=1}^N q_k \frac{n(x_m-x_{k,m})(x_n)}{|p-\iota^n_m(p_k)|^{n+2}} \\ q_1 \frac{-n(x_{m+1})(x_1-x_{1,1})}{|p-\iota^n_m(p_1)|^{n+2}} & & q_1 \frac{-n(x_{m+1})(x_1-x_{1,m})}{|p-\iota^n_m(p_1)|^{n+2}} & \sum_{k=1}^N q_k \frac{n(x_{m+1})^2-|p-\iota^n_m(p_k)|^2}{|p-\iota^n_m(p_k)|^{n+2}} & & \sum_{k=1}^N q_k \frac{n(x_{m+1})(x_n)}{|p-\iota^n_m(p_k)|^{n+2}} \\ \vdots & & & & \ddots & \vdots \\ q_1 \frac{-n(x_n)(x_1-x_{1,1})}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{-n(x_n)(x_1-x_{1,m})}{|p-\iota^n_m(p_1)|^{n+2}} & \sum_{k=1}^N q_k \frac{n(x_n)(x_{m+1})}{|p-\iota^n_m(p_k)|^{n+2}} & \cdots & \sum_{k=1}^N q_k \frac{n(x_n)^2-|p-\iota^n_m(p_k)|^2}{|p-\iota^n_m(p_k)|^{n+2}} \end{pmatrix} =$$

$$C_n \begin{pmatrix} q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_1-x_{1,1})^2}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{-n(x_1-p_{1,1})(x_m-x_{1,m})}{|p-\iota^n_m(p_1)|^{n+2}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ q_1 \frac{-n(x_m-x_{1,m})(x_1-x_{1,1})}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_m-x_{1,m})^2}{|p-\iota^n_m(p_1)|^{n+2}} & 0 & & 0 \\ 0 & & 0 & \sum_{k=1}^N \frac{-q_k|p-\iota^n_m(p_k)|^2}{|p-\iota^n_m(p_k)|^{n+2}} & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \sum_{k=1}^N \frac{-q_k|p-\iota^n_m(p_k)|^2}{|p-\iota^n_m(p_k)|^{n+2}} \end{pmatrix}$$

Therefore, $\det H(p, p_1)$ is equal to

$$(C_n)^n \begin{vmatrix} q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_1-x_{1,1})^2}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{-n(x_1-x_{1,1})(x_m-x_{1,m})}{|p-\iota^n_m(p_1)|^{n+2}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ q_1 \frac{-n(x_m-x_{1,m})(x_1-x_{1,1})}{|p-\iota^n_m(p_1)|^{n+2}} & \cdots & q_1 \frac{|p-\iota^n_m(p_1)|^2-n(x_m-x_{1,m})^2}{|p-\iota^n_m(p_1)|^{n+2}} & 0 & & 0 \\ 0 & & 0 & -\sum_{k=1}^N \frac{q_k}{|p-\iota^n_m(p_k)|^n} & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\sum_{k=1}^N \frac{q_k}{|p-\iota^n_m(p_k)|^n} \end{vmatrix}$$

$$= \frac{(C_n)^n \, (-1)^{n-m} (q_1)^m}{|p-\iota^n_m(p_1)|^{nm}} \left(\sum_{k=1}^N \frac{q_k}{|p-\iota^n_m(p_k)|^n} \right)^{n-m} D,$$

where

$$D = \begin{vmatrix} 1-\frac{n(x_1-x_{1,1})^2}{|p-\iota_m^n(p_1)|^2} & \cdots & \frac{-n(x_1-x_{1,1})(x_m-x_{1,m})}{|p-\iota_m^n(p_1)|^2} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ \frac{-n(x_m-x_{1,m})(x_1-x_{1,1})}{|p-\iota_m^n(p_1)|^2} & \cdots & 1-\frac{n(x_m-x_{1,m})^2}{|p-\iota_m^n(p_1)|^2} & 0 & & 0 \\ 0 & & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}$$

Since $q_k > 0$ for all k, we see that

$$\frac{(C_n)^n\,(-1)^{n-m}(q_1)^m}{|p-\iota_m^n(p_1)|^{nm}}\left(\sum_{k=1}^{N}\frac{q_k}{|p-\iota_m^n(p_k)|^n}\right)^{n-m} \neq 0.$$

On the other hand, we have that

$$D = \begin{vmatrix} 1-\dfrac{n(x_1-x_{1,1})^2}{|p-\iota_m^n(p_1)|^2} & \cdots & \dfrac{-n(x_1-x_{1,1})(x_m-x_{1,m})}{|p-\iota_m^n(p_1)|^2} \\ \vdots & \ddots & \vdots \\ \dfrac{-n(x_m-x_{1,m})(x_1-x_{1,1})}{|p-\iota_m^n(p_1)|^2} & \cdots & 1-\dfrac{n(x_m-x_{1,m})^2}{|p-\iota_m^n(p_1)|^2} \end{vmatrix}.$$

Now, let A be the matrix

$$A = \begin{pmatrix} x_1 - x_{1,1} \\ \vdots \\ x_i - x_{1,i} \\ \vdots \\ x_m - x_{1,m} \end{pmatrix} \in \mathcal{M}_{m,1}.$$

We can see that

$$D = \left| I_m + \frac{-n}{|p-\iota_m^n(p_1)|^2} A A^T \right|.$$

Then, by Lemma 3.3.3,

$$D = \left|1 + \frac{-n}{|p-\iota_m^n(p_1)|^2} A^T A\right| = \left|1 + \frac{-n}{|p-\iota_m^n(p_1)|^2} |p-\iota_m^n(p_1)|^2\right| = (1-n) \neq 0$$

We deduce that $\det H(p, p_1) \neq 0$, and the result follows from Lemma 3.3.2.

□

From this proposition, we can follow the same argumentation that we used in Section 3.3.1 to prove an analogous result:

Theorem 3.3.25. *For $n \geq m \geq 1$, the set of $m-$dimensional configurations of N charges in $\mathbb{R}^n$ that are Morse is generic in $\mathcal{C}^m_{n,N}$.*

Remark 3.3.26. This basically means that, given an affine subset A of $\mathbb{R}^n$, all except a "small" set of configurations whose charges lie in A are Morse.

Chapter 4

Upper Bounds for the Number of Critical Points

In this chapter, we will obtain several upper bounds for the number of critical points of V depending on certain conditions. In the first part, we study the general case and in the second part, as usual, we restrict ourselves to the equal-sign case. In both parts, we will need to separate our analysis depending on whether n is even or odd, as in even dimensions we can work with polynomials while in odd dimensions there appear square roots of polynomials, which are more difficult to work with. From this fact, we can expect that the bounds for odd dimensions will be worse. It is worth noting that all the results (the upper bounds) obtained in this chapter are new.

4.1. Arbitrary Configurations

For arbitrary configurations, we can obtain an upper bound for the number of points in $\mathrm{Cr}(V)$ whenever this set is finite. We know from previous results in the text that most electric potentials have only finitely many critical points, so this result is relevant. This bound will depend on n and N, that is, the dimension of the space $\mathbb{R}^n$ where the charges are located and the number of them. Before we start our study, we offer a few comments on a concept that will be important, the connected components of a topological space:

Definition 4.1.1 (Connected Components)**.** Let X be a topological space. Then, the inclusion $\subseteq$ defines a partial order in X. The maximal connected subsets of X with respect to that order are called the connected components of X. These sets form a partition of X, that is, they are disjoint, non-empty and their union is the whole space X.

Proposition 4.1.2. *If X is a finite subset of $\mathbb{R}^n$, the number of connected components of X (with respect to the subspace topology) equals $\#X$, its number of elements.*

Proof. The subspace topology of a finite subset of $\mathbb{R}^n$ is the discrete topology and a discrete space is connected if and only if it is a single point. Thus, the connected components of X are the singletons $\{x\}$, for each $x \in X$. □

The importance of this result is that, if we are able to give an upper bound for the number of connected components of a subset X of $\mathbb{R}^n$, and we know that this set is finite, we have an upper bound for the number of points in X. This is precisely the reasoning we will follow in this first part.

4.1.1. Even Dimension

Throughout this section, we assume that n is an even natural number. We start by noticing that, for each electric potential V, we can find n polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ whose set of common zeros consists on the set of critical points of V and the set of charges of the configuration that generates V. This will allow us to use tools from Algebraic Geometry to study the critical points of V.

Proposition 4.1.3. *Let $\{(p_k, q_k)\}_{k=1}^N$ be a configuration of N charges in $\mathbb{R}^n$ and V the electric potential it generates. There exist polynomials $f_1, \dots, f_n \in \mathbb{R}[X_1, \dots, X_n]$, all of them of degree at most $n(N-1)+1$, such that*

$$\mathcal{Z}(f_i) = \mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right) \cup \{p_1, \dots, p_N\}$$

for each $i = 1, \dots, n$. Consequently, we have that

$$\mathcal{Z}(f_1, \dots, f_n) = \mathrm{Cr}(V) \cup \{p_1, \dots, p_N\},$$

and that last union is clearly disjoint.

Proof. Take some $p \in \mathbb{R}^n \setminus \{p_1, \dots, p_N\}$. Then, for each i,

$$\frac{\partial V}{\partial x_i}(p) = 0 \iff -C_n \sum_{k=1}^N q_k \frac{x_i - x_{k,i}}{|p-p_k|^n} = 0.$$

As $p \neq p_1, \dots, p_N$, this happens if and only if

$$0 = C_n \prod_{j=1}^N |p-p_j|^n \left(\sum_{k=1}^N q_k \frac{x_i - x_{k,i}}{|p-p_k|^n}\right) = C_n \sum_{k=1}^N q_k (x_i - x_{k,i}) \prod_{j \neq k} |p-p_j|^n$$

$$= C_n \sum_{k=1}^N q_k (x_i - x_{k,i}) \prod_{j \neq k} \left(\sum_{s=1}^n (x_s - x_{j,s})^2\right)^{n/2}.$$

Noticing that $n/2 \in \mathbb{N}$, we define, for each i, the polynomial

$$f_i = C_n \sum_{k=1}^N q_k (X_i - x_{k,i}) \prod_{j \neq k} \left(\sum_{s=1}^n (X_s - x_{j,s})^2\right)^{n/2} \in \mathbb{R}[X_1, \dots, X_n].$$

Each f_i is a sum of N polynomials of the form

$$C_n q_k (X_i - x_{k,i}) \prod_{j \neq k} \left(\sum_{s=1}^n (X_s - x_{j,s})^2\right)^{n/2}.$$

We see that the polynomial $\left(\sum_{s=1}^n (X_s - x_{j,s})^2\right)^{n/2}$ has degree n, so

$$\prod_{j \neq k} \left(\sum_{s=1}^n (X_s - x_{j,s})^2\right)^{n/2}$$

has degree $n(N-1)$. Therefore, f_i is a sum of polynomials of degree $n(N-1)+1$, so f_i has degree less than or equal to $n(N-1)+1$.

By our previous discussion, we have that

$$\mathcal{Z}(f_i) \cap (\mathbb{R}^n \setminus \{p_1, \dots, p_N\}) = \mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right).$$

Besides, we can see that $p_1, \dots, p_N \in \mathcal{Z}(f_i)$, so

$$\mathcal{Z}(f_i) = (\mathcal{Z}(f_i) \cap (\mathbb{R}^n \setminus \{p_1, \dots, p_N\})) \cup \{p_1, \dots, p_N\} = \mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right) \cup \{p_1, \dots, p_N\},$$

as we wanted to prove. The second part follows directly from the fact that

$$\mathcal{Z}(f_1,\ldots,f_n)=\bigcap_{i=1}^{N}\mathcal{Z}(f_i)=\bigcap_{i=1}^{N}\left(\mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right)\cup\{p_1,\ldots,p_N\}\right)=$$

$$\left(\bigcap_{i=1}^{N}\mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right)\right)\cup\{p_1,\ldots,p_N\}=\mathrm{Cr}(V)\cup\{p_1,\ldots,p_N\}.$$

□

Corollary 4.1.4. *In the context of the previous proposition, if* $\mathrm{Cr}(V)$ *is a finite set, then* $\mathcal{Z}(f_1,\ldots,f_n)$ *is finite too and*

$$\#\,\mathrm{Cr}(V)=\#\mathcal{Z}(f_1,\ldots,f_n)-N,$$

where the symbol $\#$ *denotes the number of elements of a set.*

We have the following result due to Milnor [11] about the topology of algebraic sets, whose proof can be found in [1]:

Proposition 4.1.5 ([1], Theorem 11.5.3)**.** *Let* $Z\subseteq\mathbb{R}^n$ *be an algebraic set defined by equations of degree less than or equal to* d*. Then, the sum of the Betti numbers of* Z *is less than or equal to* $d(2d-1)^{n-1}$*.*

Remark 4.1.6. We are not going to deal with Homology in this text, but it is well-known that the 0-th Betti number $b_0(X)$ of a topological space X is equal to the number of connected components of X. Therefore, as all the Betti numbers of a space are non-negative, the bound in Proposition 4.1.5 works for the number of connected components, which is what we want.

Thus, we have:

Proposition 4.1.7. *Let* $Z\subseteq\mathbb{R}^n$ *be an algebraic set defined by equations of degree less than or equal to* d*. Then, the number of connected components of* Z*,* $b_0(Z)$*, is less than or equal to* $d(2d-1)^{n-1}$*.*

As a consequence of the previous results and Proposition 4.1.2, we get the following:

Theorem 4.1.8. *If* $\mathrm{Cr}(V)$ *is a finite set, then*

$$\#\,\mathrm{Cr}(V)\leq(n(N-1)+1)\,(2n(N-1)+1)^{n-1}-N.$$

Remark 4.1.9. For example, if $n=2$, we get the upper bound $8N^2-11M+3$, which is clearly worse than $N-1$, the one we obtained in Theorem 2.2.2, so the previous bound is not sharp.

4.1.2. Odd Dimension

Now, we assume that n is an odd natural number ($n\geq 2$). We will use different procedures in order to obtain two different upper bounds for $\#\,\mathrm{Cr}(V)$ whenever $\mathrm{Cr}(V)$ is a finite set. For both of them, we will use a method similar to the one of the previous section, but with some more work, as we do not have polynomials initially. As we could expect, the bounds for odd dimension will be worse than the one for even dimension that we obtained in the preceding section.

Remark 4.1.10. In this section we will sometimes write $\mathcal{Z}(f)$ to denote the set of zeros of some function f even when it is not real analytic, which is the type of functions for which we have defined such notation. This is done in order to facilitate the argumentation.

By what we discussed in the proof of Proposition 4.1.3, we have the following:

Proposition 4.1.11. *If we have some configuration* $\{(p_k, q_k)\}_{k=1}^N$*, let* $f_1, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}$ *be the functions defined by*

$$f_i = C_n \sum_{k=1}^{N} q_k (X_i - x_{k,i}) \sqrt{\prod_{j \neq k} \left(\sum_{s=1}^{n} (X_s - x_{j,s})^2 \right)^n},$$

Then,

$$\mathcal{Z}(f_i) = \mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right) \cup \{p_1, \ldots, p_N\}$$

for each $i = 1, \ldots, n$*. Consequently, we have that*

$$\mathcal{Z}(f_1, \ldots, f_n) = \mathrm{Cr}(V) \cup \{p_1, \ldots, p_N\},$$

and that last union is clearly disjoint.

Remark 4.1.12. The analogous to Corollary 4.1.4 is also true in this situation. That is, if $\mathrm{Cr}(V)$ is a finite set, then $\mathcal{Z}(f_1, \ldots, f_n)$ is finite too and

$$\# \mathrm{Cr}(V) = \# \mathcal{Z}(f_1, \ldots, f_n) - N,$$

Remark 4.1.13. To make things simpler, consider

$$Q_{k,i} = C_n q_k (X_i - x_{k,i}) \quad , \quad P_k = \prod_{j \neq k} \left(\sum_{s=1}^{n} (X_s - x_{j,s})^2 \right)^n,$$

which are polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, for each $i = 1 \ldots, n$ and each $k = 1, \ldots, N$. Then,

$$f_i = Q_{1,i}\sqrt{P_1} + \cdots + Q_{N,i}\sqrt{P_N}.$$

The following interaction between projections and connected components will be useful in order to obtain our upper bounds:

Proposition 4.1.14. *Let* $\Pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be the projection on the first* n *coordinates. If* X *is a subset of* $\mathbb{R}^{n+m}$*, then* $b_0(\Pi(X)) \leq b_0(X)$*. That is, the number of connected components does not increase by projection.*

Proof. The result follows from the facts that Π is continuous and that continuous mappings preserve connectedness. □

We can now get the first upper bound:

Theorem 4.1.15. *If* $\mathrm{Cr}(V)$ *is a finite set, then*

$$\# \mathrm{Cr}(V) \leq 2n(N-1)\left(4n(N-1) - 1\right)^{n+2N-1} - N = M_1(n, N).$$

Proof. First of all, let us define the polynomials

$$\begin{aligned} F_i &= Q_{1,i}Y_1 + \cdots + Q_{N,i}Y_N \quad && 1 \leq i \leq n \\ G_k &= Y_k^2 - P_k && \\ H_k &= Z_k^2 - Y_k && 1 \leq k \leq N \end{aligned}$$

in $\mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_N, Z_1, \ldots, Z_N]$ and the analytic sets

$$A_i = \mathcal{Z}\left(F_i, G_1, \ldots, G_N, H_1, \ldots, H_N\right) \subset \mathbb{R}^{n+2N},$$

$$A = \bigcap_{i=1}^{n} A_i = \mathcal{Z}\left(F_1, \ldots, F_n, G_1, \ldots, G_N, H_1, \ldots, H_N\right).$$

If $\Pi : \mathbb{R}^{n+2N} \to \mathbb{R}^n$ is the projection on the first n coordinates, then

$$\mathcal{Z}(f_1, \dots, f_n) = \Pi(\mathcal{Z}(A)),$$

so

$$b_0(\mathcal{Z}(f_1, \dots, f_n)) \le b_0(A).$$

We have that $\deg P_k = 2n(N-1) \ge 2$, so A is an algebraic subset of $\mathbb{R}^{n+2N}$ defined by polynomials of degree less that or equal to $2n(N-1)$. By Proposition 4.1.7, we have that

$$b_0(A) \le 2n(N-1)\left(4n(N-1)-1\right)^{n+2N-1}.$$

By Remark 4.1.12, since $\mathrm{Cr}(V)$ is a finite set, $\mathcal{Z}(f_1, \dots, f_n)$ is a finite set. Therefore, we have that

$$\#\,\mathrm{Cr}(V) = \#\mathcal{Z}(f_1, \dots, f_n) - N = b_0(\mathcal{Z}(f_1, \dots, f_n)) - N \le$$

$$b_0(A) - N \le 2n(N-1)\left(4n(N-1)-1\right)^{n+2N-1} - N.$$

□

We have the following result, similar to Proposition 4.1.5 but for basic closed semi-algebraic sets (see Definition C.11), also due to Milnor [11]:

Proposition 4.1.16 ([1], p. 285)**.** *If $X \subseteq \mathbb{R}^n$ is a basic closed semi-algebraic set defined by p polynomial inequalities $f_1 \ge 0, \dots, f_p \ge 0$ of degree less than or equal to d, then the sum of the Betti numbers of X is less than or equal to $\frac{1}{2}(dp+2)(dp+1)^{n-1}$.*

Again, we can use the bound only for $b_0(X)$:

Proposition 4.1.17. *If $X \subseteq \mathbb{R}^n$ is a basic closed semi-algebraic set defined by p polynomial inequalities $f_1 \ge 0, \dots, f_p \ge 0$ of degree less than or equal to d. Then, the number of connected components of X, $b_0(X)$, is less than or equal to $\frac{1}{2}(dp+2)(dp+1)^{n-1}$.*

Using this, we can get the second upper bound:

Theorem 4.1.18. *If $\mathrm{Cr}(V)$ is a finite set, then*

$$\#\,\mathrm{Cr}(V) \le \frac{1}{2}\left[2n(N-1)(2n+3N)+2\right]\left[2n(N-1)(2n+3N)+1\right]^{n+N-1} - N = M_2(n,N).$$

Proof. First of all, let us define the polynomials

$$\begin{array}{ll} F_i = Q_{1,i}Y_1 + \cdots + Q_{N,i}Y_N & 1 \le i \le n \\ G_k = Y_k^2 - P_k & 1 \le k \le N \end{array}$$

in $\mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_N]$ and the closed basic semi-algebraic set

$$A = \left\{x \in \mathbb{R}^{n+N} : F_1(x) = 0, \dots, F_n(x) = 0, G_1(x) = 0, \dots, G_N(x) = 0, Y_1(x) \ge 0, \dots, Y_N(x) \ge 0\right\}$$

$$= \left(\bigcap_{i=1}^{n} \{F_i \ge 0, -F_i \ge 0\}\right) \cap \left(\bigcap_{k=1}^{N} \{G_k \ge 0, -G_k \ge 0\}\right) \cap \left(\bigcap_{k=1}^{N} \{Y_k \ge 0\}\right)$$

If $\Pi : \mathbb{R}^{n+N} \to \mathbb{R}^n$ is the projection on the first n coordinates, then

$$\mathcal{Z}(f_1, \dots, f_n) = \Pi(\mathcal{Z}(A)),$$

so

$$b_0(\mathcal{Z}(f_1,\dots,f_n)) \leq b_0(A).$$

by Proposition 4.1.14. We have that $\deg P_k = 2n(N-1) \geq 2$, so A is a basic closed semi-algebraic subset of $\mathbb{R}^{n+N}$ defined by $2n+3N$ polynomials of degree less that or equal to $2n(N-1)$. By Proposition 4.1.17, we have that

$$b_0(A) \leq \frac{1}{2}\left[2n(N-1)(2n+3N)+2\right]\left[(2n(N-1)(2n+3N)+1)\right]^{n+N-1}.$$

By Remark 4.1.12, since $\mathrm{Cr}(V)$ is a finite set, $\mathcal{Z}(f_1,\dots,f_n)$ is a finite set. Therefore, we have that

$$\#\,\mathrm{Cr}(V) = \#\mathcal{Z}(f_1,\dots,f_n) - N = b_0(\mathcal{Z}(f_1,\dots,f_n)) - N \leq b_0(A) - N \leq$$
$$\frac{1}{2}\left[2n(N-1)(2n+3N)+2\right]\left[(2n(N-1)(2n+3N)+1)\right]^{n+N-1} - N.$$

□

Now we can study how $M_1(n,N)$ and $M_2(n,N)$ are related to see which one is a better bound for $\#\,\mathrm{Cr}(V)$. We have the following result in this regard:

Theorem 4.1.19. *For any natural number $N \geq 2$,*

$$\lim_{n\to\infty} \frac{M_1(n,N)}{M_2(n,N)} = 0.$$

Additionally, if $k \leq 2$,

$$\lim_{n\to\infty} \frac{M_1(n,kn)}{M_2(n,kn)} = \infty.$$

Proof. First, we have that

$$0 \leq \lim_{n\to\infty} \frac{M_1(n,N)}{M_2(n,N)} = \lim_{n\to\infty} \frac{2n(N-1)\left(4n(N-1)-1\right)^{n+2N-1} - N}{\frac{1}{2}\left[2n(N-1)(2n+3N)+2\right]\left[2n(N-1)(2n+3N)+\right]^{n+N-1} - N} =$$

$$\lim_{n\to\infty} \frac{4n(N-1)\left(4n(N-1)-1\right)^{n+2N-1}}{\left[2n(N-1)(2n+3N)+2\right]\left[2n(N-1)(2n+3N)+1\right]^{n+N-1}} \leq$$

$$\lim_{n\to\infty} \frac{(4n(N-1))^{n+2N}}{(2n(N-1)(2n+3N))^{n+N}} = \lim_{n\to\infty} \frac{(2n(N-1))^{n+2N}\,2^{n+2N}}{(2n(N-1))^{n+N}(2n+3N)^{n+N}} =$$

$$\lim_{n\to\infty} \frac{(2n(N-1))^{N}\,2^{n+2N}}{(2n+3N)^{n+N}} \leq \lim_{n\to\infty} \frac{(2n(N-1))^{N}\,2^{n+2N}}{(2n)^{n+N}} =$$

$$\lim_{n\to\infty} \frac{(N-1)^N 2^{n+2N}}{(2n)^n} = \lim_{n\to\infty} \frac{(N-1)^N 2^{2N}}{n^n} = 0.$$

For the second part, we have that

$$\lim_{n\to\infty} \frac{M_1(n,kn)}{M_2(n,kn)} = \lim_{n\to\infty} \frac{4n(kn-1)\left(4n(kn-1)-1\right)^{(2k+1)n-1}}{\left(2n(kn-1)(3k+2)n+2\right)\left(2n(kn-1)(3k+2)n+1\right)^{(k+1)n-1}} \geq$$

$$\lim_{n\to\infty} \frac{4n(kn-n)\left(4n(kn-n)-1\right)^{(2k+1)n-1}}{\left(2n(kn)(3k+2)n+2\right)\left(2n(kn)(3k+2)n+1\right)^{(k+1)n-1}} =$$

$$\lim_{n\to\infty} \frac{4(k-1)n^2\left(4(k-1)n^2-1\right)^{(2k+1)n-1}}{\left(2k(3k+2)n^3+2\right)\left(2k(3k+2)n^3+1\right)^{(k+1)n-1}} \geq$$

$$\lim_{n\to\infty} \frac{4(k-1)n^2\left(2(k-1)n^2\right)^{(2k+1)n-1}}{(3k(3k+2)n^3)\left(3k(3k+2)n^3\right)^{(k+1)n-1}} = \lim_{n\to\infty} \frac{2\left(2(k-1)n^2\right)^{(2k+1)n}}{\left(3k(3k+2)n^3\right)^{(k+1)n}} =$$

$$\lim_{n\to\infty} \frac{2\left(2(k-1)\right)^{(2k+1)n} n^{2(2k+1)n}}{\left(3k(3k+2)\right)^{(k+1)n} n^{3(k+1)n}} = \lim_{n\to\infty} \left(\frac{2\left(2(k-1)\right)^{(2k+1)}}{\left(3k(3k+2)\right)^{(k+1)}} n^{k-1}\right)^n = \infty.$$

□

Remark 4.1.20. The first limit of the previous result tells us that, for each number of charges N, the upper bound M_1 is increasingly better than M_2 for every odd n bigger than some natural number. Also, notice that the speed at which the limit approaches 0 is very high, so the first bound is much better than the second for n sufficiently larger than N. The second limit shows that, despite what we have just said about M_1 being better than M_2 for infinitely many values of n and N, there are, for each $k \geq 2$, infinitely many values of n such that $M_2(n, kn) \leq M_1(n, kn)$. In fact, we can find values of n that have as many values of N such that $M_2(n, N) \leq M_1(n, N)$ as we want.

4.2. Configurations with the Same Sign

Now, we assume that the charges of our configurations have the same sign. Again, we can assume that they are all positive. In this Section, we are going to obtain different bounds for $\mathrm{Cr}(V)$ depending on the dimension of the configuration that generates V, that is, the dimension of the affine hull of the set of charges. As it is reasonable, we will obtain better bounds for configurations of smaller dimensions.

Remark 4.2.1. As we did in previous occasions to simplify things without risking losing generality, we will restrict ourselves, in order to study m-dimensional configurations, to the case where

$$\mathrm{aff}(\{p_1, \dots, p_N\}) = \Pi_m^n = \{x \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0\}.$$

Thus, when we say throughout this section that some configuration of charges is m-dimensional, we will be referring to the specific case just mentioned.

What will allow us to improve the upper bounds for $\#\,\mathrm{Cr}(V)$ when working with equal-sign m-dimensional configurations is the fact that there are less partial derivatives of V involved in the description of $\mathrm{Cr}(V)$. Specifically, we have the following:

Proposition 4.2.2. *If $\{(p_k, q_k)\}_{k=1}^N$ is an m-dimensional configuration of charges, with $1 \leq m \leq n$, then*

$$\mathrm{Cr}(V) = \mathcal{Z}\left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_m}\right) \cap \Pi_m^n.$$

Proof. We know that $\mathrm{aff}(\{p_1, \dots, p_N\}) = \Pi_m^n$, so, by Theorem 3.1.16,

$$\mathrm{Cr}(V) \subset \mathrm{conv}\left(\{p_1, \dots, p_N\}\right) \subset \mathrm{aff}(\{p_1, \dots, p_N\}) = \Pi_m^n.$$

Now, take some i such that $m+1 \leq i \leq n$. Then, for any $p \in \Pi_m^n \setminus \{p_1, \dots, p_N\}$,

$$\frac{\partial V}{\partial x_i}(p) = -C_n \sum_{k=1}^N q_k \frac{x_i - x_{k,i}}{|p - p_k|^n} = -C_n \sum_{k=1}^N q_k \frac{0 - 0}{|p - p_k|^n} = 0,$$

so

$$\mathcal{Z}\left(\frac{\partial V}{\partial x_i}\right) \cap \Pi_m^n = \Pi_m^n \setminus \{p_1, \dots, p_N\}.$$

We deduce that

$$\operatorname{Cr}(V) = \operatorname{Cr}(V) \cap \Pi_m^n = \mathcal{Z}\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) \cap \Pi_m^n = \left(\bigcap_{k=1}^{N} \mathcal{Z}\left(\frac{\partial V}{\partial x_k}\right)\right) \cap \Pi_m^n =$$

$$\left(\bigcap_{k=1}^{m} \mathcal{Z}\left(\frac{\partial V}{\partial x_k}\right)\right) \cap \left(\mathcal{Z}\left(\frac{\partial V}{\partial x_{m+1}}\right) \cap \Pi_m^n\right) \cap \cdots \cap \left(\mathcal{Z}\left(\frac{\partial V}{\partial x_n}\right) \cap \Pi_m^n\right) =$$

$$\mathcal{Z}\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m}\right) \cap (\Pi_m^n \setminus \{p_1, \ldots, p_N\}) = \mathcal{Z}\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m}\right) \cap \Pi_m^n,$$

as we wanted. □

As a direct consequence, we have the following:

Corollary 4.2.3. *If $\{(p_k, q_k)\}_{k=1}^N$ is an m-dimensional configuration of charges, with $1 \leq m \leq n$, then*

$$\operatorname{Cr}(V) = \mathcal{Z}\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m}\right) \cap \mathcal{Z}\left(X_{m+1}, \ldots, X_n\right).$$

Proof. It is straightforward to see that the set $\Pi_m^n \subseteq \mathbb{R}^n$ is equal to the algebraic set defined by the polynomials $X_{m+1}, \ldots, X_n \in \mathbb{R}[X_1, \ldots, X_n]$. The result follows from Proposition 4.2.2. □

Definition 4.2.4 (Non-Degenerate Solution). Let $f_1, \ldots, f_n : U \to \mathbb{R}$ be Nash functions (see Definition C.14). We say that some $x \in U$ is a non-degenerate solution of the system $f_1 = \cdots = f_n = 0$ if $f_i(x) = 0$ for $i = 1, \ldots, n$ and

$$\operatorname{rank}\left[\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leq i,j \leq n}\right] = n,$$

that is, the Jacobian matrix of the map $(f_1, \ldots, f_n) : U \to \mathbb{R}^n$ at x is non-singular.

Lemma 4.2.5 (Bézout's Theorem for Nash Functions ([15], Theorem 3.1))**.** *Let U be a connected open semi-algebraic subset of $\mathbb{R}^n$ (see Section C.1) and $f_1, \ldots, f_n$ Nash functions of complexities $c_1, \ldots, c_n$, respectively, defined on U. Then, the number of non-degenerate solutions of the system $f_1(x) = \cdots = f_n(x) = 0$ is finite and less than or equal to $\prod_{i=1}^n c_i$.*

We want to use this result to give an upper bound for $\# \operatorname{Cr}(V)$. To do that, we need to prove several things first.

Proposition 4.2.6. *The set $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$ is semi-algebraic.*

Proof. We have that

$$\mathbb{R}^n \setminus \{p_1, \ldots, p_N\} = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : g_i(x) > 0\},$$

where

$$g_i = \sum_{j=1}^{n} (X_j - x_{i,j})^2.$$

□

Remark 4.2.7. For the rest of this section, if we have a configuration of charges $\{(p_k, q_k)\}_{k=1}^N$, we consider the functions $f_i : \mathbb{R}^n \setminus \{p_1, \ldots, p_N\} \to \mathbb{R}$, $i = 1, \ldots, n$, defined by

$$f_i(p) = \frac{\partial V}{\partial x_i}(p) \prod_{j=1}^{N} |p - p_j|^n = C_n \sum_{k=1}^{N} q_k (x_i - x_{k,i}) \prod_{j \neq k} \left(\sum_{s=1}^{n} (x_s - x_{j,s})^2\right)^{n/2}$$

Proposition 4.2.8. *For any configuration, $f_1, \dots, f_n$ are Nash functions.*

Proof. We know by Proposition 4.2.6 that $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$ is a semi-algebraic set, and it is clearly open. If n is even, $f_1, \dots, f_n$ are polynomials, so the result follows. If n is odd, we can see that the functions are of class C^∞, as the square roots in their definition are positive except at some of the points $p_1, \dots, p_N$. Now, we need to prove that they are semi-algebraic functions.

Like before, let us define the polynomials

$$\begin{aligned} F_i &= Q_{1,i}Y_1 + \cdots + Q_{N,i}Y_N - Y_0 \quad && 1 \le i \le n \\ G_k &= Y_k^2 - P_k \\ H_k &= Y_k && 1 \le k \le N \end{aligned}$$

in $\mathbb{R}[X_1, \dots, X_n, Y_0, Y_1, \dots, Y_N]$ and the semi-analytic sets

$$S_i = \left\{x \in \mathbb{R}^{n+N+1} : F_i(x) = G_1(x) = \cdots = G_N(x) = 0,\ H_1(x) \ge 0, \dots, H_N(x) \ge 0\right\}.$$

for each $i = 1, \dots, n$. If we consider the projection $\Pi : \mathbb{R}^{n+N+1} \to \mathbb{R}^{n+1}$ on the first $n+1$ coordinates, then

$$G(f_i) = \left\{(x, f_i(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \setminus \{p_1, \dots, p_N\}\right\} =$$

$$\Pi(S_i) \cap (\mathbb{R}^n \setminus \{p_1, \dots, p_N\} \times \mathbb{R}).$$

This last set is semi-algebraic in $\mathbb{R}^{n+1}$ by Propositions 4.2.6, C.10 and because the product of semi-algebraics is semi-algebraic, so f_i is a semi-algebraic function for each $i = 1, \dots, n$ and the result follows. □

Lemma 4.2.9. *Let $m \ge 1$. Now, consider the polynomial*

$$f = X_1 + \cdots + X_{m+1} \in \mathbb{R}[X_1, \dots, X_{m+1}].$$

Next, for each $\varepsilon \in \{1, -1\}^m$, define

$$f_\varepsilon = X_1 + \varepsilon_1 X_2 + \cdots + \varepsilon_m X_{m+1} = X_1 + \sum_{k=1}^{m} \varepsilon_k X_{k+1}.$$

If we call

$$F = \prod_{\varepsilon \in \{1,-1\}^m} f_\varepsilon,$$

then F is a polynomial whose monomials have only even powers of the variables $X_1, \dots, X_{m+1}$. That is, every monomial of F is of the form

$$a \prod_{k=1}^{m+1} X_k^{e(k)},$$

where $a \in \mathbb{Z}$ and $e(k)$ is an even natural number for each k (can be 0).

Proof. We can write $F = F_e + F_o$, where F_e is formed by the monomials of F with only even powers of the variables and F_o is the rest, the monomials that have some variable raised to an odd power. Notice that we want to prove that $F_o = 0$. We can see that, for every $x \in \mathbb{R}^{m+1}$ and every $k \in \{1, \dots, m+1\}$,

$$F_e(x) + F_o(x) = F(x) = F(x_1, \dots, -x_k, \dots, x_{m+1}) =$$

$$F_e(x_1, \dots, -x_k, \dots, x_{m+1}) + F_o(x_1, \dots, -x_k, \dots, x_{m+1}) =$$

$$F_e(x) + F_o(x_1, \dots, -x_k, \dots, x_{m+1}).$$

Therefore, we have that

$$F_o(x) = F_o(x_1, \dots, -x_k, \dots, x_{m+1}).$$

Assume that $F_o \neq 0$. Then, there is some $k \in \{1, \dots, m+1\}$ such that there is a nonzero monomial in F_o with an odd power of X_k. We can write F_o as an element of $\mathbb{R}[X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_{m+1}][X_k]$ and get that

$$F_o = \sum_{i=1}^{2^{m-1}} X_k^{2i-1} g_i + \sum_{i=0}^{2^{m-1}} X_k^{2i} h_i,$$

where g_i and h_i are polynomials that do not depend on the variable X_k and at least one of the g_i is nonzero. We deduce that

$$\sum_{i=1}^{2^{m-1}} x_k^{2i-1} g_i(x) + \sum_{i=0}^{2^{m-1}} x_k^{2i} h_i(x) = F_o(x) = F_o(x_1, \dots, -x_k, \dots, x_{m+1}) = -\sum_{i=1}^{2^{m-1}} x_k^{2i-1} g_i(x) + \sum_{i=0}^{2^{m-1}} x_k^{2i} h_i(x),$$

so

$$2 \sum_{i=1}^{2^{m-1}} X_k^{2i-1} g_i = 0.$$

Therefore, as $\{X_k, X_k^2, X_k^3, \dots\}$ is a basis of $\mathbb{R}[X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_{m+1}][X_k]$ as a module over the ring $\mathbb{R}[X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_{m+1}]$, we have that $g_i = 0$ for all i, contradicting our previous statement. Thus, $F_o = 0$ and the result follows. □

Proposition 4.2.10. *If n is an odd natural number, there exists, for each $i = 1, \dots, n$, a nonzero polynomial $F_i \in \mathbb{R}[X_1, \dots, X_n, Y]$ of degree at most $2^N(n(N-1)+1)$ such that*

$$G(f_i) \subset \mathcal{Z}(F_i).$$

Proof. First of all, we consider the functions $g_i : (\mathbb{R}^n \setminus \{p_1, \dots, p_N\}) \times \mathbb{R} \to \mathbb{R}$ given by

$$g_i = Q_{1,i}\sqrt{P_1} + \cdots + Q_{N,i}\sqrt{P_N} + Y,$$

where $Y : \mathbb{R}^{n+1} \to \mathbb{R}$ is the projection on the last coordinate. Next, for each $\varepsilon \in \{1, -1\}^N$, we define

$$g_{i,\varepsilon} = Q_{1,i}\sqrt{P_1} + \varepsilon_1 Q_{2,i}\sqrt{P_2} + \cdots + \varepsilon_{N-1} Q_{N,i}\sqrt{P_N} + \varepsilon_N Y,$$

the "conjugates" of g_i. Now, we construct the functions

$$F_i = \prod_{\varepsilon \in \{1,-1\}^N} g_{i,\varepsilon},$$

the products, for each i, of all conjugates of g_i. We have, by Lemma 4.2.9, that F_i depends on the squares of the square roots of the polynomials $Q_{1,i}^2 P_1, \dots, Q_{N,i}^2 P_N, Y^2$, so F_i is a polynomial in $\mathbb{R}[X_1, \dots, X_n, Y]$. It is clearly nonzero (for example, the coefficient of Y^{2^N} is 1) and has degree at most $2^N(n(N-1)+1)$, as each monomial in F_i is a product of 2^N square roots of polynomials of degree at most $2n(N-1)+2$. Also, we see that

$$g_{i,(1,\dots,1,-1)} = Q_{1,i}\sqrt{P_1} + \cdots + Q_{N,i}\sqrt{P_N} - Y = f_i - Y.$$

Therefore,

$$G(f_i) = \mathcal{Z}(g_{i,(1,\dots,1,-1)}) \subset \mathcal{Z}(F_i),$$

and we have the result. □

Corollary 4.2.11. *If $c_1, \ldots, c_n$ are the respective complexities of the Nash functions $f_1, \ldots, f_n$, we have the following:*

- *If n is even, $c_i = n(N-1)+1$.*
- *If n is odd, $c_i \leq 2^N(n(N-1)+1)$.*

We are now able use Lemma 4.2.5 in order to restrict the number of non-degenerate solutions of the system defined by the functions $f_1, \ldots, f_n$. What we want to do next is to establish a relation between the non-degenerate critical points of V and the non-degenerate solutions of that system.

Proposition 4.2.12. *Let $\{(p_k, q_k)\}_{k=1}^N$ be an m-dimensional configuration. If $p \in \mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$ is a non-degenerate critical point of the electric potential V, then p is a non-degenerate solution of the system $f_1(x) = 0, \ldots, f_m(x) = 0, X_{m+1}(x) = 0, \ldots, X_n(x) = 0$ of Nash functions defined on $\mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$, where X_k is the projection on the k-th coordinate.*

Proof. We have that, for $p \in \mathbb{R}^n \setminus \{p_1, \ldots, p_N\}$,

$$f_i(p) = \frac{\partial V}{\partial x_i}(p) \prod_{j=1}^N |p - p_j|^n.$$

If p is a critical point of V, by Corollary 4.2.3, we know that $f_1(p) = 0, \ldots, f_m(p) = 0, x_{m+1} = 0, \ldots, x_n = 0$, so p is a solution of the system. Besides, we have that

$$\frac{\partial f_i}{\partial x_l}(p) = \frac{\partial^2 V}{\partial x_i \partial x_l}(p) \prod_{j=1}^N |p - p_j|^n + \sum_{k=1}^N \left(\frac{\partial V}{\partial x_i}(p) n(x_l - x_{k,l}) |p - p_k|^{n-2} \prod_{j \neq k} |p - p_j|^n \right) = .$$

$$\frac{\partial^2 V}{\partial x_i \partial x_l}(p) \prod_{j=1}^N |p - p_j|^n,$$

because $p \in \mathrm{Cr}(V)$, so $\frac{\partial V}{\partial x_1}(p) = \cdots = \frac{\partial V}{\partial x_n}(p) = 0$. If p is non-degenerate, then

$$0 \neq \det H_V(p) = (C_n)^n \begin{vmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m}(p) & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1}(p) & & \frac{\partial^2 V}{\partial x_m \partial x_m}(p) & 0 & & 0 \\ 0 & & 0 & -\sum_{k=1}^N \frac{q_k}{|p - p_k|^n} & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\sum_{k=1}^N \frac{q_k}{|p - p_k|^n} \end{vmatrix} =$$

$$(C_n)^n \left(-\sum_{k=1}^{N} \frac{q_k}{|p-p_k|^n}\right)^{n-m} \begin{vmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_m \partial x_m}(p) \end{vmatrix}.$$

This implies that

$$\begin{vmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_m \partial x_m}(p) \end{vmatrix} \neq 0.$$

Consequently, we know that

$$0 \neq \left(\prod_{j=1}^{N} |p-p_j|^n\right)^m \begin{vmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1}(p) & \cdots & \frac{\partial^2 V}{\partial x_m \partial x_m}(p) \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1}(p) \prod_{j=1}^{N} |p-p_j|^n & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m}(p) \prod_{j=1}^{N} |p-p_j|^n \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1}(p) \prod_{j=1}^{N} |p-p_j|^n & \cdots & \frac{\partial^2 V}{\partial x_m \partial x_m}(p) \prod_{j=1}^{N} |p-p_j|^n \end{vmatrix} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_m}(p) \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) & \frac{\partial f_1}{\partial x_{m+1}}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_m}(p) & \frac{\partial f_m}{\partial x_{m+1}}(p) & & \frac{\partial f_m}{\partial x_n}(p) \\ 0 & & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

This last matrix is precisely the Jacobian matrix of $(f_1, \ldots, f_m, X_{m+1}, \ldots, X_n)$ at p. As it is non-singular, we deduce that p is a non-degenerate solution of the system, and the result follows. □

Finally, from the previous results, we can obtain the following upper bound for the number of non-degenerate critical points of V depending also on the dimension m of the configuration that generates V:

Theorem 4.2.13. *Let $\{(p_k, q_k)\}_{k=1}^N$ be an m-dimensional configuration of N charges with the same sign in $\mathbb{R}^n$. Then, the number of non-degenerate critical points of V is less than or equal to*

- $(n(N-1)+1)^m$ *if n is even.*
- $2^{Nm}(n(N-1)+1)^m$ *if n is odd.*

Proof. By Proposition 4.2.12, we know that every non-degenerate critical point of V is a non-degenerate solution of the system defined by the Nash functions $f_1, \dots, f_m, X_{m+1}, \dots, X_n$ on $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$. Since $\mathbb{R}^n \setminus \{p_1, \dots, p_N\}$ is a connected open semi-algebraic subset of $\mathbb{R}^n$ and $X_{m+1}, \dots, X_n$ are polynomials of degree 1, the result follows from Lemma 4.2.5. □

We can now obtain our upper bound for $\#\,\mathrm{Cr}(V)$.

Theorem 4.2.14. *If $\{(p_k, q_k)\}_{k=1}^N$ is an m-dimensional Morse configuration in $\mathbb{R}^n$, then V has finitely many critical points and*

- $\#\,\mathrm{Cr}(V) \leq (n(N-1)+1)^m$ *if n is even.*
- $\#\,\mathrm{Cr}(V) \leq 2^{Nm}(n(N-1)+1)^m$ *if n is odd.*

We can notice that, for $m = n$, the previous upper bounds are valid in the general case discussed in Section 4.1, not just for configurations of charges with the same sign. In particular, this means that every Morse configuration generates an electric potential with finitely many critical points, extending Proposition 3.1.5 for configurations with zero total charge.

We see that the upper bound for even n obtained in Theorem 4.2.14 is better than the one from 4.1.8 in the case of Morse configurations. In fact, we have that

$$(n(N-1)+1)^n \leq (n(N-1)+1)\,(2n(N-1)+1)^{n-1} - N$$

for every choice of $n, N \geq 2$. Besides, we have that

$$2^{Nm}(n(N-1)+1)^m \leq M_1(n, N), M_2(n, N)$$

for big N since the left side is of the order of $2^{nN}N^n$ while M_1 and M_2 have terms with order N^N.

Remark 4.2.15. In [4], they prove that, for a non-degenerate configuration,

$$\#\,\mathrm{Cr}(V) \leq 4^{N^2}(3N)^{2N}.$$

This upper bound is interesting, as it does not depend on n at all. Therefore, it is better, for big n, than all the bounds discussed in this text. However, it has a much faster growth with respect to N for a fixed n, as it has a term with N^2 in the exponent. Thus, our bounds are better for big N.

Remark 4.2.16. As a final comment to this chapter, it is worth to mention that we have obtained our upper bounds by means of very general results about polynomials and Nash functions, without taking into consideration the highly particular structure of our functions. For this reason, it is likely that the upper bounds can be refined using the methods, rather than the results, of Real Algebraic Geometry, focusing on the specific functions we have worked with.

Appendices

Appendix A

Cauchy-Kovalevskaya Theorem

The contents of this appendix have been extracted from [2] and [3]. We start with some preliminary definitions.

Definition A.1 (Hypersurface)**.** A subset $S \subset \mathbb{R}^n$ is called a hypersurface of class C^k $(1 \leq k \leq \infty)$ if for every $x_0 \in S$ there is an open set $V \subset \mathbb{R}^n$ containing x_0 and a real-valued function $\phi \in C^k(V)$ such that $\nabla\phi \neq 0$ in $S \cap V$ and

$$S \cap V = \{x \in V : \phi(x) = 0\}.$$

If S satisfies these conditions with C^k replaced by C^ω, we say that S is a real analytic hypersurface.

Remark A.2. In the situation of Definition A.1, by the Implicit Function Theorem (Lemma 3.3.11), we can solve the equation $\phi(x) = 0$ near x_0 for some coordinate x_i (for convenience, say $i = n$) to obtain

$$x_n = \psi(x_1, \ldots, x_{n-1})$$

for some C^k function ψ. An open neighborhood of x_0 in S can be mapped to a piece of the hyperplane $\{x_n = 0\} \subset \mathbb{R}^n$ by the C^k transformation

$$x \longmapsto (x_1, \ldots, x_{n-1}, x_n - \psi(x_1, \ldots, x_{n-1}))\,.$$

This neighborhood of S can be represented, in parametric form, as the image of an open subset of $\mathbb{R}^{n-1}$ under the map

$$(x_1, \ldots, x_{n-1}) \longmapsto (x_1, \ldots, x_{n-1}, \psi(x_1, \ldots, x_{n-1})).$$

The inverse of this map may be thought of as giving a local coordinate system of S near x_0.

With S, V and ϕ as above, the vector $\nabla\phi(x)$ is perpendicular to S at x for every $x \in S \cap V$. We shall always suppose that S is oriented, i.e., that we have chosen a unit vector $\nu(x)$ for each $x \in S$, varying continuously with x, which is perpendicular to S at x. The vector $\nu(x)$ will be called the normal vector to S at x. In particular, on $S \cap V$ we have

$$\nu(x) = \pm\frac{\nabla\phi(x)}{|\nabla\phi(x)|}.$$

Definition A.3 (Directional Derivative)**.** If f is a function of class C^k defined in a neighborhood of some point $x \in \mathbb{R}^n$, we define the k^{th} directional derivative of f along a vector $w \in \mathbb{R}^n$ at x by

$$\partial_w^k u(x) = \sum_{|\alpha|=k} \partial^\alpha u(x)\frac{w^\alpha}{|w|^k} = \sum_{\alpha_1+\ldots+\alpha_n=k} \frac{\partial^k u}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}(x)\frac{w_1^{\alpha_1}\ldots w_n^{\alpha_n}}{|w|^k}.$$

Remark A.4. We can see that the previous definition only depends on the direction of the vector, not on the vector itself. Then, it can be stated just in terms of unit vectors, simplifying the notation.

We consider the k^{th}-order linear differential operator

$$L = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha$$

and, for some function f, the k^{th}-order linear partial differential equation

$$Lu = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha u(x) = f(x)$$

in some region $U \subset \mathbb{R}^n$. Throughout the rest of the present section, we consider S to be an analytic hypersurface contained in U and $\nu(x)$ to be the normal vector to S in each $x \in S$. Also, let $\phi_0, \dots, \phi_{k-1}$ be functions of class C^{k-1} defined near S. We want to find solutions for the Cauchy problem

$$(P) \quad \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha u = f, \quad \partial_\nu^j u = \phi_j \text{ in } S \ (0 \leq j \leq k-1)$$

that are defined in a neighborhood of some point $x_0 \in S$. That is, we want to find a neighborhood $W \subseteq U$ of x_0, where the ϕ_j are defined, and a function $u : W \to \mathbb{R}$ such that

$$\sum_{|\alpha| \leq k} a_\alpha \partial^\alpha u(x) = f(x) \quad \text{for } x \in W \quad \text{and} \quad \partial_\nu^j u(x) = \phi_j(x) \quad \text{in } S \cap W \ (0 \leq j \leq k-1).$$

In order for the Cauchy problem to be well-behaved, we need S to be non-characteristic for the operator L. This condition is described in the following definitions.

Definition A.5 (Characteristic Form, Vector and Variety)**.** Let $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$ be a linear partial differential operator of order k in some region $U \subset \mathbb{R}^n$. Its characteristic form at some $x \in U$ is the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$\chi_L(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x)\xi^\alpha \quad (\xi \in \mathbb{R}^n).$$

A nonzero vector $\xi \in \mathbb{R}^n$ is called characteristic for L at x if $\chi_L(x, \xi) = 0$, and the set of such ξ is called the characteristic variety of L at x and is denoted by

$$\text{char}_x(L) = \{\xi \neq 0 : \chi_L(x, \xi) = 0\}.$$

Definition A.6 (Non-characteristic hypersurface)**.** We say that a hypersurface S is non-characteristic for the operator L if $\nu(x) \notin \text{char}_x(L)$ for all $x \in S$, i.e., if

$$\sum_{|\alpha| = k} a_\alpha(x)\nu(x)^\alpha \neq 0$$

for all $x \in S$.

Definition A.7 (Elliptic Operator)**.** Assume that we are in the context of the previous section. A linear differential operator L is said to be elliptic at a point $x \in U$ if $\text{char}_x(L) = \emptyset$, that is, if every nonzero vector is not characteristic for L at x. Also, L is said to be elliptic on U if it is elliptic at every $x \in U$.

The following result is a direct consequence of the definitions of elliptic operator and non-characteristic hypersurface:

Proposition A.8. If L is an elliptic operator, then every hypersurface in $\mathbb{R}^n$ is non-characteristic for L.

We are now in a position to state the Cauchy-Kovalevskaya Theorem. The proof can be found, for example, in Section 4.6.3. of [2] or in Section 1.D of [3].

Theorem A.9 (Cauchy-Kovalevskaya Theorem)**.** If $\phi_0, \dots, \phi_{k-1}, f, a_\alpha(|\alpha| \leq k)$ are analytic functions near x_0 and the hypersurface S is non-characteristic for L, there is a neighborhood of x_0 on which the Cauchy problem (P) has a unique analytic solution.

Appendix B

Measure Theory

B.1. Measure Spaces

Definition B.1 (Sigma Algebra)**.** Let X be a set. A σ-algebra on X is a collection $\mathcal{B}$ of subsets of X that satisfies the following properties:

(I) (Empty set) $\emptyset \in \mathcal{B}$.

(II) (Complement) If $E \in \mathcal{B}$, then the complement $E^c := X \setminus E$ also lies in $\mathcal{B}$.

(III) (Countable unions) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of elements in $\mathcal{B}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}$.

We refer to the pair $(X, \mathcal{B})$ of a set X together with a σ-algebra on that set as a measurable space. We say that a subset $E \subset X$ is $\mathcal{B}$-measurable, or just measurable, if $E \in \mathcal{B}$.

Example B.2 (Borel σ-algebra)**.** Let X be a topological space. The Borel σ-algebra $\mathcal{B}[X]$ of X is defined to be the σ-algebra generated by the open subsets of X, i.e., the intersection of all σ-algebras that contain the open subsets of X. Elements of $\mathcal{B}[X]$ will be called Borel measurable.

Definition B.3 (Outer Measure)**.** Let X be a set and $\mathcal{P}(X)$ the power set of X. An outer measure on X is a map $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ that satisfies the following properties:

(I) (Empty set) $\mu^*(\emptyset) = 0$.

(II) (Monotonicity) If $E \subset F$, then $\mu^*(E) \leq \mu^*(F)$.

(III) (Countable subadditivity) If $\{E_n\}_{n=1}^{\infty}$ is sequence of subsets of X, then $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

Definition B.4 (Measure)**.** Let $(X, \mathcal{B})$ be a measurable space. A measure μ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, +\infty]$ that satisfies the following properties:

(I) (Empty set) $\mu(\emptyset) = 0$.

(II) (Countable additivity) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$.

A triplet $(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B})$ is a measurable space and $\mu : \mathcal{B} \to [0, +\infty]$ is a measure is called a measure space.

Proposition B.5. Let $(X, \mathcal{B}, \mu)$ be a measure space. Then,

(I) (Countable subadditivity) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of measurable sets, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$.

(II) (Upwards monotone convergence) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of measurable sets such that $E_n \subset E_{n+1}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n) = \sup_{n \in \mathbb{N}} \mu(E_n).$$

(III) (Downwards monotone convergence) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of measurable sets such that $E_n \supset E_{n+1}$ and $\mu(E_k) < +\infty$ for at least one k, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \inf_{n\in\mathbb{N}} \mu(E_n).$$

B.2. Lebesgue Measure

Definition B.6 (boxes). A box in $\mathbb{R}^n$ is a Cartesian product $B := I_1 \times \ldots \times I_n$ of n bounded intervals. The volume $|B|$ of such a box B is defined as $|B| := |I_1| \ldots |I_n|$, where $|I_k|$ is the length of the interval I_k.

Definition B.7 (Lebesgue Outer Measure). Let E be a subset of $\mathbb{R}^n$. We define the Lebesgue outer measure of E by

$$m^*(E) := \inf\left\{\sum_{k=1}^{\infty} |B_k| : (B_k)_{k\in\mathbb{N}} \text{ is a sequence of boxes such that } E \subset \bigcup_{k=1}^{\infty} B_k\right\}$$

The map $m^* : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ satisfies the properties of Definition B.3.

Definition B.8 (Lebesgue Measurability). A set $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable if, for every $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ containing E such that $m^*(U \setminus E) \leq \varepsilon$. If E is Lebesgue measurable, we refer to $m(E) := m^*(E)$ as the Lebesgue measure of E. The collection of Lebesgue measurable subsets of $\mathbb{R}^n$ is a σ-algebra on $\mathbb{R}^n$. We will call it the Lebesgue σ-algebra and denote it by $\mathcal{L}[\mathbb{R}^n]$. The map $m : \mathcal{L}[\mathbb{R}^n] \to [0, +\infty]$ satisfies the properties of Definition B.4 on. Unless stated differently, we will just say a set is measurable when referring to the fact that it is Lebesgue measurable.

There is a direct consequence of this definition:

Corollary B.9. Every open subset U of $\mathbb{R}^n$ is measurable and if $U \neq \emptyset$, then $m(E) > 0$. In fact, every Borel subset of $\mathbb{R}^n$ is measurable.

Definition B.10. (Null Sets) Every subset of $\mathbb{R}^n$ of Lebesgue outer measure zero is measurable. Such sets are called null sets.

Lebesgue measure is unique in the following sense:

Proposition B.11. The Lebesgue measure $E \mapsto m(E)$ is the only measure defined on $\mathcal{L}[\mathbb{R}^n]$ that satisfies the properties:

(I) (Translation Invariance) If E is measurable and $x \in \mathbb{R}^n$, then $m(E + x) = m(E)$, where we define $E + x := \{y + x : y \in E\} \subset \mathbb{R}^n$.

(II) (Normalisation) $m([0, 1]^n) = 1$.

Definition B.12 (Almost Everywhere). A property $P(x)$ that affects points $x \in \mathbb{R}^n$ is said to hold almost everywhere in some nonempty open subset U of $\mathbb{R}^n$, or for almost all $x \in U$, if the set of $x \in U$ for which $P(x)$ fails has Lebesgue measure zero (i.e. P is true outside of a null set).

Appendix C

Real Algebraic Geometry

C.1. Algebraic and Semi-Algebraic Sets

Definition C.1 (Algebraic Set)**.** Let $\mathbb{R}[X_1,\dots,X_n]$ be the ring of polynomials on n variables with coefficients in $\mathbb{R}$. Let B be a subset of $\mathbb{R}[X_1,\dots,X_n]$. Denote

$$\mathcal{Z}(B) = \{x \in \mathbb{R}^n : \forall f \in B \ f(x) = 0\}.$$

The elements of $\mathcal{Z}(B)$ are the zeros of the set B. An algebraic subset of $\mathbb{R}^n$ is the set of zeros of some $B \subseteq \mathbb{R}[X_1,\dots,X_n]$. If $B = \{f_1,\dots,f_k\}$ is a finite set, we denote

$$\mathcal{Z}(f_1,\dots,f_k) = \mathcal{Z}(B) = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}.$$

We can see that the algebraic set defined by some $B \subseteq \mathbb{R}[X_1,\dots,X_n]$ only depends on the ideal it generates in $\mathbb{R}[X_1,\dots,X_n]$.

Proposition C.2. Let $B \subseteq \mathbb{R}[X_1,\dots,X_n]$ and $I = (B)$ the ideal it generates in $\mathbb{R}[X_1,\dots,X_n]$. Then,

$$\mathcal{Z}(B) = \mathcal{Z}(I).$$

Proof. As B generates I, we have that $B \subseteq I$. Then,

$$\mathcal{Z}(I) = \bigcap_{f\in I} \mathcal{Z}(f) \subseteq \bigcap_{f\in B} \mathcal{Z}(f) = \mathcal{Z}(B).$$

On the other hand, let $x \in \mathcal{Z}(B)$ and $f \in I$. There exist $f_1,\dots,f_k \in B$ and $g_1,\dots,g_k \in \mathbb{R}[X_1,\dots,X_n]$ such that

$$f = \sum_{j=1}^{k} g_j f_j.$$

Then,

$$f(x) = \sum_{j=1}^{k} g_j(x) f_j(x) = \sum_{j=1}^{k} g_j(x)\cdot 0 = 0,$$

so $x \in \mathcal{Z}(I)$. The result follows. □

As a result, the set $\mathcal{Z}(I)$ equals $\mathcal{Z}(G)$ for every set G that generates I. Then, every algebraic set can be described as the set of zeros of an ideal in $\mathbb{R}[X_1,\dots,X_n]$.

Proposition C.3. Given an algebraic subset V of $\mathbb{R}^n$, there exists $f \in \mathbb{R}[X_1,\dots,X_n]$ such that $V = \mathcal{Z}(f)$.

Proof. There is some ideal $I \subseteq \mathbb{R}[X_1,\dots,X_n]$ such that $V = \mathcal{Z}(I)$. Besides, the ring $\mathbb{R}[X_1,\dots,X_n]$ is Noetherian (all its ideals are finitely generated) by Hilbert's Basissatz. Thus, there are some polynomials $f_1,\dots,f_k$ that generate I, so $V = \mathcal{Z}(f_1,\dots,f_k)$. If we take $f = f_1^2 + \cdots + f_k$, we have the result. □

Proposition C.4 (Zariski Topology)**.** We have the following properties of algebraic sets in $\mathbb{R}[X_1, \ldots, X_n]$:

(I) $\emptyset = \mathcal{Z}(\mathbb{R}[X_1, \ldots, X_n])$ and $\mathbb{R}^n = \mathcal{Z}(0)$, where $0 \in \mathbb{R}[X_1, \ldots, X_n]$ is the zero polynomial.

(II) Let $\{\mathcal{Z}(I_j) : j \in J\}$ be a family of algebraic sets. Their intersection is an algebraic set too. In fact,

$$\bigcap_{j \in J} \mathcal{Z}(I_j) = \mathcal{Z}\left(\sum_{j \in j} I_j\right),$$

where $\sum_{j \in j} I_j$ is the sum ideal.

(III) Let $\mathcal{Z}(I_1)$ and $\mathcal{Z}(I_2)$ be two algebraic sets. Their union is an algebraic set too. In fact,

$$\mathcal{Z}(I_1) \cup \mathcal{Z}(I_2) = \mathcal{Z}(I_1 I_2),$$

where $I_1 I_2$ is the product ideal.

As a consequence of these properties, the set $\{\mathcal{Z}(B) : B \subseteq \mathbb{R}^n\}$ of algebraic subsets of $\mathbb{R}^n$ defines a topology on $\mathbb{R}^n$ where this set corresponds to the class of closed sets. It is called the Zariski topology on $\mathbb{R}^n$.

Definition C.5. Given a set S of $\mathbb{R}^n$, denote by

$$\mathcal{I}(S) := \{f \in \mathbb{R}[X_1, \ldots, X_n] : \forall x \in S \ f(x) = 0\}$$

the ideal of $\mathbb{R}[X_1, \ldots, X_n]$ of polynomials that vanish on S.

Proposition C.6. If $\{S_j : j \in J\}$ is a family of subsets of $\mathbb{R}^n$, then

$$\mathcal{I}\left(\bigcup_{j \in J} S_i\right) = \bigcap_{j \in J} \mathcal{I}(S_j).$$

Proposition C.7. If S is a subset of $\mathbb{R}^n$, then

$$\overline{S} = \mathcal{Z}(\mathcal{I}(S)),$$

where $\overline{S}$ corresponds to the closure of the set S in the Zariski topology on $\mathbb{R}^n$. Furthermore,

$$\mathcal{I}(S) = \mathcal{I}\left(\overline{S}\right).$$

Definition C.8. (Semi-Algebraic Set) A semi-algebraic subset of $\mathbb{R}^n$ is a set of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : f_{i,j} *_{i,j} 0\},$$

where $f_{i,j} \in \mathbb{R}[X_1, \ldots, X_n]$ and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

Remark C.9. Clearly, every algebraic set is semi-algebraic.

By their definition, semi-algebraic sets are stable under finite union, finite intersection and taking complements. In fact, they are also stable under projection.

Proposition C.10 ([1], Theorem 2.2.1)**.** Let S be a semi-algebraic subset of $\mathbb{R}^{n+1}$ and $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection on the first n coordinates. Then, $\Pi(S)$ is a semi-algebraic subset of $\mathbb{R}^n$.

Definition C.11 (Basic Semi-Algebraic Sets)**.** A basic open semi-algebraic subset of $\mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n : f_1(x) > 0, \ldots, f_k(x) > 0\},$$

where $f_1, \ldots, f_k \in \mathbb{R}[X_1, \ldots, X_n]$.

A basic closed semi-algebraic subset of $\mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n : f_1(x) \geq 0, \ldots, f_k(x) \geq 0\},$$

where $f_1, \ldots, f_k \in \mathbb{R}[X_1, \ldots, X_n]$.

C.2. Nash Functions

Now, we introduce the concept of Nash functions and some of their basic properties.

Definition C.12 (Graph of a Map)**.** Let X and Y be two sets and $f : X \to Y$ a map. The graph of f is the set

$$G(f) = \{(x, f(x)) \in X \times Y : x \in X\} .$$

Definition C.13 (Semi-Algebraic Mapping)**.** Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be two semi-algebraic sets. A mapping $f : A \to B$ is said to be semi-algebraic if its graph $G(f)$ is semi-algebraic in $\mathbb{R}^{n+m}$.

Definition C.14 (Nash Function)**.** Let U be an open semi-algebraic subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is said to be a Nash function if it is a semi-algebraic function of class C^∞. The ring of Nash functions on U is denoted by $\mathcal{N}(U)$.

Proposition C.2.1. *If $f : U \to \mathbb{R}$ is a Nash function then it is real analytic on U.*

The following result about Nash functions can be found in [15] as Lemma 2.1:

Lemma C.15. If $f : U \to \mathbb{R}$ is a Nash function, there exists a nonzero polynomial $P \in \mathbb{R}[X_1, \dots, X_n, Y]$ such that $P(x, f(x)) = 0$ for all $x \in U$.

Definition C.16 (Complexity)**.** Let $f : U \to \mathbb{R}$ be a Nash function.. The complexity of f, denoted $c(f)$, is the minimum degree of all nonzero polynomials $P \in \mathbb{R}[X_1, \dots, X_n, Y]$ such that $P(x, f(x)) = 0$ for all $x \in U$. That is,

$$c(f) = min \{\deg P : P \in \mathbb{R}[X_1, \dots, X_n, Y] \setminus \{0\} \text{ and } P(x, f(x)) = 0 \text{ for all } x \in U\} .$$

Clearly, $c(f) \geq 1$.

Remark C.17. The definition of complexity makes sense by Proposition C.15. Besides, it is a generalization of polynomial degree since all polynomials are Nash functions and, if f is a polynomial, we have that $c(f) = \max \{\deg(f), 1\}$.

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