ON THE LOCAL AND GLOBAL EXISTENCE OF SOLUTIONS TO 1D TRANSPORT EQUATIONS WITH NONLOCAL VELOCITY

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Abstract. We consider the 1D transport equation with nonlocal velocity field:

$$\theta_t + u\theta_x + \nu\Lambda^\gamma\theta = 0,$$
$$u = \mathcal{N}(\theta),$$

where $\mathcal{N}$ is a nonlocal operator. In this paper, we show the existence of solutions of this model locally and globally in time for various types of nonlocal operators.

1. Introduction

In this paper, we study transport equations with nonlocal velocity. One of the most well-known equation is the two dimensional Euler equation in vorticity form,

$$\omega_t + u \cdot \nabla\omega = 0,$$

where the velocity u is recovered from the vorticity ω through

$$u = \nabla^\perp(-\Delta)^{-1}\omega \quad \text{or equivalently} \quad \widehat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2}\widehat{\omega}(\xi).$$

Other nonlocal and quadratically nonlinear equations, such as the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magneto-geostrophic equation in multi-dimensions, have been studied intensively as one can see in [1, 2, 5, 6, 7, 8, 9, 12, 15, 16, 18, 19, 21] and references therein.

We here consider the 1D transport equations with nonlocal velocity field of the form

$$\theta_t + u\theta_x + \nu\Lambda^\gamma\theta = 0, \tag{1.1a}$$
$$u = \mathcal{N}(\theta), \tag{1.1b}$$

where $\mathcal{N}$ is typically expressed by a Fourier multiplier. The study of (1.1) is mainly motivated by [11] where Córdoba, Córdoba, and Fontelos proposed the following 1D model

$$\theta_t + u\theta_x = 0, \tag{1.2a}$$
$$u = -\mathcal{H}\theta, \quad (\mathcal{H} \text{ being the Hilbert transform}) \tag{1.2b}$$

for the 2D surface quasi-geostrophic equation and proved the finite time blow-up of smooth solutions. In this paper, we deal with (1.2) and its variations with the following objectives.

(1) The existence of weak solution with *rough initial data*. The existence of global-in-time solutions is possible even if strong solutions blow up in finite time, as in the case of the Burgers' equation.
(2) The existence of strong solution when the velocity u is more singular than θ. We intend to see the competitive relationship between nonlinear terms and viscous terms.

More specifically, the topics covered in this paper can be summarized as follows.

• **The model 1: $\mathcal{N} = -\mathcal{H}$ and $\nu = 0$.** We first show the existence of local-in-time solution in a critical space under the scaling $\theta_0(x) \mapsto \theta_0(\lambda x)$. We then introduce the notion of a weak super-solution and obtain a global-in-time weak super-solution with $\theta_0 \in L^1 \cap L^\infty$ and $\theta_0 \geq 0$.

Date: March 28, 2022.
2010 *Mathematics Subject Classification.* 35A01, 35D30, 35Q35, 35Q86.
Key words and phrases. Fluid equations, 1D models, Global weak solution.

• **The model 2:** $\mathcal{N} = -\mathcal{H}(\partial_{xx})^{-\alpha}$**,** $\alpha > 0$**,** $\nu = 1$**, and** $\gamma > 0$**.** This is a regularized version of (1.2) which is also closely related to many equations as mentioned in [3]. In this case, we show the existence of weak solutions globally in time under weaker conditions on α and γ compared to [3].

• **The model 3:** $\mathcal{N} = -\mathcal{H}(\partial_{xx})^{\beta}$**,** $\beta > 0$**,** $\nu = 1$**, and** $\gamma > 0$**.** Since $\beta > 0$, the velocity field is more singular than the previous two models. In this case, we show the existence of strong solutions locally in time in two cases: (1) $0 < \beta \leq \frac{\gamma}{4}$ when $0 < \gamma < 2$ and (2) $0 < \beta < 1$ when $\gamma = 2$. We also show the existence of strong solutions for $0 < \beta < \frac{1}{2}$ and $\gamma = 2$ with rough initial data. We finally show the existence of strong solutions globally in time with $0 < \beta < \frac{1}{4}$ and $\gamma = 2$.

We will give detailed statements and proofs of our results in Section 3–5.

2. Preliminaries

All constants will be denoted by C that is a generic constant. In a series of inequalities, the value of C can vary with each inequality. We use following notation: for a Banach space X,

$$C_T X = C([0,T] : X), \quad L^p_T X = L^p(0,T : X).$$

The Hilbert transform is defined as

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

We will use the BMO space (see e.g. [4] for the definition) and its dual which is the Hardy space $\mathcal{H}^1$ which consists of those f such that f and $\mathcal{H}f$ are integrable. We will use the following formula

$$2\mathcal{H}(f\mathcal{H}f) = (\mathcal{H}f)^2 - f^2$$

which implies that $g = f\mathcal{H}f \in \mathcal{H}^1$ and for any $f \in L^2$,

$$\|g\|_{\mathcal{H}^1} \leq \|f\|^2_{L^2}. \tag{2.1}$$

The differential operator $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ is defined by the action of the following kernels [10]:

$$\Lambda^\gamma f(x) = c_\gamma \text{p.v.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\gamma}} dy, \tag{2.2}$$

where $c_\gamma > 0$ is a normalized constant. Alternatively, we can define $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ as a Fourier multiplier: $\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi)$. When $\gamma = 1$, $\Lambda f(x) = \mathcal{H}f_x(x)$.

We finally introduce Simon's compactness.

Lemma 2.1. [22] *Let X_0, X_1, and X_2 be Banach spaces such that X_0 is compactly embedded in X_1 and X_1 is a subset of X_2. Then, for $1 \leq p < \infty$, the set $\left\{v \in L^p_T X_0 : \frac{\partial v}{\partial t} \in L^1_T X_2\right\}$ is compactly embedded in $L^p_T X_1$.*

3. The model 1

We now study (1.1) with $\mathcal{N} = -\mathcal{H}$ and $\nu = 0$ which is nothing but (1.2):

$$\theta_t - (\mathcal{H}\theta)\,\theta_x = 0, \tag{3.1a}$$

$$\theta(0,x) = \theta_0(x). \tag{3.1b}$$

3.1. **Local well-posedness.** The local well-posedness of (3.1) is established in H^2 ([2]) and $H^{\frac{3}{2}-\gamma}$ with the viscous term $\Lambda^\gamma\theta$ ([13]). To improve these results, we first notice that (3.1) has the following scaling invariant property: if $\theta(t,x)$ is a solution of (3.1), then so is $\theta_\lambda(t,x)=\theta(\lambda t,\lambda x)$. So, we take initial data in a space whose norm is closely invariant under the scaling: $\theta_0(x)\mapsto\theta_{\lambda 0}(x)=\theta_0(\lambda x)$. In this paper, we take the space $\dot{B}^{\frac{3}{2}}_{2,1}$ because there is a constant C such that

$$C^{-1}\left\|\theta_{\lambda 0}\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\leq\left\|\theta_0\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\leq C\left\|\theta_{\lambda 0}\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}}.$$

The mathematical tools needed to prove the local well-posedness of (3.1), such as the Littlewood-Paley decomposition and Besov spaces, are provided in the appendix. We also need the following commutator estimate [4, Lemma 2.100, Remark 2.101].

Lemma 3.1 (Commutator estimate)**.** *For $f,g\in\mathcal{S}$*

$$\left\|[f,\Delta_j]g_x\right\|_{L^2}\leq Cc_j2^{-\frac{3}{2}j}\left\|f_x\right\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\left\|g\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}},\qquad\sum_{j=-\infty}^{\infty}c_j\leq 1.$$

The first result in this paper the following theorem.

Theorem 3.1. *For any $\theta_0\in\dot{B}^{\frac{3}{2}}_{2,1}$, there exists $T=T(\|\theta_0\|)$ such that a unique solution of (3.1) exists in $C_T\dot{B}^{\frac{3}{2}}_{2,1}$.*

Proof. We only provide a priori estimates of θ in the space stated in Theorem 3.1. The other parts, including the approximation procedure, are rather standard.

We apply Δ_j to (3.1), multiply by $\Delta_j\theta$, and integrate the resulting equation over $\mathbb{R}$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Delta_j\theta\right\|_{L^2}^2&=\int_{\mathbb{R}}\Delta_j\left((\mathcal{H}\theta)\theta_x\right)\Delta_j\theta dx\\&=\int_{\mathbb{R}}\left((\mathcal{H}\theta)\Delta_j\theta_x\right)\Delta_j\theta dx+\int_{\mathbb{R}}\left[\Delta_j,\mathcal{H}\theta\right]\Delta_j\theta_x\Delta_j\theta dx\\&=-\frac{1}{2}\int_{\mathbb{R}}(\mathcal{H}\theta)_x\left|\Delta_j\theta\right|^2dx+\int_{\mathbb{R}}\left[\Delta_j,\mathcal{H}\theta\right]\Delta_j\theta_x\Delta_j\theta dx.\end{aligned}\tag{3.2}$$

By the Bernstein inequality, we have

$$\left\|\mathcal{H}\theta_x\right\|_{L^\infty}\leq C\|\theta\|_{\dot{B}^{\frac{3}{2}}_{2,1}}.\tag{3.3}$$

We then apply Lemma 3.1 to the second term in the right-hand side of (3.2) to obtain

$$\int_{\mathbb{R}}\left[\Delta_j,\mathcal{H}\theta\right]\Delta_j\theta_x\Delta_j\theta dx\leq Cc_j2^{-\frac{3}{2}j}\|\theta\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\left\|\Delta_j\theta\right\|_{L^2}.\tag{3.4}$$

By (3.2), (3.3), and (3.4), we have

$$\frac{d}{dt}\|\theta\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\leq C\|\theta\|^3_{\dot{B}^{\frac{3}{2}}_{2,1}},$$

from which we deduce

$$\|\theta(t)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\leq\frac{\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}}{1-Ct\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}}\leq 2\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\quad\text{for all } t\leq T=\frac{1}{2C\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}}.$$

This completes the proof. □

3.2. **Global weak super-solution.** We next consider (3.1) with rough initial data. More precisely, we assume that θ_0 satisfies the following conditions

$$\theta_0 \geq 0, \quad \theta_0 \in L^1 \cap L^\infty. \tag{3.5}$$

Since θ satisfies the transport equation, we have

$$\theta(t,x) \geq 0, \quad \theta \in L^\infty(\mathbb{R}) \quad \text{for all time.} \tag{3.6}$$

If we follow the usual weak formulation of (3.1), for all $\phi \in C_c^\infty([0,\infty) \times \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t - (\mathcal{H}\theta)\,\theta\psi_x + (\Lambda\theta)\,\theta\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx. \tag{3.7}$$

For $\theta_0 \geq 0$, there is gain of a half derivative from the structure of the nonlinearity, that is

$$\|\theta(t)\|_{L^1} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta(s)\right\|_{L^2}^2 ds = \|\theta_0\|_{L^1}. \tag{3.8}$$

So, we can rewrite the left-hand side of (3.7) as

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t - (\mathcal{H}\theta)\,\theta\psi_x + \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}},\psi\right]\theta + \left|\Lambda^{\frac{1}{2}}\theta\right|^2\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx.$$

However, the $\dot{H}^{\frac{1}{2}}$ regularity derived from (3.8) is not enough to pass to the limit in

$$\int_0^T \int_{\mathbb{R}} \left|\Lambda^{\frac{1}{2}}\theta^\epsilon\right|^2 \psi dxdt$$

from the ϵ-regularized equations described below. So, we introduce a new notion of solution. Let

$$\mathcal{A}_T = L_T^\infty\left(L^1 \cap L^\infty\right) \cap L_T^2 H^{\frac{1}{2}}.$$

Definition 3.2. *We say θ is a weak super-solution of (3.1) on the time interval $[0,T]$ if $\theta(t,x) \geq 0$ for all $t \in [0,T]$, $\theta \in \mathcal{A}_T$, and for each nonnegative $\psi \in C_c^\infty([0,T] \times \mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t - (\mathcal{H}\theta)\,\theta\psi_x + \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}},\psi\right]\theta + \left|\Lambda^{\frac{1}{2}}\theta\right|^2\psi\right] dxdt \geq \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx. \tag{3.9}$$

To prove Theorem 3.3, we need to estimate a commutator term involving $\Lambda^{\frac{1}{2}}$:

$$\left[\Lambda^{1/2},\psi\right](f-g) \in L^6$$

which is proved in [3].

Lemma 3.2. *For $f \in L^{\frac{3}{2}}$, $g \in L^{\frac{3}{2}}$ and $\psi \in W^{1,\infty}$, we have*

$$\left\|\left[\Lambda^{\frac{1}{2}},\psi\right]f - \left[\Lambda^{\frac{1}{2}},\psi\right]g\right\|_{L^6} \leq C\|\psi\|_{W^{1,\infty}}\|f-g\|_{L^{\frac{3}{2}}}.$$

The second result in our paper is the following theorem.

Theorem 3.3. *For any θ_0 satisfying (3.5), there exists a weak super-solution of (3.1) in $\mathcal{A}_T$.*

Proof. We first regularize initial data as $\theta_0^\epsilon = \rho_\epsilon * \theta_0$ where ρ_ϵ is a standard mollifier that preserve the positivity of the regularized initial data. We then regularize the equation by introducing the Laplacian term with a coefficient $\epsilon > 0$, namely

$$\theta_t^\epsilon - \mathcal{H}\theta^\epsilon\theta_x^\epsilon = \epsilon\theta_{xx}^\epsilon. \tag{3.10}$$

For the proof of the existence of a global-in-time smooth solution we refer to [17]. Moreover, θ^ϵ satisfies that $\theta^\epsilon \geq 0$ and

$$\|\theta^\epsilon(t)\|_{L^1} + \|\theta^\epsilon(t)\|_{L^\infty} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon(s)\right\|_{L^2}^2 ds \leq \|\theta_0\|_{L^1} + \|\theta_0\|_{L^\infty}.$$

Therefore, (θ_ϵ) is bounded in $\mathcal{A}_T$ uniformly in $\epsilon > 0$.

From this, we have uniform bounds

$$\mathcal{H}\theta^\epsilon \in L^4_T L^2, \quad \theta^\epsilon \in L^2_T L^2, \quad \left(\left(\mathcal{H}\theta^\epsilon\right)\theta^\epsilon\right)_x \in L^{\frac{4}{3}}_T H^{-2}, \quad \epsilon\theta^\epsilon_{xx} \in L^2_T H^{-2}.$$

Moreover, for any $\phi \in H^2$,

$$\int_{\mathbb{R}} |\theta^\epsilon \Lambda\theta^\epsilon \phi|\, dx \le \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon\right\|^2_{L^2} \|\phi\|_{L^\infty} + \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon\right\|_{L^2} \|\theta^\epsilon\|_{L^\infty} \left\|\Lambda^{\frac{1}{2}}\phi\right\|_{L^\infty}$$

which implies that

$$\theta^\epsilon \Lambda\theta^\epsilon \in L^1_T H^{-2}.$$

Combining all together, we obtain

$$\theta^\epsilon_t = \mathcal{H}\theta^\epsilon \theta^\epsilon_x + \epsilon\theta^\epsilon_{xx} = \left(\mathcal{H}\theta^\epsilon\theta^\epsilon\right)_x - \theta^\epsilon\Lambda\theta^\epsilon + \epsilon\theta^\epsilon_{xx} \in L^1_T H^{-2}.$$

To pass to the limit into the weak super-solution formulation, we extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{A}_T$ such that

$$\begin{aligned} &\theta^\epsilon \overset{\star}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty_T\left(L^p \cap H^{\frac{1}{2}}\right) \quad \text{for all } p \in (1,\infty), \\ &\theta^\epsilon \rightharpoonup \theta \quad \text{in} \quad L^2_T H^{\frac{1}{2}}, \\ &\theta^\epsilon \to \theta \quad \text{in } L^2_T L^p \text{ for all } 1 < p < \infty, \end{aligned} \tag{3.11}$$

where we use Lemma 2.1 for the strong convergence with

$$X_0 = L^2_T H^{\frac{1}{2}}, \quad X_1 = L^2_T L^p, \quad X_2 = L^1_T H^{-2}.$$

We now multiply (3.10) by a test function $\psi \in \mathcal{C}^\infty_c([0,T) \times \mathbb{R})$ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned} &\int_0^T \int \Big[\theta^\epsilon\psi_t - \underbrace{\left(\mathcal{H}\theta^\epsilon\right)\theta^\epsilon\psi_x}_{\text{I}} + \epsilon\theta^\epsilon\psi_{xx}\Big]\, dxdt - \int \theta^\epsilon_0(x)\psi(0,x)dx \\ &= -\int_0^T \int \underbrace{\Lambda^{\frac{1}{2}}\theta^\epsilon\left[\Lambda^{\frac{1}{2}},\psi\right]\theta^\epsilon}_{\text{II}}\, dxdt - \int_0^T \int \underbrace{\left|\Lambda^{\frac{1}{2}}\theta^\epsilon\right|^2 \psi}_{\text{III}}\, dxdt. \end{aligned} \tag{3.12}$$

We note that we are able to rearrange terms in the usual weak formulation into (3.12) since θ^ϵ is smooth. By the strong convergence in (3.11), we can pass to the limit to I. Moreover, since

$$\left[\Lambda^{\frac{1}{2}},\psi\right]\theta^\epsilon \to \left[\Lambda^{\frac{1}{2}},\psi\right]\theta$$

strongly in $L^2_T L^6$ by Lemma 3.2 and the strong convergence in (3.11), we can pass to the limit to II. Lastly, by Fatou's lemma,

$$\lim_{\epsilon\to 0}\int_0^T \int \left|\Lambda^{\frac{1}{2}}\theta^\epsilon\right|^2 \psi dxdt \ge \int_0^T \int \left|\Lambda^{\frac{1}{2}}\theta\right|^2 \psi dxdt.$$

Combining all the limits together, we obtain that

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t - \left(\mathcal{H}\theta\right)\theta\psi_x + \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}},\psi\right]\theta + \left|\Lambda^{\frac{1}{2}}\theta\right|^2\psi\right] dxdt \ge \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx. \tag{3.13}$$

This completes the proof. □

4. The model 2

We now consider the following equation:

$$\theta_t - \left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta_x + \Lambda^\gamma\theta = 0, \tag{4.1a}$$

$$\theta(0,x) = \theta_0(x), \tag{4.1b}$$

where $\alpha, \gamma > 0$. In this case, we focus on the existence of weak solutions under some conditions of (α, γ). As before, we assume that θ_0 satisfies the following conditions

$$\theta_0 \geq 0, \quad \theta_0 \in L^1 \cap L^\infty. \tag{4.2}$$

Let

$$\mathcal{B}_T = L^\infty_T\left(L^1 \cap L^\infty\right) \cap L^2_T H^{\frac{\gamma}{2}}.$$

Definition 4.1. *We say θ is a weak solution of (4.1) on the time interval $[0,T]$ if $\theta(t,x) \geq 0$ for all $t \in [0,T]$, $\theta \in \mathcal{B}_T$, and for each $\psi \in C^\infty_c([0,T]\times\mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} \left[\theta\psi_t - \left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta\psi_x - \Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta\Lambda^{\frac{\gamma}{2}}(\theta\psi) - \theta\Lambda^\gamma\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx.$$

The third result in the paper is the following.

Theorem 4.2. *Suppose that two positive numbers α and γ satisfy*

$$0 < \gamma < 1, \quad \alpha \geq \frac{1}{2} - \frac{\gamma}{2}. \tag{4.3}$$

Then, for any θ_0 satisfying (4.2), there exists a weak solution of (4.1) in $\mathcal{B}_T$ for all $T > 0$.

Proof. As in the proof of Theorem 3.3, we regularize θ_0 and the equation as

$$\theta^\epsilon_0 = \rho_\epsilon * \theta_0, \quad \theta^\epsilon_t - \left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\right)\theta^\epsilon_x + \Lambda^\gamma\theta^\epsilon = \epsilon\theta^\epsilon_{xx}. \tag{4.4}$$

Then, the corresponding θ^ϵ satisfies

$$\theta^\epsilon(t,x) \geq 0, \quad \|\theta^\epsilon(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} \quad \text{for all time} \tag{4.5}$$

and

$$\|\theta^\epsilon(t)\|_{L^1} + \int_0^t \left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(s)\right\|^2_{L^2} ds \leq \|\theta_0\|_{L^1}. \tag{4.6}$$

We next multiply (4.4) by θ^ϵ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta^\epsilon(t)\|^2_{L^2} + \left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|^2_{L^2} + \epsilon\|\theta^\epsilon_x\|^2_{L^2} = -\frac{1}{2}\int_{\mathbb{R}}\left\{\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\}(\theta^\epsilon(t))^2 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}}\left\{\Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\}\Lambda^{\frac{\gamma}{2}}(\theta^\epsilon(t))^2 dx \\
&\leq C\left\|\Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\|_{L^2}\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}\|\theta^\epsilon(t)\|_{L^\infty} \\
&\leq \frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|^2_{L^2} + C\left\|\Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\|^2_{L^2}\|\theta^\epsilon(t)\|^2_{L^\infty}.
\end{aligned}$$

By (4.3), (4.5) and (4.6), we obtain

$$\|\theta^\epsilon(t)\|^2_{L^2} + \int_0^t \left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(s)\right\|^2_{L^2} ds + \epsilon\int_0^t \|\theta^\epsilon_x(s)\|^2_{L^2} ds \leq C\|\theta_0\|^2_{L^1}\|\theta_0\|^2_{L^\infty}. \tag{4.7}$$

Therefore, (θ_ϵ) is bounded in $\mathcal{B}_T$ uniformly in $\epsilon > 0$.

From this, we have uniform bounds

$$\left\{\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta\right\}_x \in L^2_T L^2, \quad \Lambda^\gamma\theta^\epsilon + \epsilon\theta^\epsilon_{xx} \in L^2_T H^{-2}.$$

Moreover, the condition (4.3) implies that

$$\left(\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon\right)\theta^\epsilon \in L^1_T H^{-1}.$$

Combining all together, we also derive that

$$\theta^\epsilon_t \in L^1_T H^{-2}.$$

We now multiply (4.4) by a test function $\psi \in \mathcal{C}^\infty_c\left([0,T)\times\mathbb{R}\right)$ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}&\int_0^T\int\Big[\theta^\epsilon\psi_t - \underbrace{\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\right)\theta^\epsilon\psi_x}_{\mathrm{I}} + \Lambda^\gamma\theta^\epsilon + \epsilon\theta^\epsilon\psi_{xx}\Big]dxdt - \int\theta^\epsilon_0(x)\psi(0,x)dx\\ &= \int_0^T\int\underbrace{\Lambda^{1-\frac{\gamma}{2}}\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\Lambda^{\frac{\gamma}{2}}(\theta^\epsilon\psi)}_{\mathrm{II}}\,dxdt.\end{aligned}\tag{4.8}$$

To pass the limit to this formulation, we extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{B}_T$ such that

$$\begin{aligned}&\theta^\epsilon \overset{\star}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty_T\left(L^p\cap H^{\frac{1}{2}}\right) \quad \text{for all } p\in(1,\infty),\\ &\theta^\epsilon \rightharpoonup \theta \quad \text{in} \quad L^2_T H^{\frac{\gamma}{2}},\\ &\theta^\epsilon \to \theta \quad \text{in } L^2_T H^{1-\frac{\gamma}{2}-2\alpha}\cap L^2_T L^p \text{ for all } 1<p<\infty,\end{aligned}\tag{4.9}$$

where we use Lemma 2.1 for the strong convergence with the condition (4.3) and

$$X_0 = L^2_T H^{\frac{\gamma}{2}}, \quad X_1 = L^2_T H^{1-\frac{\gamma}{2}-2\alpha}\cap L^2_T L^p, \quad X_2 = L^1_T H^{-2}.$$

By the strong convergence in (4.9), we can pass to the limit to I and II in (4.8). Therefore, we obtain

$$\int_0^T\int_{\mathbb{R}}\left[\theta\psi_t - \left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta\psi_x - \Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta\Lambda^{\frac{\gamma}{2}}(\theta\psi) - \theta\Lambda^\gamma\psi\right]dxdt = \int_{\mathbb{R}}\theta_0(x)\psi(x,0)dx.$$

This completes the proof of Theorem 4.2. □

Remark. Theorem 4.2 improves Theorem 1.4 in [3], where (α,γ) is assumed to satisfy $\alpha \geq \frac{1}{2}-\frac{\gamma}{4}$. The main idea of taking weaker regularization in (4.1) is that the Hilbert transform in front of $(1-\partial_{xx})^{-\alpha}$ gives (4.6) which makes to obtain (4.7). We choose $\alpha > \frac{1}{2}-\frac{\gamma}{2}$ instead of $\alpha \geq \frac{1}{2}-\frac{\gamma}{2}$ to apply compactness argument when we pass to the limit to ϵ-regularized equations.

5. The model 3

In this section, we consider the following equation

$$\theta_t - \left(\mathcal{H}(\partial_{xx})^\beta\theta\right)\theta_x + \Lambda^\gamma\theta = 0, \tag{5.1a}$$

$$\theta(0,x) = \theta_0(x) \tag{5.1b}$$

where $\beta,\gamma > 0$. Depending on the range of β and γ, we will have four different results.

5.1. **Local well-posedness.** We begin with the local well-posedness result.

Theorem 5.1. *Let $0<\gamma<2$ and $0<\beta\leq\frac{\gamma}{4}$. For $\theta_0\in H^2(\mathbb{R})$ there exists $T=T(\|\theta_0\|_{H^2})$ such that a unique solution of (5.8) exists in $C\left([0,T);H^2(\mathbb{R})\right)$. Moreover, we have the following blow-up criterion:*

$$\limsup_{t\nearrow T^*}\|\theta(t)\|_{H^2} = \infty \quad \textit{if and only if} \quad \int_0^{T^*}\Big(\|u_x(s)\|_{L^\infty} + \|\theta_x(s)\|_{L^\infty}\Big)ds = \infty. \tag{5.2}$$

Proof. Let $u=-\mathcal{H}(\partial_{xx})^{\beta}\theta$. Operating ∂_x^l on (5.8), taking its L^2 inner product with $\partial_x^l\theta$, and summing over $l=0,1,2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{H^2}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2&=-\sum_{l=0}^{2}\int\partial_x^l(u\theta_x)\partial_x^l\theta dx\\&=-\sum_{l=0}^{2}\int\left(\partial_x^l(u\theta_x)-u\partial_x^l\theta_x\right)\partial_x^l\theta dx-\sum_{l=0}^{2}\int u\partial_x^l\theta_x\partial_x^l\theta dx=\mathrm{I}_1+\mathrm{I}_2.\end{aligned}\tag{5.3}$$

Using the commutator estimate in [14]

$$\sum_{|l|\leq 2}\left\|D^l(fg)-fD^lg\right\|_{L^2}\leq C\left(\|\nabla f\|_{L^\infty}\|Dg\|_{L^2}+\left\|D^2f\right\|_{L^2}\|g\|_{L^\infty}\right),$$

we have

$$\begin{aligned}\mathrm{I}_1\leq\sum_{l=0}^{2}\left\|\partial_x^l(u\theta_x)-u\partial_x^l\theta_x\right\|_{L^2}\|\theta\|_{H^2}&\leq C\left(\|u_x\|_{L^\infty}\|\theta\|_{H^2}+\|u\|_{H^2}\|\theta_x\|_{L^\infty}\right)\|\theta\|_{H^2}\\&\leq C_\kappa\left(\|u_x\|_{L^\infty}+\|\theta_x\|_{L^\infty}^2\right)\|\theta\|_{H^2}^2+\kappa\|u\|_{H^2}^2.\end{aligned}\tag{5.4}$$

And by integration by parts,

$$\mathrm{I}_2=-\frac{1}{2}\sum_{l=0}^{2}\int u\partial_x\left|\partial_x^l\theta\right|^2dx=\frac{1}{2}\sum_{l=0}^{2}\int u_x\left|\partial_x^l\theta\right|^2dx\leq C\|u_x\|_{L^\infty}\|\theta\|_{H^2}^2.\tag{5.5}$$

Since $\beta\leq\frac{\gamma}{4}$, for a sufficiently small $\kappa>0$

$$\kappa\|u\|_{H^2}^2\leq\frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2.$$

By (5.4) and (5.5), we obtain

$$\frac{d}{dt}\|\theta\|_{H^2}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2\leq C\left(\|u_x\|_{L^\infty}+\|\theta_x\|_{L^\infty}^2\right)\|\theta\|_{H^2}^2\leq C\|\theta\|_{H^2}^3+C\|\theta\|_{H^2}^4,\quad\beta\leq\frac{\gamma}{4}\tag{5.6}$$

from which we deduce that there is $T=T(\|\theta_0\|_{H^2})$ such that

$$\|\theta(t)\|_{H^2}\leq 2\|\theta_0\|_{H^2}\quad\text{for all } t<T.$$

(5.6) also implies (5.2).

To show the uniqueness, let θ_1 and θ_2 be two solutions of (5.8), and let $\theta=\theta_1-\theta_2$ and $u=u_1-u_2$. Then, (θ,u) satisfies the following equations

$$\theta_t+u_1\theta_x-u\theta_{2x}=-\Lambda^\gamma\theta,\quad u=-\mathcal{H}(\partial_{xx})^\beta\theta,\quad\theta(0,x)=0.$$

By taking the L^2 product of the equation with θ,

$$\frac{d}{dt}\|\theta\|_{L^2}^2+2\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2\leq C\left(\|u_{1x}\|_{L^\infty}+\|\theta_{2x}\|_{L^\infty}\right)\|\theta\|_{L^2}^2\leq C\left(\left\|\Lambda^{\frac{\gamma}{2}}\theta_1\right\|_{H^2}+\|\theta_2\|_{H^2}\right)\|\theta\|_{L^2}^2.$$

So, $\theta=0$ in L^2 and thus a solution is unique. This completes the proof of Theorem 5.1. □

Theorem 5.1 provides a local existence result for $\beta\nearrow\frac{1}{2}$ as $\gamma\nearrow 2$. But, we can increase the range of β when we deal with (5.8) directly with $\gamma=2$ because we can do the integration by parts.

Theorem 5.2. *Let $\gamma=2$ and $0<\beta<1$. For $\theta_0\in H^2(\mathbb{R})$ there exists $T=T(\|\theta_0\|_{H^2})$ such that a unique solution of (5.8) exists in $C\left([0,T);H^2(\mathbb{R})\right)$.*

Proof. We begin the L^2 bound:

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 \le \|\theta\|_{L^\infty}\left\|\mathcal{H}(\partial_{xx})^\beta\theta\right\|_{L^2}\|\theta_x\|_{L^2} \le C\|\theta\|_{H^2}^3.$$

We next estimate θ_{xx}. Indeed, after several integration parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^2}^2 + \|\theta\|_{\dot{H}^3}^2 = -\int\left\{\mathcal{H}(\partial_{xx})^\beta\theta_x\right\}\theta_x\theta_{xxx}dx + \frac{1}{2}\int\left\{\mathcal{H}(\partial_{xx})^\beta\theta_x\right\}\theta_{xx}\theta_{xx}dx = \mathrm{I}_1 + \mathrm{I}_2.$$

When $0<\beta<1$,

$$\begin{aligned}|\mathrm{I}_1| &\le \|\theta_x\|_{L^\infty}\left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xxx}\|_{L^2} = \|\theta_x\|_{L^\infty}\left\|\Lambda^{2\beta+1}\theta\right\|_{L^2}\|\theta_{xxx}\|_{L^2}\\ &\le C\|\theta\|_{H^2}\|\theta_x\|_{L^2}^{1-\beta}\|\theta_{xxx}\|_{L^2}^{1+\beta} \le C\|\theta\|_{H^2}^4 + C\|\theta\|_{H^2}^{\frac{4-2\beta}{1-\beta}} + \frac{1}{4}\|\theta_{xxx}\|_{L^2}^2.\end{aligned}$$

And

$$|\mathrm{I}_2| \le \left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xx}\|_{L^4}^2 \le C\left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xx}\|_{L^2}^{\frac{3}{2}}\|\theta_{xxx}\|_{L^2}^{\frac{1}{2}} \le C\|\theta\|_{H^2}^4 + \frac{1}{4}\|\theta_{xxx}\|_{L^2}^2.$$

Therefore, we obtain

$$\frac{d}{dt}\|\theta\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 \le C\|\theta\|_{H^2}^4 + C\|\theta\|_{H^2}^{\frac{4-2\beta}{1-\beta}}. \tag{5.7}$$

This implies that there exists $T = T(\|\theta_0\|_{H^2})$ such that there exists a unique solution of (5.8) in $C\left([0,T);H^2(\mathbb{R})\right)$. □

We may lower the regularity of the initial data to prove a local existence result of a weak solution by considering initial data in $\dot{H}^{\frac{1}{2}}$. The main tools to achieve this will be the use of the Hardy-BMO duality together with interpolation arguments. However, in order to simplify the computation, we consider an equivalent equation by changing the sign of the nonlinearity:

$$\theta_t + \left(\mathcal{H}(-\partial_{xx})^\beta\theta\right)\theta_x + \Lambda^\gamma\theta = 0, \tag{5.8a}$$

$$\theta(0,x) = \theta_0(x) \tag{5.8b}$$

This can be obtained from (5.8) via $\theta\mapsto-\theta$. For this equation, we do $\dot{H}^{\frac{1}{2}}$ estimates and prove that there exists a local existence of a unique solution in that special case.

Theorem 5.3. *Let $\gamma=2$ and $0<\beta<\frac{1}{2}$. For any $\theta_0\in\dot{H}^{\frac{1}{2}}(\mathbb{R})$, there exists $T=T(\|\theta_0\|_{\dot{H}^{\frac{1}{2}}})$ such that there exists a unique local-in-time solution in $C([0,T);\dot{H}^{\frac{1}{2}}(\mathbb{R}))\cap L^2\left([0,T);H^{\frac{3}{2}}(\mathbb{R})\right)$.*

Proof. By recalling that $\Lambda^{2\beta}=(-\partial_{xx})^\beta$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|_{L^2}^2 &= -\int\Lambda^{\frac{1}{2}}\theta\Lambda^{\frac{1}{2}}\left\{\left(\mathcal{H}(-\partial_{xx})^\beta\theta\right)\theta_x\right\}dx\\ &= -\int\theta_x\Lambda\theta\ \mathcal{H}(-\partial_{xx})^\beta\theta dx = -\int\theta_x\mathcal{H}\theta_x\mathcal{H}(-\partial_{xx})^\beta\theta dx.\end{aligned}$$

We now use the $\mathcal{H}^1$-BMO duality to estimate the right hand side of the last equality. By using the estimate (2.1) and $\dot{H}^{\frac{1}{2}}\hookrightarrow BMO$, we obtain

$$\|\theta_x\mathcal{H}\theta_x\|_{\mathcal{H}^1} \le \|\theta\|_{\dot{H}^1}^2, \quad \left\|\mathcal{H}(-\partial_{xx})^\beta\theta\right\|_{L^2} \le C\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}}$$

and thus we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|_{L^2}^2 \le C\|\theta\|_{\dot{H}^1}^2\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}}.$$

By fixing $\gamma = 2$ and by using the interpolation inequalities

$$\|\theta\|_{\dot{H}^1}^2 \leq \|\theta\|_{\dot{H}^{\frac{3}{2}}}\|\theta\|_{\dot{H}^{\frac{1}{2}}}, \quad \|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}} \leq \|\theta\|_{\dot{H}^{\frac{3}{2}}}^{2\beta}\|\theta\|_{\dot{H}^{\frac{1}{2}}}^{1-2\beta},$$

where we use $\frac{1}{2} \leq 2\beta + \frac{1}{2} \leq \frac{3}{2}$ for $\beta \in \left(0, \frac{1}{2}\right)$ to get the second inequality. Hence, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\Lambda^{\frac{3}{2}}\theta\right\|_{L^2}^2 &\leq \|\theta\|_{\dot{H}^1}^2\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}} \\ &\leq \|\theta\|_{\dot{H}^{\frac{3}{2}}}^{1+2\beta}\|\theta\|_{\dot{H}^{\frac{1}{2}}}^{2-2\beta} \leq \frac{1}{2}\|\theta\|_{\dot{H}^{\frac{3}{2}}}^2 + 2\|\theta\|_{\dot{H}^{\frac{1}{2}}}^{4\frac{1-\beta}{1-2\beta}},\end{aligned}$$

where we use the condition $\beta \in \left(0, \frac{1}{2}\right)$ again to derive the inequality. This implies local existence of a unique solution up to some time $T = T(\|\theta_0\|_{\dot{H}^{\frac{1}{2}}})$. ☐

5.2. **Global well-posedness.** We finally deal with (5.8) with $\gamma = 2$.

Theorem 5.4. *Let $\gamma = 2$ and $\beta < \frac{1}{4}$. For any $\theta_0 \in H^2(\mathbb{R})$, there exists a unique global-in-time solution in $C\left([0, \infty); H^2(\mathbb{R})\right)$.*

Proof. By Theorem 5.1, we only need to control the quantities in (5.2). Let $u = -\mathcal{H}(\partial_{xx})^\beta\theta$. We first note that (5.8) satisfies the maximum principle and so

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} \leq C\|\theta_0\|_{H^2}.$$

We take the L^2 inner product of (5.8) with θ. Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 = -\int u\theta_x\theta dx \leq \|\theta_0\|_{L^\infty}\|u\|_{L^2}\|\theta_x\|_{L^2}. \tag{5.9}$$

Since

$$\|u\|_{L^2} \leq C\|\theta\|_{L^2}^{1-2\beta}\|\theta_x\|_{L^2}^{2\beta} \quad \text{for } \beta < \frac{1}{2},$$

we have

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\theta_x(s)\|_{L^2}^2 ds \leq C\left(t, \|\theta_0\|_{H^2}\right). \tag{5.10}$$

We next take ∂_x to (5.8), take its L^2 inner product with θ_x, and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta_x\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2 = \int u\theta_x\theta_{xx}dx \leq 2\|u\|_{L^\infty}^2\|\theta_x\|_{L^2}^2 + \frac{1}{2}\|\theta_{xx}\|_{L^2}^2.$$

Since

$$\|u\|_{L^\infty}^2 \leq C\|\theta\|_{L^2}^2 + C\|\theta_x\|_{L^2}^2 \quad \text{when } \beta < \frac{1}{4},$$

we obtain

$$\|\theta_x(t)\|_{L^2}^2 + \int_0^t \|\theta_{xx}(s)\|_{L^2}^2 ds \leq C\left(t, \|\theta_0\|_{L^1}, \|\theta_0\|_{H^2}\right) \quad \text{when } \beta < \frac{1}{4}. \tag{5.11}$$

By (5.10) and (5.11), we finally obtain

$$\int_0^t \left(\|\theta_x(s)\|_{L^\infty} + \|u_x(s)\|_{L^\infty}\right) ds \leq C\int_0^t \left(\|\theta_x(s)\|_{L^2} + \|\theta_{xx}(s)\|_{L^2}\right) ds \leq C\left(t, \|\theta_0\|_{L^1}, \|\theta_0\|_{H^2}\right)$$

and so we complete the proof of Theorem 5.4. ☐

6. Appendix

This appendix is briefly written based on [4]. We first provide notation and definitions in the Littlewood-Paley theory. Let $\mathcal{C}$ be the ring of center 0, of small radius $\frac{3}{4}$ and great radius $\frac{8}{3}$. We take smooth radial functions (χ, ϕ) with values in $[0,1]$ that are supported on the ball $B_{\frac{4}{3}}(0)$ and $\mathcal{C}$, respectively, and satisfy

$$
\begin{aligned}
&\chi(\xi)+\sum_{j=0}^{\infty}\phi\left(2^{-j}\xi\right)=1 \quad \forall\, \xi \in \mathbb{R}^d,\\
&\sum_{j=-\infty}^{\infty}\phi\left(2^{-j}\xi\right)=1 \quad \forall\, \xi \in \mathbb{R}^d\setminus\{0\},\\
&\left|j-j^{'}\right|\geq 2 \implies \text{supp } \phi\left(2^{-j}\cdot\right)\bigcap \text{supp } \phi\left(2^{-j^{'}}\cdot\right)=\emptyset,\\
&j\geq 1 \implies \text{supp } \chi\bigcap \text{supp } \phi\left(2^{-j}\cdot\right)=\emptyset.
\end{aligned}
\tag{6.1}
$$

From now on, we use the notation

$$\phi_j(\xi)=\phi\left(2^{-j}\xi\right).$$

We define dyadic blocks and lower frequency cut-off functions.

$$
\begin{aligned}
&h=\mathcal{F}^{-1}\phi, \quad \widetilde{h}=\mathcal{F}^{-1}\chi,\\
&\Delta_j f=\phi_j\left(D\right)f=2^{jd}\int_{\mathbb{R}^d}h\left(2^j y\right)f(x-y)dy,\\
&S_j f=\chi\left(2^{-j}D\right)f=2^{jd}\int_{\mathbb{R}^d}\widetilde{h}\left(2^j y\right)f(x-y)dy,\\
&\Delta_{-1}f=\chi\left(D\right)f=\int_{\mathbb{R}^d}\widetilde{h}\left(y\right)f(x-y)dy.
\end{aligned}
\tag{6.2}
$$

Then, the homogeneous Littlewood-Paley decomposition is given by

$$f=\sum_{j\in\mathbb{Z}}\Delta_j f \quad \text{in} \quad \mathcal{S}_h^{'},$$

where $\mathcal{S}_h^{'}$ is the space of tempered distributions $u\in\mathcal{S}^{'}$ such that

$$\lim_{j\to-\infty}S_j u=0 \quad \text{in } \mathcal{S}'.$$

We now define the homogeneous Besov spaces:

$$\dot{B}^s_{p,q}=\left\{f\in\mathcal{S}_h^{'}:\ \|f\|_{\dot{B}^s_{p,q}}=\left\|\left\|2^{js}\left\|\Delta_j f\right\|_{L^p}\right\|_{l^q(\mathbb{Z})}\right\|<\infty\right\}.$$

We also recall Bernstein's inequality in 1D : for $1\leq p\leq q\leq\infty$ and $k\in\mathbb{N}$,

$$\sup_{|\alpha|=k}\left\|\partial^\alpha\Delta_j f\right\|_{L^p}\leq C2^{jk}\left\|\Delta_j f\right\|_{L^p}, \quad \left\|\Delta_j f\right\|_{L^q}\leq C2^{j\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\Delta_j f\right\|_{L^p}. \tag{6.3}$$

Acknowledgments

H.B. was supported by the National Research Foundation of Korea (NRF-2015R1D1A1A01058892).

R.G.B. was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR), and by the Universidad de Cantabria.

O.L. was partially supported by the Marie-Curie Grant, acronym: TRANSIC, from the FP7-IEF program and by the ERC through the Starting Grant project H2020-EU.1.1.-63922.

Both O. L. and R.G.B. were partially supported by the Grant MTM2014-59488-P from the former Ministerio de Economía y Competitividad (MINECO, Spain).

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