WELL-POSEDNESS OF A WATER WAVE MODEL WITH VISCOUS EFFECTS

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ABSTRACT. Starting from the paper by Dias, Dyachenko and Zakharov (*Physics Letters A, 2008*) on viscous water waves, we derive a model that describes water waves with viscosity moving in deep water with or without surface tension effects. This equation takes the form of a nonlocal fourth order wave equation and retains the main contributions to the dynamics of the free surface. Then, we prove the well-posedness in Sobolev spaces of such equation.

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1. INTRODUCTION

The motion of the free surface of an incompressible fluid has been studied for centuries [35]. Many of these works examine the case where the fluid is inviscid and irrotational and thus they have to study the Euler equations. However, there are situations where the viscous damping cannot be neglected and should be considered. The first works in this direction date back to Boussinesq [5] and Lamb [26]. Then, Ruvinsky & Freidman [34] formulated a system of equations for weakly damped water waves in deep water (see also [25, 32, 33]). Similar models were also proposed and studied by Longuet-Higgins [27], Jiang, Ting, Perlin & Schultz [20], Joseph & Wang [21, Equation (6.7) and (6.8)], Wang & Joseph [36] and Wu, Liu & Yue [37]. The interested reader can refer to [17] for more details on these contributions.

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More recently, Dias, Dyachenko & Zakharov [9] proposed the following system

$$\Delta\phi = 0 \qquad \text{in } \Omega(t), \tag{1a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) = -2\mu\partial_2^2\phi \qquad \text{on } \Gamma(t), \tag{1b}$$

$$h_t = \nabla\phi\cdot(-\partial_1 h, 1) + 2\frac{\mu}{\rho}\partial_1^2 h \qquad \text{on } \Gamma(t), \tag{1c}$$

as a model of viscous water waves. Here the $t \in [0, T]$ is the time while $\Omega(t)$ and $\Gamma(t)$ are the the region occupied by the fluid and the water wave. Furthermore, h denotes the height of the interface and ϕ is the velocity potential.

This model obtained a considerable amount of attention and was also considered by several other authors. For instance, Dutykh & Dias [12] deduced a new system taking into account the effects of the bottom topography (see also [10, 10, 11, 13]). Kakleas & Nicholls [22], using the analytic dependence of the Dirichlet-Neumann operator, derived and asymtotic model of (1). This asymptotic model takes the form of a system of two equations and is the viscous analog of the classical Craig-Sulem WW2 model [7]. For the well-posedness of these systems we refer to the works Ambrose, Bona & Nicholls [3], Ngom & Nicholls [31] or the new preprint by the authors [18]. In the very recent paper [17], the authors derived two different asymptotic models in the regime of small steepness. In particular, starting from the model for water waves with viscosity proposed by Dias, Dyachenko and Zakharov (1), the authors obtained a nonlocal wave equation for the free surface. This wave equation contains terms at different scales. The purpose of this paper is to propose a model that retains only the main contributions and to study whether this new model is well-posed in the sense of Hadamard. In particular, this note is devoted to the derivation and mathematical study of the following nonlocal wave equation

$$\begin{aligned} f_{tt} + 2\delta\Lambda^2 f_t + \Lambda f + \beta\Lambda^3 f + \delta^2\Lambda^4 f = -\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_x[\![\mathcal{H}, f]\!]\Lambda f \\ + \beta\partial_x[\![\mathcal{H}, f]\!]\Lambda^3 f + \delta\partial_x[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_x^2 f + \delta\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_x^2 f\right) \\ - \delta\partial_x\left[\!\left[\partial_x^2, f\right]\!\right]\mathcal{H}f_t, \end{aligned} \tag{2}$$

where $(x, t) \in \mathbb{S}^1 \times [0, T]$, $\mathbb{S}^1$ denotes the one-dimensional flat torus (identified with $[-\pi, \pi]$ equipped with periodic boundary conditions), $\delta > 0$ is a dimensionless parameter reflecting the viscous damping of water waves, $\beta \geq 0$ is the Bond number, the operators $\mathcal{H}$ and Λ stand for the Hilbert transform and the Calderón operator

$$\widehat{\mathcal{H}f}(k) = -i\text{sgn}(k)\hat{f}(k)\,, \quad \widehat{\Lambda f}(k) = |k|\hat{f}(k)\,, \quad \widehat{\Lambda^s f}(k) = |k|^s\hat{f}(k)\,, \tag{3}$$

and

$$[\![A, B]\!]f = A(Bf) - B(Af),$$

is the commutator between two operators acting on the function f. Together with (2), we have to consider the initial data

$$f(x, 0) = f_0(x), \qquad\qquad f_t(x, 0) = f_1(x). \tag{4}$$

From this point onwards and without losing generality, we will assume that the initial data have zero mean. In fact, this equation is obtained as a simplified mathematical model of water waves with viscosity (see below for a more detailed explanation). The inviscid analog of equation (2) was studied by different authors: Matsuno [28–30], Akers & Milewski [1] and Cheng, Granero-Belinchón, Shkoller and Wilkening [4] derived the inviscid model. Furthermore, Cheng, Granero-Belinchón, Shkoller and Wilkening also proved that it is well-posed in a class of analytic functions and performed numerical simulations. Finally, let us mention that Akers & Nicholls [2] studied the inviscid model under the traveling solitary wave ansatz. Finally, we note that there is a striking similarity between the inviscid version of (2) and models describing water waves in a porous media [15, 16].
The main result of this work is the following theorem

Theorem 1. *Fix $\delta > 0$ and $\beta \geq 0$. Let $(f_0, f_1) \in H^{5.5} \times H^{3.5}$ be the arbitrary initial data (4) for (2). Then, there exist $0 < T \leq \infty$ such that there is a unique solution*

$$f \in C\left([0, T], H^{5.5}\right),$$
$$f_t \in C\left([0, T], H^{3.5}\right) \cap L^2(0, T; H^{4.5}).$$

for (2) for a short enough lifespan $0 < T \leq \infty$.

The proof of this theorem relies on appropriate energy estimates and a Picard iteration method for a mollified version of (2) (see [19] for more details). We would like to emphasize that, in order the initial data can be arbitrarily big, the energy estimates need to exploit the fine structure of some terms in the nonlinearity of (2). Let us also explain the reason behind the use of the space $H^{5.5}$ (once this spaces is fixed, the other ones are also fixed due to scaling): as we are interested in classical solutions, we need at least H^α with $\alpha > 4.5$. That means testing against $\Lambda^\sigma f_t$ with $\sigma > 5$ (σ and α are related via $\alpha = (\sigma + 4)/2$). For the sake of simplicity, we consider

$\sigma = 7$ and that leads to the space $H^{5.5}$ in the statement. However, with more sophisticates estimates we could also reach the space H^5.

2. Motivation

Let us briefly sketch the derivation of (2). Starting from the model for water waves with viscosity system proposed by Dias, Dyachenko, and Zakharov [9], we derived the following nonlocal wave equation [17]

$$\begin{aligned} f_{tt} - (\alpha_1+\alpha_2)\partial_1^2 f_t + \Lambda f + \beta\Lambda^3 f + \alpha_1\alpha_2\partial_1^4 f \\ = \varepsilon\Big\{ -\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_x[\![\mathcal{H}, f]\!]\Lambda f + \beta\partial_x[\![\mathcal{H}, f]\!]\Lambda^3 f \\ + \alpha_2\partial_x[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_x^2 f + \alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_x^2 f\right) + \alpha_1\alpha_2\partial_x\left[\!\left[\partial_x^2, f\right]\!\right]\Lambda\partial_1 f \\ - \alpha_1\partial_x\left[\!\left[\partial_x^2, f\right]\!\right]\mathcal{H}f_t - \alpha_2\alpha_2\partial_x\left[\!\left[\mathcal{H}, \partial_1^2 f\right]\!\right]\partial_1^2 f\Big\}. \end{aligned} \tag{5}$$

Roughly speaking, some typical values of the dimensionless parameters are (see [17]):

$$\varepsilon \approx O(10^{-2}), \qquad \beta \approx O(10^{-5}), \qquad \alpha_1, \alpha_2 \approx O(10^{-4}). \tag{6}$$

Thus, we see that the smallest terms are those with $\varepsilon\alpha_2\alpha_2, \varepsilon\alpha_2\alpha_1$. These terms are $O(10^{-10})$. It is reasonable to think that they are not crucial for the evolution of the interface and that, even neglecting them, we still can describe most of the dynamics. As a consequence, we obtain the new model (where $\alpha_1 = \alpha_2 = \delta$)

$$\begin{aligned} f_{tt} + 2\delta\Lambda^2 f_t + \Lambda f + \beta\Lambda^3 f + \delta^2\Lambda^4 f = \varepsilon\Big\{ -\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_x[\![\mathcal{H}, f]\!]\Lambda f \\ + \beta\partial_x[\![\mathcal{H}, f]\!]\Lambda^3 f + \delta\partial_x[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_x^2 f + \delta\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_x^2 f\right) \\ - \delta\partial_x\left[\!\left[\partial_x^2, f\right]\!\right]\mathcal{H}f_t\Big\}, \ (x,t) \in \mathbb{S}^1 \times [0,T], \end{aligned} \tag{7}$$

As our analysis is independent of the steepness, without losing generality, we take the steepness parameter to be identically one, *i.e.* $\varepsilon = 1$. As a consequence, we recover (2).

3. Proof of Theorem 1

We are going to find appropriate *a priori* bounds for the energy

$$\mathfrak{E}(t) = \max_{0\le s\le t}\left\{\|f(s)\|^2_{\dot{H}^4} + \beta\|f(s)\|^2_{\dot{H}^5} + \delta^2\|f(s)\|^2_{\dot{H}^{5.5}} + \|f_t(s)\|^2_{\dot{H}^{3.5}}\right\}. \tag{8}$$

It is also convenient to define the dissipation

$$\mathfrak{D}(t) = 2\delta\|f_t(t)\|^2_{\dot{H}^{4.5}}. \tag{9}$$

Our goal is to find a polynomial inequality of the form

$$\mathfrak{E}(t) \leq \mathcal{M}_0(f_0, f_1) + t^\alpha P(\mathfrak{E}(t)),$$

for certain $\alpha > 0$, constant $\mathcal{M}_0(f_0, f_1)$ and polynomial P. Testing (2) against $\Lambda^7 f_t$ and integrating by parts, we find that

$$\frac{1}{2}\mathfrak{E}(t) + \int_0^t \mathfrak{D}(s)\, \mathrm{d}s = \frac{1}{2}\mathfrak{E}(0) + \sum_{i=1}^{6} I_i,$$

with

$$I_1 = -\int_0^t \int_{\mathbb{S}^1} \Lambda\left((\mathcal{H}f_t)^2\right)\Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s \tag{10a}$$

$$I_2 = \int_0^t \int_{\mathbb{S}^1} \partial_x[\![\mathcal{H}, f]\!]\Lambda f \Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s \tag{10b}$$

$$I_3 = \beta\int_0^t \int_{\mathbb{S}^1} \partial_x[\![\mathcal{H}, f]\!]\Lambda^3 f \Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s \tag{10c}$$

$$I_4 = \delta\int_0^t \int_{\mathbb{S}^1} \partial_x[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_x^2 f \Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s \tag{10d}$$

$$I_5 = \delta\int_0^t \int_{\mathbb{S}^1} \Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_x^2 f\right)\Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s \tag{10e}$$

$$I_6 = -\delta\int_0^t \int_{\mathbb{S}^1} \partial_x\left[\!\left[\partial_x^2, f\right]\!\right]\mathcal{H}f_t\Lambda^7 f_t \,\mathrm{d}x\,\mathrm{d}s. \tag{10f}$$

Using the self-adjointness of the operator Λ together with Hölder's inequality and the Sobolev embedding

$$\|g\|_{L^4} \leq C\|g\|_{H^{0.25}},$$

we find that

$$\begin{aligned}
I_1 &= -\int_0^t \int_{\mathbb{S}^1} \left((\mathcal{H}f_t)^2\right)\Lambda^8 f_t \,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t \int_{\mathbb{S}^1} \left((\mathcal{H}f_t)^2\right)\partial_x^4\Lambda^4 f_t \,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t \int_{\mathbb{S}^1} \partial_x^4\left((\mathcal{H}f_t)^2\right)\Lambda^4 f_t \,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t \int_{\mathbb{S}^1} \left(2\mathcal{H}f_t\Lambda\partial_x^3 f_t + 6(\Lambda\partial_x f_t)^2 + 8\Lambda f_t\partial_x^2\Lambda f_t\right)\Lambda^4 f_t \,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\int_0^t \|f_t\|_{\dot{H}^4}\left(\|f_t\|_{\dot{H}^4}\|\mathcal{H}f_t\|_{L^\infty} + \|f_t\|_{\dot{H}^{2.25}}^2 + \|f_t\|_{\dot{H}^3}\|\Lambda f_t\|_{L^\infty}\right)\,\mathrm{d}s \\
&\leq C\int_0^t \|f_t\|_{\dot{H}^{4.5}}\|f_t\|_{\dot{H}^{3.5}}\|f_t\|_{\dot{H}^{2.25}}\,\mathrm{d}s.
\end{aligned} \tag{11}$$

Furthermore, using interpolation between Sobolev spaces, the embedding

$$H^s \subset H^r\,, r \leq s,$$

the continuity of the Hilbert transform

$$\|\mathcal{H}F\|_{L^p} \leq C\|F\|_{L^p}, \qquad 1<p<\infty,$$

and Young's inequality, we can further compute that

$$I_1 \leq \frac{C}{\delta} t\mathfrak{E}^2 + \frac{\delta}{4}\int_0^t \mathfrak{D} \ \mathrm{d}s \tag{12}$$

We recall the following commutator estimate (see equation (1.13) in [8])

$$\|\partial_x^\ell [\![\mathcal{H}, u]\!] \partial_x^m v\|_{L^p} \leq C\|\partial_x^{\ell+m} u\|_{L^\infty}\|v\|_{L^p}, \quad p\in(1,\infty), \quad \ell, m\in\mathbb{N}. \tag{13}$$

Equipped with (13), we can estimate I_2 as follows

$$\begin{aligned}
I_2 &= \int_0^t\int_{\mathbb{S}^1} \Lambda^3\partial_x [\![\mathcal{H}, f]\!] \Lambda f \Lambda^4 f_t \ \mathrm{d}x \ \mathrm{d}s\\
&\leq \int_0^t \|\partial_x^4 [\![\mathcal{H}, f]\!] \Lambda f\|_{L^2}\|\Lambda^4 f_t\|_{L^2} \ \mathrm{d}s\\
&\leq C\int_0^t \|\partial_x^4 f\|_{L^\infty}\|\Lambda f\|_{L^2}\|\Lambda^4 f_t\|_{L^2} \ \mathrm{d}s\\
&\leq C\int_0^t \|\partial_x^5 f\|_{L^2}\|\Lambda f\|_{L^2}\|\Lambda^4 f_t\|_{L^2} \ \mathrm{d}s\\
&\leq \frac{C}{\delta^3} t\mathfrak{E}^2(t) + \frac{\delta}{4}\int_0^t \mathfrak{D}(s) \ \mathrm{d}s.
\end{aligned} \tag{14}$$

Using (13), we have that I_3 can be estimated as follows:

$$\begin{aligned}
I_3 &= \int_0^t\int_{\mathbb{S}^1} \beta\Lambda^3\partial_x [\![\mathcal{H}, f]\!] \Lambda^3 f \Lambda^4 f_t \ \mathrm{d}x \ \mathrm{d}s\\
&\leq C\beta\int_0^t \|f_t\|_{\dot{H}^4}\|\partial_x^4 f\|_{L^\infty}\|f\|_{\dot{H}^3} \ \mathrm{d}s\\
&\leq \frac{C}{\delta} t\mathfrak{E}^2 + \frac{1}{4}\int_0^t \mathfrak{D} \ \mathrm{d}s.
\end{aligned} \tag{15}$$

We can decompose I_4 as follows

$$\begin{aligned}
I_4 &= \delta\int_0^t\int_{\mathbb{S}^1} \left[\Lambda(\mathcal{H}f_t\Lambda\partial_x f) + \partial_x(\mathcal{H}f_t\partial_x^2 f)\right]\Lambda^7 f_t \ \mathrm{d}x \ \mathrm{d}s\\
&= \delta\int_0^t\int_{\mathbb{S}^1} \left[\Lambda(\mathcal{H}f_t\Lambda\partial_x f) - \partial_x(\mathcal{H}f_t\Lambda^2 f)\right]\Lambda^7 f_t \ \mathrm{d}x \ \mathrm{d}s\\
&= J_1^4 + J_2^4,
\end{aligned}$$

with

$$\begin{aligned}
J_1^4 &= \delta\int_0^t\int_{\mathbb{S}^1} \Lambda(\mathcal{H}f_t\Lambda\partial_x f)\Lambda^7 f_t \ \mathrm{d}x \ \mathrm{d}s\\
J_2^4 &= -\delta\int_0^t\int_{\mathbb{S}^1} \partial_x(\mathcal{H}f_t\Lambda^2 f)\Lambda^7 f_t \ \mathrm{d}x \ \mathrm{d}s.
\end{aligned}$$

We will use the fractional Leibniz rule (see [14, 23, 24]):

$$\|\Lambda^s(uv)\|_{L^p} \leq C\left(\|\Lambda^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|\Lambda^s v\|_{L^{p_3}}\|u\|_{L^{p_4}}\right),$$

which holds whenever

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \qquad \text{where } 1/2 < p < \infty, 1 < p_i \leq \infty,$$

and $s > \max\{0, 1/p - 1\}$. Recalling the inequality (see for instance [6] for a proof)

$$\|uv\|_{H^{0.5}} \leq C_\sigma \|u\|_{H^{0.5}}\|v\|_{H^{0.5+\sigma}}, \ \forall \sigma > 0, \tag{16}$$

using the duality pairing $H^{0.5}-H^{-0.5}$, the Sobolev embedding, the fractional Leibniz rule and the self-adjointness of the operator Λ, we compute

$$\begin{aligned}
J_1^4 &= \delta \int_0^t \int_{\mathbb{S}^1} \Lambda^3(\mathcal{H}f_t\Lambda\partial_x f)\Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \delta C \int_0^t \|\Lambda^{3.5}(\mathcal{H}f_t\Lambda\partial_x f)\|_{L^2}\|\Lambda^5 f_t\|_{H^{-0.5}} \, \mathrm{d}s \\
&\leq \delta C \int_0^t (\|f_t\|_{\dot{H}^1}\|f\|_{\dot{H}^{5.5}} + \|f_t\|_{\dot{H}^{3.5}}\|f\|_{\dot{H}^3})\|f_t\|_{H^{4.5}} \, \mathrm{d}s \\
&\leq \frac{C}{\delta} t\mathfrak{E}^2 + \frac{\delta}{4}\int_0^t \mathfrak{D} \, \mathrm{d}s.
\end{aligned}$$

The terms J_2^4 and $I_5 = J_1^4$ can be estimated in a similar way and we find that

$$I_4 + I_5 \leq \frac{C}{\delta} t\mathfrak{E}^2(t) + \frac{\delta}{2}\int_0^t \mathfrak{D}(s) \, \mathrm{d}s. \tag{17}$$

Now we are left with I_6. This is the more delicate term due to the high number of derivatives present in the terms involving f_t. In fact, if we just use the $H^{0.5} - H^{-0.5}$ duality, we would get an estimate of the form

$$I^6 \leq 2\delta \int_0^t \|f\|_{\dot{H}^2}\|f_t\|_{\dot{H}^{4.5}}^2 \, \mathrm{d}s + \text{l.o.t.} \leq 2\delta\sqrt{\mathfrak{E}}\int_0^t \mathfrak{D} \, \mathrm{d}s + \text{l.o.t.}.$$

Thus, in order to absorb this term with the linear part we would have to assume a size condition on the initial data. Instead of doing this, we are going to exploit the structure of this integrand. In particular, after appropriately splitting I_6, we will find a perfect derivative and, via an integration by parts, this will allow us to improve the estimates and avoid any size restriction. We compute

$$\begin{aligned}
I_6 &= -\delta \int_0^t \int_{\mathbb{S}^1} \partial_x\left[\partial_x^2 f\mathcal{H}f_t + 2\partial_x f\Lambda f_t\right]\Lambda^7 f_t \, \mathrm{d}x \, \mathrm{d}s \\
&= \delta \int_0^t \int_{\mathbb{S}^1} \partial_x^3\left[\partial_x^2 f\mathcal{H}f_t + 2\partial_x f\Lambda f_t\right]\Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s \\
&= J_1^6 + J_2^6 + J_3^6 + J_4^6 + J_5^6,
\end{aligned}$$

with

$$J_1^6 = \delta \int_0^t \int_{\mathbb{S}^1} \partial_x^5 f \mathcal{H} f_t \Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s$$
$$J_2^6 = 7\delta \int_0^t \int_{\mathbb{S}^1} \partial_x^2 f \Lambda \partial_x^2 f_t \Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s$$
$$J_3^6 = 9\delta \int_0^t \int_{\mathbb{S}^1} \partial_x^3 f \Lambda \partial_x f_t \Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s$$
$$J_4^6 = 5\delta \int_0^t \int_{\mathbb{S}^1} \partial_x^4 f \Lambda f_t \Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s$$

and

$$J_5^6 = 2\delta \int_0^t \int_{\mathbb{S}^1} \partial_x f \Lambda \partial_x^3 f_t \Lambda^5 f_t \, \mathrm{d}x \, \mathrm{d}s.$$

To bound the term J_1^6 we use the $H^{0.5} - H^{-0.5}$ duality together with the Sobolev embeddging and inequality (16):

$$J_1^6 \leq \delta \int_0^t \|f\|_{\dot{H}^{5.5}} \|f_t\|_{\dot{H}^1} \|f_t\|_{\dot{H}^{4.5}} \, \mathrm{d}s$$
$$\leq \frac{C}{\delta} t \mathfrak{E}^2 (t) + \frac{\delta}{20} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s.$$

Similarly,

$$J_2^6 \leq 7\delta \int_0^t \|f\|_{\dot{H}^3} \|f_t\|_{\dot{H}^{3.5}} \|f_t\|_{\dot{H}^{4.5}} \, \mathrm{d}s$$
$$\leq \frac{C}{\delta} t \mathfrak{E}^2 (t) + \frac{\delta}{20} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s,$$

$$J_3^6 \leq 9\delta \int_0^t \|f\|_{\dot{H}^{3.5}} \|f_t\|_{\dot{H}^3} \|f_t\|_{\dot{H}^{4.5}} \, \mathrm{d}s$$
$$\leq \frac{C}{\delta} t \mathfrak{E}^2 (t) + \frac{\delta}{20} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s,$$

$$J_4^6 \leq 5\delta \int_0^t \|f\|_{\dot{H}^{4.5}} \|f_t\|_{\dot{H}^2} \|f_t\|_{\dot{H}^{4.5}} \, \mathrm{d}s$$
$$\leq \frac{C}{\delta} t \mathfrak{E}^2 (t) + \frac{\delta}{20} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s.$$

Now we use that

$$\Lambda^4 f_t = \partial_x^4 f_t$$

to write

$$J_5^6 = 2\delta \int_0^t \int_{\mathbb{S}^1} \partial_x f \Lambda \partial_x^3 f_t \Lambda \partial_x^4 f_t \, \mathrm{d}x \, \mathrm{d}s.$$

Integrating by parts, we find

$$\begin{aligned} J_5^6 &= -\delta \int_0^t \int_{\mathbb{S}^1} \partial_x^2 f (\Lambda \partial_x^3 f_t)^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\leq \delta \int_0^t \|f\|_{\dot{H}^3} \|f_t\|_{\dot{H}^4}^2 \, \mathrm{d}s \\ &\leq \delta \int_0^t \|f\|_{\dot{H}^3} \|f_t\|_{\dot{H}^{3.5}} \|f_t\|_{\dot{H}^{4.5}} \, \mathrm{d}s \\ &\leq Ct\mathfrak{E}^2(t) + \frac{\delta}{20} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s. \end{aligned}$$

Thus,

$$I_6 \leq \frac{C}{\delta} t\mathfrak{E}^2(t) + \frac{\delta}{4} \int_0^t \mathfrak{D}(s) \, \mathrm{d}s. \tag{18}$$

Collecting the previous estimates for I^j (12),(14),(15),(17) and (18), we find

$$\mathfrak{E}(t) \leq \mathfrak{E}(0) + c(\delta) t (\mathfrak{E}(t))^2.$$

This last inequality implies the existence of a uniform time such that

$$\mathfrak{E}(t) \leq 4\mathfrak{E}(0).$$

Once the estimates are uniform in time, we can mollify (2) as in [19] to perform a Picard iteration scheme and conclude the existence of a solution

$$f \in L^\infty \left(0, T; H^{5.5}\right),$$

$$f_t \in L^\infty \left(0, T; H^{3.5}\right) \cap L^2(0, T; H^{4.5}).$$

To obtain the endpoint continuity in time, we observe that one can prove that the solution is continuous in time with respect to the weak topologies. Then, we can perform similar estimates and we obtain a differential inequality that implies the desired continuity in time (see [19] for more details). Uniqueness of solutions follows from a standard contradiction argument using the L^2-type energy estimate, and we omit the details.This finishes the proof.

Remark 1. We observe that, to reduce the regularity of the initial data from $H^{5.5}$ to H^5, we would have to estimate the term

$$I = \int_0^t \int_{\mathbb{S}^1} \partial_x f \Lambda \partial_x f_t \Lambda^6 f_t \, \mathrm{d}x \, \mathrm{d}s.$$

Here, the previous strategy of looking for a perfect derivative in order to integrate by parts will not work. Instead, we would have to use a hidden commutator. Indeed, using that

$$\int_{\mathbb{S}^1} uv\mathcal{H}v \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{S}^1} uv\mathcal{H}v \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{S}^1} \mathcal{H}(uv)v \, \mathrm{d}x = -\frac{1}{2} \int_{\mathbb{S}^1} [\![\mathcal{H}, u]\!] vv \, \mathrm{d}x,$$

the term

$$J = \int_0^t \int_{\mathbb{S}^1} \partial_x f \mathcal{H} \partial_x^4 f_t \partial_x^4 f_t \, \mathrm{d}x \, \mathrm{d}s$$

has a commutator structure that we can exploit.

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