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Infinite Horizon Optimal Control for a General Class of Semilinear Parabolic Equations

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Abstract

A class of infinite horizon optimal control problems subject to semilinear parabolic equations is investigated. First and second order optimality conditions are obtained, in the presence of constraints on the controls, which can be either pointwise in space-time, or pointwise in time and L^2 in space. These results rely on a new L^∞ estimate for nonlinear parabolic equations in an essential manner.

Keywords Infinite horizon optimal control · Semilinear parabolic equations · First and second order optimality conditions · L^∞ regularity

Mathematics Subject Classification 35K58 · 49J20 · 49J52 · 49K20

1 Introduction

We study the optimal control problem

$$\text{(P)} \qquad \min_{u \in U_{ad}} J(u) = \frac{1}{2}\int_0^\infty \int_\Omega (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int_0^\infty \int_\omega u^2 \,\mathrm{d}x\,\mathrm{d}t,$$

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where $\nu > 0$, $y_d \in L^2(\Omega \times (0,\infty)) \cap L^p(0,\infty; L^2(\Omega))$ with $p \in (\frac{4}{4-n}, \infty]$, and

$$U_{ad} = \{u \in L^2(0,\infty; L^2(\omega)) : u(t) \in K_{ad} \text{ for a.a. } t \in (0,\infty)\}.$$

Above K_{ad} denotes a closed, convex, and bounded set in $L^2(\omega)$, and y_u is the solution of the following parabolic equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g + u\chi_\omega \text{ in } Q = \Omega \times (0,\infty), \\ \partial_n y = 0 \text{ on } \Sigma = \Gamma \times (0,\infty), \ \ y(0) = y_0 \text{ in } \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $1 \le n \le 3$, with a Lipschitz boundary Γ, and Ω is an interval if $n = 1$, ω is a measurable subset of Ω with positive Lebesgue measure, χ_ω denotes the characteristic function of ω, $a \in L^\infty(\Omega)$, $0 \le a \not\equiv 0$, $g \in L^2(Q)$, and additionally $g \in L^p(0,\infty; L^2(\Omega))$ with $p \in (\frac{4}{4-n}, \infty]$ if $n = 2$ or 3, and $y_0 \in L^\infty(\Omega)$. For every $u \in U_{ad}$, the symbol $u\chi_\omega$ is defined as follows:

$$(u\chi_\omega)(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in Q_\omega = \omega \times (0,\infty), \\ 0 & \text{otherwise.} \end{cases}$$

Possible choices for K_{ad} include

$$K_{ad} = B_\gamma = \{v \in L^2(\omega) : \|v\|_{L^2(\omega)} \le \gamma\}, \ \ 0 < \gamma < \infty, \tag{1.2}$$

$$K_{ad} = \{v \in L^2(\omega) : \alpha \le v(x) \le \beta \text{ for a.a. } x \in \omega\}, \ \ -\infty < \alpha < \beta < \infty. \tag{1.3}$$

Concerning the nonlinearity $f : Q \times \mathbb{R} \to \mathbb{R}$ we assume that it is a Carathéodory function of class C^1 with respect to the last variable satisfying the following properties:

$$f(x,t,0) = 0, \tag{1.4}$$

$$\exists M_f \ge 0 \text{ such that } \frac{\partial f}{\partial y}(x,t,y) \ge 0 \text{ and } f(x,t,y)y \ge 0 \ \forall |y| \ge M_f, \tag{1.5}$$

$$\forall M > 0 \ \exists C_M \text{ such that } \left|\frac{\partial f}{\partial y}(x,t,y)\right| \le C_M \ \forall |y| \le M, \tag{1.6}$$

for almost all $(x,t) \in Q$. Let us observe that (1.5) and (1.6) imply

$$\frac{\partial f}{\partial y}(x,t,y) \ge -C_{M_f} \ \forall y \in \mathbb{R} \text{ and for a.a. } (x,t) \in Q. \tag{1.7}$$

Moreover, (1.4) and (1.6) along with the mean value theorem yield

$$|f(x,t,y)| = \left|\frac{\partial f}{\partial y}(x,t,\theta(x,t)y)y\right| \le C_M M \ \forall |y| \le M \text{ and for a.a. } (x,t) \in Q. \tag{1.8}$$

The following generalized Poincaré inequality will frequently be used

$$C_a\|y\|_{H^1(\Omega)} \le \left(\int_\Omega [|\nabla y|^2 + ay^2]\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.9}$$

All along this paper we will assume that

$$p \in \left(\frac{4}{4-n}, \infty\right] \text{ if } n = 2 \text{ or } 3 \text{ and } p \in [2, \infty] \text{ if } n = 1. \tag{1.10}$$

Remark 1.1 The operator $-\Delta$ can be replaced by any uniformly elliptic operator with $L^\infty(\Omega)$ coefficients. The assumption (1.4) can be relaxed by assuming that $f(\cdot, \cdot, 0) \in L^2(Q) \cap L^\infty(0, \infty; L^2(\Omega))$ and then redefining f and g as $f(x, t, y) - f(x, t, 0)$ and $g(x, t) - f(x, t, 0)$, respectively.

By investigating (P) we continue our efforts on studying infinite horizon optimal control problems with semilinear parabolic equations as constraints. In [8] the nonlinearities were chosen of polynomial type, no constraints were enforced on the controls, and the focus was put on nonsmooth, sparsity enhancing control costs, which entail that the controls settle down at zero once the states enter into a neighborhood of a stable equilibrium. Later, in [9] the nonlinearity was not restricted to be a polynomial and the conditions on f were very similar to those imposed in the present paper. The same type of control constraints were imposed as well. The major step forward in the current paper compared to [9] consists in an $L^\infty(Q)$ estimate of the states for feasible controls, i.e. for controls with the property that the associated states y_u are in $L^2(Q)$. Utilizing this property, well-posedness and C^2 regularity of the control-to-state mapping, associating the infinite horizon controls to the infinite horizon states, can be guaranteed, and a second order analysis of (P) becomes possible. This was not the case in [9], where the first order conditions of the infinite horizon problem were obtained as the limit of the associated finite horizon problems, and no second order analysis was carried out. The authors are not aware about the availability of the second order analysis for optimal control problems with constraints as in (1.2) even in the finite horizon case. Along a related, but different line of research we also investigated infinite horizon optimal control problems with a discount factor on the state, [11] and [12]. This allows to treat a larger class of nonlinearities at the expense of less information of the optimal states as time increases.

Most of the literature on infinite horizon problems is carried out for ordinary differential equations. Let us mention some of these contributions. In [7] the importance of infinite horizon problems in applications is stressed. In general, when formulating optimal control problems, the time horizon can be subject to ambiguity. In such cases the choice as infinite horizon problem can be a valuable choice. The first article, focusing on infinite horizon problems may be [15]. More recent contributions all in the context of ordinary differential equations are available for instance in [1, 2, 4]. Concerning the literature, pointwise constraints as in (1.3) have received considerably more attention than norm constraints as in (1.2). However, from a practical point of view (1.2) appears to be equally important. In the case of optimal control of Navier–Stokes equations the suitability of this type of constraints was discussed in [14]. The use of the $L^1(\omega)$ norm replacing the $L^2(\omega)$ was studied in [10]. The last two references were devoted to final horizon control problems.

Briefly, the paper is structured in the following way. In Sect. 2, the existence of optimal controls and first order optimality conditions are established. Necessary and sufficient second order conditions for the two choices of K_{ad} in (1.2) and (1.3) are obtained in Sect. 3. Section 4 is devoted to convergence results for the finite horizon problems associated to (P), to the infinite horizon problem. This is not only of intrinsic interest but also of relevance for numerical realization. In the Appendix the relevant results for the state equation, and the associated linearized and adjoint equations are established. The $L^\infty(Q)$ regularity result for the state equation, already mentioned above, may be of interest beyond its application in optimal control.

2 Existence of an optimal control and first order optimality conditions

In this section, we prove the existence of an optimal solution of (P) and derive the first order optimality conditions satisfied by any local minimizer. For this purpose we will also address the issue of differentiability of the relation control-to-state and of the cost functional J. Let us observe that Theorem A.2 implies the existence of a unique state y_u for every control $u \in U_{ad}$. However, it could happen that $y_u \notin L^2(Q)$ and, consequently, $J(u) = \infty$. Therefore, the assumption about the existence of a control $u_0 \in U_{ad}$ such that $J(u_0) < \infty$ is needed. This issue will not be addressed in this paper, the reader is referred, for instance, to [3] and [8] for this question. We will say that u is a feasible control if $u \in U_{ad}$ and $J(u) < \infty$.

For $0 < T \le \infty$ we set $W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2(0,T;H^1(\Omega)^*)\}$ with $\|y\|_{W(0,T)} = \left(\|y\|^2_{L^2(0,T;H^1(\Omega))} + \left\|\frac{\partial y}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega)^*)}\right)^{\frac{1}{2}}$ as norm. It is well known that $(W(0,T), \|\cdot\|_{W(0,T)})$ is a Banach space. In fact, it is a Hilbert space because $\|\cdot\|_{W(0,T)}$ is a Hilbertian norm. Furthermore, the embedding $W(0,T) \subset C([0,T];L^2(\Omega))$ is continuous for $T \le \infty$ and $W(0,T)$ is compactly embedded in $L^2(0,T;L^2(\Omega))$ if $T < \infty$.

Theorem 2.1 *Let us assume that there exists a feasible control u_0. Then, (P) has at least one solution.*

Proof Let $\{u_k\}_{k=1}^\infty \subset U_{ad}$ be a minimizing sequence of feasible controls with associated states $\{y_{u_k}\}_{k=1}^\infty$. Since $J(u_k) \to \inf\,(\mathrm{P}) < \infty$, then the boundedness of $\{u_k\}_{k=1}^\infty$ and $\{y_{u_k}\}_{k=1}^\infty$ in $L^2(Q_\omega)$ and $L^2(Q)$, respectively, follows. Then, taking subsequences we can assume that $(u_k, y_{u_k}) \rightharpoonup (\bar u, \bar y)$ in $L^2(Q_\omega) \times L^2(Q)$. Since U_{ad} is a closed and convex subset of $L^2(Q_\omega)$, we infer that $\bar u \in U_{ad}$. Due to the weak lower semicontinuity of J with respect to (y,u) in $L^2(Q) \times L^2(Q_\omega)$, it is enough to establish that $\bar y$ is the state associated to $\bar u$ to conclude the proof. For this purpose we have to show that $\bar y$ satisfies (A.2) with $g + \chi_\omega \bar u$ on the right hand side for every $T < \infty$. The only delicate point in this respect is to prove the convergence of $f(x,t,y_{u_k}) \to f(x,t,\bar y)$ in $L^2(Q_T)$ for every $T > 0$, where $Q_T = \Omega \times (0,T)$. Using the boundedness of $\{(u_k, y_{u_k})\}_{k=1}^\infty$ in

$[L^2(Q_\omega)\cap L^\infty(0,\infty; L^2(\omega))]\times L^2(Q)$ we deduce from (A.4)–(A.6) the boundedness of $\{y_{u_k}\}_{k=1}^\infty$ in $W(0,\infty)\cap L^\infty(Q)$ and $\{f(\cdot,\cdot,y_{u_k})\}_{k=1}^\infty$ in $L^\infty(Q)\cap L^2(Q)$. Hence, using the compactness of the embedding $W(0,T)\subset L^2(Q_T)$ the desired convergence follows. ◻

Hereafter, the following additional hypothesis on f is assumed:

$$\begin{cases} \exists m_f>0,\ \exists\delta_f\in[0,1),\ \text{and } \exists C_f>0 \text{ such that}\\ \dfrac{\partial f}{\partial y}(x,t,s)\geq -C_f|s|-\delta_f a(x,t)\ \forall |s|\leq m_f \text{ and for a.a. } (x,t)\in Q. \end{cases} \tag{2.1}$$

Let us denote for every p satisfying (1.10)

$$\mathcal{U}_p=\{u\in L^2(Q_\omega)\cap L^p(0,\infty;L^2(\omega)) \text{ such that } y_u\in L^2(Q)\},$$
$$Y_p=\{y\in W(0,\infty)\cap L^\infty(Q):\frac{\partial y}{\partial t}-\Delta y+ay\in L^2(Q)\cap L^p(0,\infty;L^2(\Omega))\},$$

and by $G_p:\mathcal{U}_p\longrightarrow Y$ the mapping $G_p(u)=y_u$, where y_u is the solution of (1.1). Y_p is a Banach space when endowed with the associated graph norm. We observe that $\mathcal{U}_\infty\subset\mathcal{U}_p$ and G_∞ is the restriction of G_p to $\mathcal{U}_\infty$.

Theorem 2.2 *Let us assume that $\mathcal{U}_p$ is not empty. Then, $\mathcal{U}_p$ is an open subset of $L^2(Q_\omega)\cap L^p(0,\infty;L^2(\omega))$ and the mapping G_p is of class C^1. Moreover, given $u\in\mathcal{U}_p$ and $v\in L^2(Q_\omega)\cap L^p(0,\infty;L^2(\omega))$, $z_v=DG_p(u)v$ is the unique solution of*

$$\begin{cases} \dfrac{\partial z}{\partial t}-\Delta z+az+\dfrac{\partial f}{\partial y}(x,t,y_u)z=v\chi_\omega \text{ in } Q,\\ \partial_n z=0 \text{ on } \Sigma,\ z(0)=0 \text{ in } \Omega. \end{cases} \tag{2.2}$$

Proof The proof will be based on the implicit function theorem. For this purpose we define the mapping

$$\mathcal{F}_p: Y_p\times L^2(Q_\omega)\cap L^p(0,\infty;L^2(\omega))\longrightarrow L^2(Q)\cap L^p(0,\infty;L^2(\Omega))\times L^\infty(\Omega)$$
$$\mathcal{F}_p(y,u)=\Big(\frac{\partial y}{\partial t}-\Delta y+ay+f(\cdot,\cdot,y)-g-\chi_\omega u,\ y(0)-y_0\Big).$$

By definition of Y_p and using (1.8), we deduce that $\mathcal{F}_p$ is well defined and is of class C^1. Further, we have that $\mathcal{F}_p(y_u,u)=(0,0)$ for every $u\in\mathcal{U}_p$ and

$$\frac{\partial\mathcal{F}_p}{\partial y}(y_u,u): Y_p\longrightarrow L^2(Q)\cap L^p(0,\infty;L^2(\Omega))\times L^\infty(\Omega)$$
$$\frac{\partial\mathcal{F}_p}{\partial y}(y,u)z=\Big(\frac{\partial z}{\partial t}-\Delta z+az+\frac{\partial f}{\partial y}(\cdot,\cdot,y_u)z,\ z(0)\Big).$$

Then, $\frac{\partial \mathcal{F}_p}{\partial y}(y_u, u)$ is an isomorphism if and only if the equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \dfrac{\partial f}{\partial y}(x, t, y_u)z = h \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma, \ z(0) = z_0 \text{ in } \Omega \end{cases} \tag{2.3}$$

has a unique solution in Y_p for every $(h, z_0) \in L^2(Q) \cap L^p(0, \infty; L^2(\Omega)) \times L^\infty(\Omega)$ with continuous dependence. This is an immediate consequence of Theorem A.3 with $d(x, t, s) = \frac{\partial f}{\partial y}(x, t, s)$ and $y = y_u \in L^\infty(Q)$. Finally, the theorem follows by applying the implicit function theorem. □

As a consequence of the above theorem, we have that $J : \mathcal{U}_p \longrightarrow \mathbb{R}$ is well defined. The next theorem establishes its differentiability.

Theorem 2.3 *Assuming that $\mathcal{U}_p$ is not empty, the functional J is of class C^1 and for every $u \in \mathcal{U}_p$ and $v \in L^2(Q_\omega) \cap L^p(0, \infty; L^2(\omega))$ its derivative is given by*

$$J'(u)v = \int_Q (y_u - y_d)z_{u,v} \, \mathrm{d}x \, \mathrm{d}t + \nu \int_{Q_\omega} uv \, \mathrm{d}x \, \mathrm{d}t = \int_{Q_\omega} (\varphi_u + \nu u)v \, \mathrm{d}x \, \mathrm{d}t, \tag{2.4}$$

where $z_{u,v} = G_p'(u)v$ and $\varphi_u \in W(0, \infty) \cap L^\infty(Q)$ satisfies

$$\begin{cases} -\dfrac{\partial \varphi_u}{\partial t} - \Delta \varphi_u + a\varphi_u + \dfrac{\partial f}{\partial y}(x, t, y_u)\varphi_u = y_u - y_d \textit{ in } Q, \\ \partial_n \varphi_u = 0 \textit{ on } \Sigma, \ \lim_{t\to\infty} \|\varphi_u(t)\|_{L^2(\Omega)} = 0. \end{cases} \tag{2.5}$$

The fact that J is of class C^1 is an immediate consequence of Theorem 2.2 and the chain rule. Formula (2.4) is deduced in the standard way from Eqs. (2.2) and (2.5). Concerning the well posedness of (2.5) we refer to Theorem A.4.

We conclude this section establishing the first order optimality conditions satisfied by every local minimizer of (P) and deducing some consequences from them. In this paper, a local minimizer $\bar{u}$ is understood in the $L^2(Q_\omega)$ sense and it is assumed that $\bar{u} \in \mathcal{U}_\infty \cap U_{ad}$.

Theorem 2.4 *Let $\bar{u}$ be a local minimizer of (P). Then, there exist $\bar{y}, \bar{\varphi} \in W(0, \infty) \cap L^\infty(Q)$ such that*

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} - \Delta \bar{y} + a\bar{y} + f(x, t, \bar{y}) = g + \bar{u}\chi_\omega \textit{ in } Q, \\ \partial_n \bar{y} = 0 \textit{ on } \Sigma, \ \bar{y}(0) = y_0 \textit{ in } \Omega, \end{cases} \tag{2.6}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} + a\bar{\varphi} + \dfrac{\partial f}{\partial y}(x, t, \bar{y})\bar{\varphi} = \bar{y} - y_d \textit{ in } Q, \\ \partial_n \bar{\varphi} = 0 \textit{ on } \Sigma, \ \lim_{t\to\infty} \|\bar{\varphi}(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{2.7}$$

$$\int_{Q_\omega} (\bar{\varphi} + \nu \bar{u})(u - \bar{u})\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall u \in U_{ad}. \tag{2.8}$$

This theorem is an immediate consequence of Theorem 2.3 and the inequality $J'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in U_{ad}$.

Corollary 2.1 *Let $\bar{\varphi}$ and $\bar{u}$ satisfy (2.7) and (2.8). If K_{ad} is given by (1.2), then the following properties hold for almost all $t \in (0, \infty)$*

$$\int_{\omega} (\bar{\varphi}(t) + \nu \bar{u}(t))(v - \bar{u}(t))\,\mathrm{d}x \geq 0 \quad \forall v \in B_\gamma, \tag{2.9}$$

$$\text{if } \|\bar{u}(t)\|_{L^2(\omega)} < \gamma \Rightarrow \bar{\varphi}(t) + \nu \bar{u}(t) = 0 \text{ in } \omega, \tag{2.10}$$

$$\text{if } \|\bar{u}(t)\|_{L^2(\omega)} = \gamma \Rightarrow \bar{u}(x,t) = -\gamma \frac{\bar{\varphi}(x,t)}{\|\bar{\varphi}(t)\|_{L^2(\omega)}}, \tag{2.11}$$

$$\|\bar{u}\|_{L^\infty(Q_\omega)} \leq \frac{1}{\nu}\|\bar{\varphi}\|_{L^\infty(Q_\omega)}. \tag{2.12}$$

In the case that K_{ad} is given by (1.3), then we have

$$\bar{u}(x,t) = \mathrm{Proj}_{[\alpha,\beta]}\Big(-\frac{1}{\nu}\bar{\varphi}(x,t)\Big). \tag{2.13}$$

In both cases we have that $\bar{u} \in L^\infty(Q_\omega)$.

Proof For the proof of (2.9) and (2.10) the reader is referred to [9, Lemma 3.2]. Let us prove (2.11). First, we assume that $\|\bar{\varphi}(t) + \nu \bar{u}(t)\|_{L^2(\omega)} \neq 0$. Then, again from [9, Lemma 3.2] we obtain

$$\bar{u}(x,t) = -\gamma \frac{\bar{\varphi}(x,t) + \nu \bar{u}(x,t)}{\|\bar{\varphi}(t) + \nu \bar{u}(t)\|_{L^2(\omega)}} \text{ for a.a. } x \in \omega.$$

This yields

$$\Big(\nu\gamma + \|\bar{\varphi}(t) + \nu \bar{u}(t)\|_{L^2(\omega)}\Big)\bar{u}(x,t) = -\gamma \bar{\varphi}(x,t) \text{ for a.a. } x \in \omega. \tag{2.14}$$

Taking the norm in $L^2(\omega)$ in the above expression and using that $\|\bar{u}(t)\|_{L^2(\omega)} = \gamma$ we infer

$$\nu\gamma + \|\bar{\varphi}(t) + \nu \bar{u}(t)\|_{L^2(\omega)} = \|\bar{\varphi}(t)\|_{L^2(\omega)}. \tag{2.15}$$

Identities (2.14) and (2.15) imply (2.11). In the case $\|\bar{\varphi}(t) + \nu \bar{u}(t)\|_{L^2(\omega)} = 0$ and $\|\bar{u}(t)\|_{L^2(\omega)} = \gamma$ we have that

$$\bar{u}(x,t) = -\frac{1}{\nu}\bar{\varphi}(x,t) \text{ for a.a. } x \in \omega \text{ and } \nu\gamma = \|\bar{\varphi}(t)\|_{L^2(\omega)}. \tag{2.16}$$

Therefore, (2.11) also holds. Let us prove (2.12). If $\|\bar{u}(t)\|_{L^2(\omega)} < \gamma$, then (2.10) implies that $|\bar{u}(x,t)| = \frac{1}{\nu}|\bar{\varphi}(x,t)| \le \frac{1}{\nu}\|\bar{\varphi}\|_{L^\infty(Q_\omega)}$. If $\|\bar{u}(t)\|_{L^2(\omega)} = \gamma$, the inequality $\|\bar{\varphi}(t)\|_{L^2(\omega)} \ge \gamma\nu$ follows from (2.15) and (2.16). Then, (2.11) implies that $|\bar{u}(x,t)| \le \frac{1}{\nu}\|\bar{\varphi}\|_{L^\infty(Q_\omega)}$.

Finally, the identity (2.13) is well known.

□

3 Second order optimality conditions

In this section we address the second order optimality conditions for (P). For this purpose, in addition to assumptions (1.4)–(1.7) we impose the following hypotheses: $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 with respect to the second variable and satisfies

$$\exists\delta_f \in [0,1) \text{ such that } \frac{\partial f}{\partial y}(x,t,0) \ge -\delta_f a(x,t), \tag{3.1}$$

$$\forall M > 0\, \exists C_M \text{ such that } \left|\frac{\partial^2 f}{\partial y^2}(x,t,y)\right| \le C_M\ \forall |y| \le M, \tag{3.2}$$

$$\begin{cases} \forall\varepsilon > 0 \text{ and } \forall M > 0\ \exists\rho_{\varepsilon,M} \text{ such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x,t,y_2) - \dfrac{\partial^2 f}{\partial y^2}(x,t,y_1)\right| \le \varepsilon\ \forall |y_1|, |y_2| \le M \text{ with } |y_2 - y_1| \le \rho_{\varepsilon,M}, \end{cases} \tag{3.3}$$

for almost all $(x,t) \in Q$. We observe that (3.1) and (3.2) imply (2.1). Indeed, it is enough to select

$$m_f = 1 \text{ and } C_f = \max_{|s|\le 1}\left|\frac{\partial^2 f}{\partial y^2}(x,t,y)\right|.$$

Then, using the mean value theorem we infer for almost all $(x,t) \in Q$

$$\frac{\partial f}{\partial y}(x,t,s) = \frac{\partial^2 f}{\partial y^2}(x,t,\theta(x,t)s)s + \frac{\partial f}{\partial y}(x,t,0) \ge -C_f|s| - \delta_f a(x,t)\ \forall |s| \le m_f.$$

Theorem 3.1 *Under assumptions* (1.4)–(1.7) *and* (3.1)–(3.2) *and supposing that* $\mathcal{U}_p$ *is not empty,* $G_p : \mathcal{U}_p \longrightarrow Y_p$ *is of class* C^2*. Moreover, given* $u \in \mathcal{U}_p$ *and* $v_1, v_2 \in L^2(Q_\omega) \cap L^p(0,\infty; L^2(\omega))$*, then* $z_{v_1,v_2} = G_p''(u)(v_1,v_2)$ *is the solution of the equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + \dfrac{\partial f}{\partial y}(x,t,y_u)z = -\dfrac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega, \end{cases} \tag{3.4}$$

where $z_{v_i} = G_p'(u)v_i$ *for* $i = 1, 2$.

The C^2 differentiability of G follows from the implicit function theorem applied to the mapping $\mathcal{F}_p$ introduced in the proof of Theorem 2.2. It is enough to observe that now $\mathcal{F}_p$ is of class C^2. The Eq. (3.4) follows differentiating the identity $\mathcal{F}_p(G_p(u), u) = 0$ twice.

As a consequence of Theorem 3.1 and the chain rule we have the following corollary.

Corollary 3.1 *If $\mathcal{U}_p$ is not empty, then the function $J : \mathcal{U}_p \longrightarrow \mathbb{R}$ is of class C^2 and we have*

$$J''(u)(v_1, v_2) = \int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x, t, y_u)\varphi_u\Big] z_{v_1} z_{v_2}\,\mathrm{d}x\,\mathrm{d}t + \nu \int_{Q_\omega} v_1 v_2\,\mathrm{d}x\,\mathrm{d}t \tag{3.5}$$

for every $u \in \mathcal{U}$ and $v_1, v_2 \in L^2(Q_\omega) \cap L^p(0, \infty; L^2(\omega))$.

Remark 3.1 Under assumptions (1.4)–(1.7) and (3.1)–(3.2), for every $u \in \mathcal{U}_p$ the linear form $J'(u) : L^2(Q_\omega) \cap L^p(0, \infty; L^2(\omega)) \longrightarrow \mathbb{R}$ as well as the bilinear form $J''(u) : [L^2(Q_\omega) \cap L^p(0, \infty; L^2(\omega))]^2 \longrightarrow \mathbb{R}$ can be extended to continuous linear and bilinear forms $J'(u) : L^2(Q_\omega) \longrightarrow \mathbb{R}$ and $J''(u) : L^2(Q_\omega)^2 \longrightarrow \mathbb{R}$ given by the same expressions (2.4) and (3.5), respectively. Indeed, this is an immediate consequence of Theorem A.3 along with the $L^\infty(Q) \cap L^2(Q)$ regularity of the adjoint states established in Theorem A.4.

The analysis of second order optimality conditions is carried out in the next two subsections, where we consider the cases with K_{ad} given by (1.2) or (1.3).

3.1 Case I: $K_{ad} = B_\gamma = \{v \in L^2(\omega) : \|v\|_{L^2(\omega)} \le \gamma\}$.

For this case we consider the Lagrange function

$$\mathcal{L} : \mathcal{U}_p \times L^\infty(0, \infty) \longrightarrow \mathbb{R}, \ \ \mathcal{L}(u, \lambda) = J(u) + \frac{1}{2\gamma} \int_0^\infty \lambda(t) \|u(t)\|^2_{L^2(\omega)}\,\mathrm{d}t.$$

Theorem 2.3 and Corollary 3.1 imply that $\mathcal{L}$ is of class C^2 and we have the expressions

$$\frac{\partial \mathcal{L}}{\partial u}(u, \lambda)v = \int_{Q_\omega} (\varphi_u + \nu u)v\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma}\int_0^\infty \lambda \int_\omega uv\,\mathrm{d}x\,\mathrm{d}t, \tag{3.6}$$

$$\begin{aligned}&\frac{\partial^2 \mathcal{L}}{\partial u^2}(u, \lambda)(v_1, v_2)\\ &= \int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x, t, y_u)\varphi_u\Big] z_{v_1} z_{v_2}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty (\nu + \frac{1}{\gamma}\lambda) \int_\omega v_1 v_2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.7}$$

The identities (3.6) and (3.7) define continuous linear and bilinear forms on $L^2(Q_\omega)$ and $L^2(Q_\omega)^2$, respectively.

Let $\bar{u} \in U_{ad} \cap \mathcal{U}_\infty$ satisfy the first oder optimality conditions (2.6)–(2.8). Associated with $\bar{u}$ we define $\bar{\lambda}(t) = \|\bar{\varphi}(t) + \nu\bar{u}(t)\|_{L^2(\omega)}$. From Theorem 2.4 and (2.12) we get that $\bar{\lambda} \in L^\infty(0,\infty) \cap L^2(0,\infty)$. We also set

$$I_\gamma = \{t \in (0,\infty) : \|\bar{u}(t)\|_{L^2(\omega)} = \gamma\} \text{ and } I_\gamma^+ = \{t \in I_\gamma : \bar{\lambda}(t) \neq 0\}.$$

The choice of $\bar{\lambda}$ as Lagrange multiplier associated with the control constraint is suggested by (2.10). Actually, next lemma confirms that this is the correct choice.

Lemma 3.1 *Let $\bar{u}$ and $\bar{\varphi}$ satisfy (2.7) and (2.8). Then we have $\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})v = 0$ for every $v \in L^2(Q_\omega)$.*

Proof Using (3.6), (2.10), (2.11), and (2.15) we infer

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})v &= \int_{Q_\omega} (\bar{\varphi} + \nu\bar{u})v \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma}\int_0^\infty \bar{\lambda}(t)\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{I_\gamma^+}\int_\omega (\bar{\varphi} + \nu\bar{u})v \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma}\int_{I_\gamma^+} \bar{\lambda}(t)\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{I_\gamma^+}\int_\omega \Big(\bar{\varphi} - \nu\gamma\frac{\bar{\varphi}}{\|\bar{\varphi}(t)\|_{L^2(\omega)}}\Big)v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{I_\gamma^+} \bar{\lambda}(t)\int_\omega \frac{\bar{\varphi}}{\|\bar{\varphi}(t)\|_{L^2(\omega)}} v(t)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

□

In order to formulate the second order optimality conditions we introduce the cone of critical directions associated with $\bar{u}$:

$$C_{\bar{u}} = \{v \in L^2(Q_\omega) : J'(\bar{u})v = 0 \text{ and } \int_\omega \bar{u}(t)v(t)\,\mathrm{d}x \begin{cases} \leq 0 \text{ if } t \in I_\gamma \\ = 0 \text{ if } t \in I_\gamma^+ \end{cases}\}.$$

Then we have the following second order necessary optimality conditions.

Theorem 3.2 *If $\bar{u}$ is a local minimizer of (P), then $\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})v^2 \geq 0$ for all $v \in C_{\bar{u}}$.*

Proof Since $\bar{u}$ is a local minimizer of (P), there exists $\varepsilon > 0$ such that $J(\bar{u}) \leq J(u)$ for all $u \in U_{ad} \cap B_\varepsilon(\bar{u})$, where $B_\varepsilon(\bar{u}) = \{u \in L^2(Q_\omega) : \|u - \bar{u}\|_{L^2(Q_\omega)} < \varepsilon\}$. Due to $\bar{u} \in \mathcal{U}_\infty$ and since $\mathcal{U}_\infty$ is an open subset of $L^2(Q_\omega) \cap L^\infty(0,\infty; L^2(\omega))$, we can select ε small enough so that every control $u \in B_\varepsilon(\bar{u})$ satisfying $\|u - \bar{u}\|_{L^\infty(0,\infty;L^2(\omega))} < \varepsilon$ belongs to $\mathcal{U}_\infty$.

Let $v \in C_{\bar{u}} \cap L^\infty(0,\infty; L^2(\omega))$. The assumption $v \in L^\infty(0,\infty; L^2(\omega))$ will be removed later. Let us fix an integer

$$k_0 > \max\left\{\sqrt{\frac{2\max\{\|\bar{u}\|_{L^2(Q_\omega)}, \|\bar{u}\|_{L^\infty(0,\infty;L^2(\omega))}\}}{\gamma^4\varepsilon}}, \frac{1}{\gamma^2}\right\},$$

and set

$$v_k(x,t) = \begin{cases} 0 & \text{if } \gamma^2 - \dfrac{1}{k} < \|\bar{u}(t)\|^2_{L^2(\omega)} < \gamma^2 \\ v(x,t) & \text{otherwise} \end{cases} \quad \forall k \geq k_0.$$

It is obvious that $\{v_k\}_{k\geq k_0} \subset L^2(Q_\omega) \cap L^\infty(0,\infty; L^2(\omega))$. Moreover, the convergence $v_k \to v$ in $L^2(Q_\omega)$ follows from Lebesgue's dominated convergence theorem.

For fixed $k \geq k_0$, we define

$$\alpha_k = \min\left\{ \frac{\min\{1,\gamma\}\varepsilon}{2\max\{\|v\|_{L^2(Q_\omega)}, \|v\|_{L^\infty(0,\infty;L^2(\omega))}\}}, \frac{\gamma - \sqrt{\gamma^2 - \frac{1}{k}}}{\|v\|_{L^\infty(0,\infty;L^2(\omega))}} \right\}$$

and $\phi_k : (-\alpha_k, +\alpha_k) \longrightarrow L^2(Q_\omega) \cap L^\infty(0,\infty; L^2(\omega))$ by

$$\phi_k(\rho) = \sqrt{1 - \frac{\rho^2}{\gamma^2}\|v_k(t)\|^2_{L^2(\omega)}}\,\bar{u} + \rho v_k.$$

By definition of α_k we have $\frac{\rho^2}{\gamma^2}\|v_k(t)\|^2_{L^2(\omega)} < 1$ for all $k \geq k_0$, $|\rho| < \alpha_k$, and almost all $t \in (0,\infty)$. Moreover, $|\phi_k(\rho)| \leq |\bar{u}| + \frac{\varepsilon}{2\|v\|_{L^2(Q_\omega)}}|v| \in L^2(Q_\omega) \cap L^\infty(0,\infty; L^2(\omega))$. Hence, the mapping ϕ_k is well defined and it is of class C^∞. Let us prove some properties of this mapping.

I - $\phi_k(\rho) \in U_{ad}$ for all $\rho \in [0, +\alpha_k)$. Let us set $u_\rho = \phi_k(\rho)$. Then, we have for almost all $t \in (0,\infty)$

$$\begin{aligned} \|u_\rho(t)\|^2_{L^2(\omega)} &= \left[1 - \frac{\rho^2}{\gamma^2}\|v_k(t)\|^2_{L^2(\omega)}\right]\|\bar{u}(t)\|^2_{L^2(\omega)} + \rho^2\|v_k(t)\|^2_{L^2(\omega)} \\ &\quad + 2\rho\sqrt{1 - \frac{\rho^2}{\gamma^2}\|v_k(t)\|^2_{L^2(\omega)}}\int_\omega \bar{u}(t)v_k(t)\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

In the case $t \in I_\gamma$, we have $v_k(t) = v(t)$. Then, using that $v \in C_{\bar{u}}$ we deduce that the last integral in the above inequality is less than or equal to zero and, consequently, (3.8) leads to $\|u_\rho(t)\|^2_{L^2(\omega)} \leq \gamma^2$. If $\gamma^2 - \frac{1}{k} < \|\bar{u}(t)\|^2_{L^2(\omega)} < \gamma^2$, then we have $v_k(t) = 0$ by definition and, hence, (3.8) implies that $\|u_\rho(t)\|^2_{L^2(\omega)} \leq \gamma^2$. Finally, we assume that $\|\bar{u}(t)\|^2_{L^2(\omega)} \leq \gamma^2 - \frac{1}{k}$. Then, we infer from the definition of α_k

$$\|u_\rho(t)\|_{L^2(\omega)} \leq \sqrt{\gamma^2 - \frac{1}{k}} + \alpha_k\|v\|_{L^\infty(0,\infty;L^2(\omega))} \leq \gamma.$$

II - $\|\phi_k(\rho) - \bar{u}\|_{L^2(Q_\omega)} \leq \varepsilon$. From the definition of ϕ_k we get

$$\|\phi_k(\rho) - \bar{u}\|_{L^2(Q_\omega)} \leq \left|1 - \sqrt{1 - \frac{\rho^2}{\gamma^2}\|v_k(t)\|^2_{L^2(\omega)}}\right| \|\bar{u}\|_{L^2(Q_\omega)} + |\rho|\|v_k\|_{L^2(Q_\omega)}$$
$$\leq \frac{\alpha_k^2}{\gamma^2}\|v\|^2_{L^\infty(0,\infty;L^2(\omega))}\|\bar{u}\|_{L^2(Q_\omega)} + \alpha_k\|v\|_{L^2(Q_\omega)}.$$

From the definition of α_k and $k \geq k_0 > \frac{1}{\gamma^2}$ we obtain

$$\alpha_k \leq \frac{\gamma - \sqrt{\gamma^2 - \frac{1}{k}}}{\|v\|_{L^\infty(0,\infty;L^2(\omega))}} \leq \frac{1}{k\gamma\|v\|_{L^\infty(0,\infty;L^2(\omega))}}.$$

Moreover, $\alpha_k \leq \frac{\varepsilon}{2\|v\|_{L^2(Q_\omega)}}$ holds. Then, we have

$$\|\phi_k(\rho) - \bar{u}\|_{L^2(Q_\omega)} \leq \frac{\|\bar{u}\|_{L^2(Q_\omega)}}{k^2\gamma^4} + \frac{\varepsilon}{2} < \varepsilon.$$

The last inequality is consequence of $k \geq k_0 > \sqrt{\frac{2\|\bar{u}\|_{L^2(Q_\omega)}}{\gamma^4\varepsilon}}$.

III - $\phi_k(\rho) \in \mathcal{U}_\infty$. Arguing as in the previous step and using again the definition of α_k and k_0 with $\|\bar{u}\|_{L^2(Q_\omega)}$ and $\|v\|_{L^2(Q_\omega)}$ replaced by $\|\bar{u}\|_{L^\infty(0,\infty;L^2(\omega))}$ and $\|v\|_{L^\infty(0,\infty;L^2(\omega))}$, respectively, we infer that $\|\phi_k(\rho) - \bar{u}\|_{L^\infty(0,\infty;L^2(\omega))} < \varepsilon$. Due to the choice of ε this implies that $\phi_k(\rho) \in \mathcal{U}_\infty$.

Now we define the function $\psi_k : (-\alpha_k, +\alpha_k) \longrightarrow \mathbb{R}$ by $\psi_k(\rho) = J(\phi_k(\rho))$. From the local optimality of $\bar{u}$ and the established properties of ϕ_k we infer that $\psi_k(0) = J(\bar{u}) \leq J(\phi_k(\rho)) = \psi_k(\rho)$ for every $\rho \in [0, +\alpha_k)$. Since ψ_k is of class C^2, and $\psi_k'(0) = 0$ then $\psi_k''(0) \geq 0$. Hence, we get

$$0 \leq \psi_k''(0) = J''(\phi_k(0))\phi_k'(0)^2 + J'(\phi_k(0))\phi_k''(0) = J''(\bar{u})v_k^2 + J'(\bar{u})\phi_k''(0)$$
$$= \int_Q \Big[1 - \bar{\varphi}\frac{\partial f}{\partial y}(x,t,\bar{y})\Big] z_{v_k}^2 \,\mathrm{d}x\,\mathrm{d}t + \nu \int_{Q_\omega} v_k^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{1}{\gamma^2}\int_0^\infty \|v_k(t)\|^2_{L^2(\omega)} \int_\omega (\bar{\varphi} + \nu\bar{u})\bar{u} \,\mathrm{d}x\,\mathrm{d}t.$$

Using (2.10), (2.11), (2.15), and (2.16) we obtain that

$$\int_0^\infty \|v_k(t)\|^2_{L^2(\omega)} \int_\omega (\bar{\varphi} + \nu\bar{u})\bar{u} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{I_\gamma} \|v_k(t)\|^2_{L^2(\omega)} \Big(- \int_\omega \gamma \frac{\bar{\varphi}^2(t)}{\|\bar{\varphi}(t)\|_{L^2(\omega)}} \,\mathrm{d}x + \nu\|\bar{u}(t)\|^2_{L^2(\omega)}\Big) \,\mathrm{d}t$$
$$= \gamma \int_{I_\gamma} \|v_k(t)\|^2_{L^2(\omega)} \Big(- \|\bar{\varphi}(t)\|_{L^2(\omega)} + \nu\gamma\Big) \,\mathrm{d}t$$

$$= -\gamma \int_{I_\gamma} \|v_k(t)\|^2_{L^2(\omega)} \|\bar\varphi(t) + \nu \bar u(t)\|_{L^2(\omega)} \,\mathrm{d}t = -\gamma \int_{I_\gamma} \bar\lambda(t) \|v_k(t)\|^2_{L^2(\omega)} \,\mathrm{d}t$$
$$= -\gamma \int_0^\infty \bar\lambda(t) \|v_k(t)\|^2_{L^2(\omega)} \,\mathrm{d}t.$$

Inserting this in the above inequality we infer with (3.7)

$$0 \le \psi_k''(0) = \frac{\partial^2 \mathcal{L}}{\partial u}(\bar u, \bar\lambda) v_k^2.$$

Now, the convergence $v_k \to v$ in $L^2(Q)$ implies

$$\frac{\partial^2 \mathcal{L}}{\partial u}(\bar u, \bar\lambda) v^2 = \lim_{k\to\infty} \frac{\partial^2 \mathcal{L}}{\partial u}(\bar u, \bar\lambda) v_k^2 \ge 0.$$

Finally, we remove the assumption $v \in L^\infty(0,\infty; L^2(\omega))$. Given $v \in C_{\bar u}$, we define $v_k(x,t) = \frac{v(x,t)}{1+\frac{1}{k}\|v(t)\|_{L^2(\omega)}}$ for every integer $k \ge 1$. Then, we have $\{v_k\}_{k=1}^\infty \subset L^\infty(0,\infty; L^2(\omega)) \cap L^2(Q_\omega)$ and $v_k \to v$ in $L^2(Q_\omega)$. Using that $v \in C_{\bar u}$ we get

$$\int_\omega \bar u(t) v_k(t)\,\mathrm{d}x = \frac{1}{1+\frac{1}{k}\|v(t)\|_{L^2(\omega)}} \int_\omega \bar u(t) v(t)\,\mathrm{d}x \begin{cases} \le 0 \text{ if } t \in I_\gamma \\ = 0 \text{ if } t \in I_\gamma^+ \end{cases}$$

Identity (2.11) implies

$$\int_\omega \bar\varphi(t) v_k(t)\,\mathrm{d}x = -\frac{\|\bar\varphi(t)\|_{L^2(\omega)}}{\gamma} \int_\omega \bar u(t) v_k(t)\,\mathrm{d}t = 0 \text{ for a.a. } t \in I_\gamma^+.$$

Therefore from (2.10) and the above relations we deduce

$$J'(\bar u) v_k = \int_{I_\gamma^+} \int_\omega (\bar\varphi(t) + \nu \bar u(t)) v_k(t)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence, $\{v_k\}_{k=1}^\infty \subset C_{\bar u} \cap L^\infty(0,\infty; L^2(\omega))$ holds and, consequently, $\frac{\partial^2 L}{\partial u^2}(\bar u, \bar\lambda) v_k^2 \ge 0$ for all $k \ge 1$. Finally, passing to the limit as $k \to \infty$ we conclude that $\frac{\partial^2 L}{\partial u^2}(\bar u, \bar\lambda) v^2 \ge 0$. □

Now, we give a second order sufficient optimality condition.

Theorem 3.3 *Let $\bar u \in U_{ad} \cap \mathcal{U}_\infty$ satisfy the first order optimality conditions (2.6)–(2.8) and the second order condition $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \mu) v^2 > 0$ for every $v \in C_{\bar u} \setminus \{0\}$. Then, there exists $\kappa > 0$ and $\varepsilon > 0$ such that*

$$J(\bar u) + \frac{\kappa}{2}\|u - \bar u\|^2_{L^2(Q_\omega)} \le J(u) \ \ \forall u \in U_{ad} \text{ with } \|u - \bar u\|_{L^2(Q_\omega)} \le \varepsilon. \tag{3.9}$$

Proof We argue by contradiction and assume that (3.9) does not hold. Then, for every integer $k \ge 1$ there exists a control $u_k \in U_{ad}$ such that

$$\rho_k = \|u_k - \bar{u}\|_{L^2(Q_\omega)} < \frac{1}{k} \text{ and } J(u_k) < J(\bar{u}) + \frac{1}{2k}\|u_k - \bar{u}\|^2_{L^2(Q_\omega)}. \quad (3.10)$$

We define $v_k = \frac{1}{\rho_k}(u_k - \bar{u})$. Since $\|v_k\|_{L^2(Q_\omega)} = 1$ for every k, taking a subsequence, we can assume that $v_k \rightharpoonup v$ in $L^2(Q_\omega)$. From (3.10) we deduce that $\{y_{u_k}\}_{k=1}^\infty$ is a bounded sequence in $L^2(Q)$, hence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_\infty$. Moreover, given $p \in (\frac{4}{4-n}, \infty)$ we have

$$\|u_k - \bar{u}\|_{L^p(0,\infty;L^2(\Omega))} \le \|u_k - \bar{u}\|^{\frac{p-2}{p}}_{L^\infty(0,\infty;L^2(\Omega))}\|u_k - \bar{u}\|^{\frac{2}{p}}_{L^2(0,\infty;L^2(\Omega))} \to 0 \text{ as } k \to \infty.$$

Then, $y_{u_k} = G_p(u_k) \to G_p(\bar{u}) = \bar{y}$ in Y_p. Consequently, there exists a ball $B_r(\bar{u}) \subset L^2(Q_\omega) \cap L^p(0, \infty; L^2(\omega))$ and $k_0 \ge 1$ such that $\{u_k\}_{k \ge k_0} \subset \mathcal{U}_p$. The rest of the proof is split into three steps.

Step I $v \in C_{\bar{u}}$. From (2.4) and (2.8) we infer that

$$0 \le J'(\bar{u})v_k = \int_{Q_\omega} (\bar{\varphi} + \nu\bar{u})v_k \,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_\omega} (\bar{\varphi} + \nu\bar{u})v \,\mathrm{d}x\,\mathrm{d}t = J'(\bar{u})v. \quad (3.11)$$

Using the differentiability of the mapping $J : \mathcal{U}_p \longrightarrow \mathbb{R}$ we infer with the mean value theorem and (3.10)

$$\int_{Q_\omega} (\varphi_{\theta_k} + \nu u_{\theta_k})v_k \,\mathrm{d}x\,\mathrm{d}t = J'(u_{\theta_k})v_k = \frac{J(u_k) - J(\bar{u})}{\rho_k} < \frac{\rho_k}{2k} \to 0,$$

where $\theta_k \in [0, 1]$, $u_{\theta_k} = \bar{u} + \theta_k(u_k - \bar{u})$, and φ_{θ_k} is the adjoint state corresponding to u_{θ_k}. Since $y_{\theta_k} = G_p(u_{\theta_k}) \to G_p(\bar{u}) = \bar{y}$ in Y_p, we deduce from Theorem A.4 that $\varphi_{\theta_k} \to \bar{\varphi}$ in Y_p as $k \to \infty$. Then, it is straightforward to pass to the limit in the above expression and to get $J'(\bar{u})v \le 0$. This inequality and (3.11) imply that $J'(\bar{u})v = 0$.

Next, taking into account that $\|u_k(t)\|_{L^2(\omega)} \le \gamma$ for almost all $t > 0$, we have for almost every $t \in I_\gamma$

$$\int_\omega \bar{u}(t)v_k(t)\,\mathrm{d}t = \frac{1}{\rho_k}\Big[\int_\omega \bar{u}(t)u_k(t)\,\mathrm{d}t - \int_\omega \bar{u}^2(t)\,\mathrm{d}t\Big] \le \frac{1}{\rho_k}\gamma\Big[\|u_k(t)\|_{L^2(\omega)} - \gamma\Big] \le 0.$$

We define the function $\phi \in L^\infty(0, \infty)$ by $\phi(t) = 1$ if $\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x > 0$ and 0 otherwise. Then, from the convergence $v_k \rightharpoonup v$ in $L^2(Q_\omega)$ and the fact that $\phi\bar{u} \in L^2(Q_\omega)$ we infer from the above inequality

$$\int_{I_\gamma} \phi(t)\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x\,\mathrm{d}t = \lim_{k\to\infty}\int_{I_\gamma} \phi(t)\int_\omega \bar{u}(t)v_k(t)\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

This is possible if and only if $\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x \le 0$ for almost all $t \in I_\gamma$. Finally, we prove that this integral is 0 if $t \in I_\gamma^+$. For this purpose we use Lemma 3.1, (3.6), and the fact that $J'(\bar{u})v = 0$ as follows

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})v = J'(\bar{u})v + \frac{1}{\gamma}\int_0^\infty \bar{\lambda}(t)\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{\gamma}\int_{I_\gamma} \bar{\lambda}(t)\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which implies that $\int_\omega \bar{u}(t)v(t)\,\mathrm{d}x = 0$ for almost all $t \in I_\gamma^+$, and thus $v \in C_{\bar{u}}$.

Step II $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})v^2 \leq 0$. First we observe that

$$\begin{aligned} &\int_0^\infty \bar{\lambda}(t)\|u_k(t)\|^2_{L^2(\omega)}\,\mathrm{d}t = \int_{I_\gamma} \bar{\lambda}(t)\|u_k(t)\|^2_{L^2(\omega)}\,\mathrm{d}t \\ &\leq \int_{I_\gamma} \bar{\lambda}(t)\|\bar{u}(t)\|^2_{L^2(\omega)}\,\mathrm{d}t = \int_0^\infty \bar{\lambda}(t)\|\bar{u}(t)\|^2_{L^2(\omega)}\,\mathrm{d}t. \end{aligned}$$

This inequality and (3.10) imply

$$\mathcal{L}(u_k, \bar{\lambda}) < \mathcal{L}(\bar{u}, \bar{\lambda}) + \frac{1}{2k}\|u_k - \bar{u}\|^2_{L^2(Q_\omega)}.$$

Performing a Taylor expansion and using again Lemma 3.1 we infer for some $\vartheta_k \in [0, 1]$

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda})(u_k - \bar{u})^2 \\ &= \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\lambda})(u_k - \bar{u}) + \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda})(u_k - \bar{u})^2 \\ &= \mathcal{L}(u_k, \bar{\lambda}) - \mathcal{L}(\bar{u}, \bar{\lambda}) < \frac{1}{2k}\|u_k - \bar{u}\|^2_{L^2(Q_\omega)}. \end{aligned}$$

Dividing the above inequality by $\frac{\rho_k^2}{2}$ we get

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda})v_k^2 \leq \frac{1}{k}. \tag{3.12}$$

Denoting by $u_{\vartheta_k} = \bar{u} + \vartheta_k(u_k - \bar{u})$, y_{ϑ_k} its associated state, and φ_{ϑ_k} the corresponding adjoint state, we get from (3.7)

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda})v_k^2 &= \int_Q \left[1 - \frac{\partial^2 f}{\partial y^2}(x, t, y_{\vartheta_k})\varphi_{\vartheta_k}\right] z^2_{\vartheta_k, v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \nu\|v_k\|^2_{L^2(Q_\omega)} + \frac{1}{\gamma}\int_0^\infty \bar{\lambda}(t)\|v_k(t)\|^2_{L^2(\omega)}\,\mathrm{d}t, \end{aligned} \tag{3.13}$$

where z_{ϑ_k,v_k} satisfies the equation

$$\begin{cases} \dfrac{\partial z_{\vartheta_k,v_k}}{\partial t} - \Delta z_{\vartheta_k,v_k} + a z_{\vartheta_k,v_k} + \dfrac{\partial f}{\partial y}(x,t,y_{\vartheta_k}) z_{\vartheta_k,v_k} = v_k \chi_\omega \text{ in } Q, \\ \partial_n z_{\vartheta_k,v_k} = 0 \text{ on } \Sigma,\ z_{\vartheta_k,v_k}(0) = 0 \text{ in } \Omega. \end{cases} \tag{3.14}$$

Now, we study the lower limit of (3.12). From Theorem A.3 and the boundedness of $\{v_k\}_{k=1}^\infty$ and $\{y_{\vartheta_k}\}_{k=1}^\infty$ in $L^2(Q_\omega)$ and $L^\infty(Q)$, respectively, we infer the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$. Therefore, we can extract a subsequence, that we denote in the same way, such that $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ converges weakly in $W(0,\infty)$. Moreover, the convergence $u_{\vartheta_k} \to \bar{u}$ in $L^p(0,\infty;L^2(\omega))$ implies $y_{\vartheta_k} = G_p(u_{\vartheta_k}) \to G_p(\bar{u}) = \bar{y}$ in Y_p. Using this and the convergence $v_k \rightharpoonup v$ in $L^2(Q_\omega)$, it is straightforward to pass to the limit in (3.14) and to deduce that $z_{\vartheta_k,v_k} \rightharpoonup z_v$ in $W(0,\infty)$, where z_v is the solution of (2.2). Further, the convergence of $y_{\vartheta_k} \to \bar{y}$ in Y_p implies the convergence in $L^p(0,\infty;L^2(\Omega)) \cap L^\infty(Q)$. Then, from Theorem A.4 we infer that $\varphi_{\vartheta_k} \to \bar{\varphi}$ in $W(0,\infty) \cap L^\infty(Q)$. Indeed, subtracting the equations satisfied by φ_{ϑ_k} and $\bar{\varphi}$ we get for $\psi_k = \varphi_{\vartheta_k} - \bar{\varphi}$

$$\begin{cases} -\dfrac{\partial \psi_k}{\partial t} - \Delta \psi_k + a\psi_k + \dfrac{\partial f}{\partial y}(x,t,\bar{y})\psi_k \\ \qquad\qquad = y_{\vartheta_k} - \bar{y} + \Big[\frac{\partial f}{\partial y}(x,t,\bar{y}) - \frac{\partial f}{\partial y}(x,t,y_{\vartheta_k})\Big]\varphi_{\vartheta_k} \text{ in } Q, \\ \partial_n \psi_k = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\psi_k(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

Then, using (3.3), the established convergence $y_{\vartheta_k} \to \bar{y}$, (A.21), and (A.22) we get the claimed convergence of $\{\varphi_{\vartheta_k}\}_{k=1}^\infty$ to $\bar{\varphi}$.

Now, we take the lower limit in (3.12). For this purpose we take into account that $z_{\vartheta_k,v_k} \rightharpoonup z_v$ in $L^2(Q)$, $v_k \rightharpoonup v$ in $L^2(Q_\omega)$, and $\bar{\lambda} \in L^\infty(Q)$ with $\bar{\lambda}(t) \geq 0$ for almost all $t \in (0,\infty)$. Hence, we get by (3.12)

$$\begin{aligned} 0 &\geq \liminf_{k\to\infty} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda}) v_k^2 \\ &\geq \liminf_{k\to\infty} \|z_{\vartheta_k,v_k}\|^2_{L^2(Q)} + \liminf_{k\to\infty} \int_Q -\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \liminf_{k\to\infty} \nu\|v_k\|^2_{L^2(Q_\omega)} + \liminf_{k\to\infty} \frac{1}{\gamma}\int_0^\infty \bar{\lambda}(t)\|v_k(t)\|^2_{L^2(\omega)}\,\mathrm{d}t \\ &\geq \|z_v\|^2_{L^2(Q)} + \liminf_{k\to\infty} \int_Q -\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \nu\|v\|^2_{L^2(Q_\omega)} + \frac{1}{\gamma}\int_0^\infty \bar{\lambda}(t)\|v(t)\|^2_{L^2(\omega)}\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

Below we prove that

$$\lim_{k\to\infty} \int_Q \frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k}\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi} z_v^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.16}$$

Thus, (3.7) and (3.15)–(3.16) yield $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \lambda)v^2 \leq 0$.

Let us prove (3.16). Given $\varepsilon > 0$, (2.7) implies the existence of $T_\varepsilon > 0$ such that $\|\bar{\varphi}(t)\|_{L^2(\Omega)} < \varepsilon$ for every $t \geq T_\varepsilon$. Further, the convergence $z_{\vartheta_k,v_k} \rightharpoonup z_v$ in $W(0,\infty)$ implies the convergence $z_{\vartheta_k,v_k} \to z_v$ in $L^2(Q_{T_\varepsilon})$. Using these properties and (3.2) with $M = \|\bar{y}\|_{L^\infty(Q)}$ we get

$$
\begin{aligned}
&\int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} z^2_{\vartheta_k,v_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi} z_v^2\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi}\Big| z^2_{\vartheta_k,v_k} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_{T_\varepsilon}} \Big|\frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi}\Big| |z^2_{\vartheta_k,v_k} - z_v^2| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{T_\varepsilon}^\infty \int_\Omega \Big|\frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi}\Big| |z^2_{\vartheta_k,v_k} - z_v^2| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Big\|\frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_k})\varphi_{\vartheta_k} - \frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\bar{\varphi}\Big\|_{L^\infty(Q)} \|z_{\vartheta_k,v_k}\|^2_{L^2(Q)} \\
&\quad + C_M \|\bar{\varphi}\|_{L^\infty(Q)} \|z_{\vartheta_k,v_k} - z_v\|_{L^2(Q_{T_\varepsilon})} \|z_{\vartheta_k,v_k} + z_v\|_{L^2(Q_{T_\varepsilon})} \\
&\quad + C_M \varepsilon \int_{T_\varepsilon}^\infty \|z_{\vartheta_k,v_k} - z_v\|_{L^2(\Omega)} \|z_{\vartheta_k,v_k} + z_v\|_{L^2(\Omega)} \,\mathrm{d}t = I_1 + I_2 + I_3
\end{aligned}
$$

The convergence $(y_{\vartheta_k}, \varphi_{\vartheta_k}) \to (\bar{y}, \bar{\varphi})$ in $L^\infty(Q)^2$ and the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$ imply that $I_1 \to 0$ as $k \to \infty$. The convergence $z_{\vartheta_k,v_k} \to z_v$ in $L^2(Q_{T_\varepsilon})$ implies that $I_2 \to 0$ as well. For I_3 we have

$$
\begin{aligned}
|I_3| &\leq C_1 C_M \varepsilon \int_{T_\varepsilon}^\infty \|z_{\vartheta_k,v_k} - z_v\|_{L^2(\Omega)} \|z_{\vartheta_k,v_k} + z_v\|_{L^2(\Omega)} \,\mathrm{d}t \\
&\leq C_1 C_M \varepsilon \|z_{\vartheta_k,v_k} - z_v\|_{L^2(Q)} \|z_{\vartheta_k,v_k} + z_v\|_{L^2(Q)} \leq C_2 \varepsilon,
\end{aligned}
$$

where we have used again the boundedness of $\{z_{\vartheta_k,v_k}\}_{k=1}^\infty$ in $W(0,\infty)$. Since $\varepsilon > 0$ is arbitrarily small, we deduce the convergence $I_3 \to 0$ as $k \to \infty$.

Step III—Final contradiction The facts proved in Steps I and II along with the assumption $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$ lead to $v = 0$ and $z_v = 0$. Therefore, looking at the relations (3.15) we obtain with (3.16) and $\|v_k\|_{L^2(Q_\omega)} = 1$

$$
0 \geq \liminf_{k\to\infty} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u} + \vartheta_k(u_k - \bar{u}), \bar{\lambda})v_k^2 \geq \liminf_{k\to\infty} \nu \|v_k\|^2_{L^2(Q_\omega)} = \nu,
$$

which contradicts the assumption $\nu > 0$. $\square$

3.2 Case II: $K_{ad} = \{v \in L^2(\omega) : \alpha \le v(x) \le \beta \text{ for a.a. } x \in \omega\}$.

In this case, the cone of critical directions is defined by

$$C_{\bar u} = \{v \in L^2(Q_\omega) : J'(\bar u)v = 0 \text{ and } v(x,t) \begin{cases} \ge 0 \text{ if } \bar u(x,t) = \alpha \\ \le 0 \text{ if } \bar u(x,t) = \beta \end{cases} \}.$$

Analogously to Theorem 3.2 we have the following result.

Theorem 3.4 *If $\bar u$ is a local minimizer of (P), then $J''(\bar u)v^2 \ge 0$ for all $v \in C_{\bar u}$.*

Proof Since $\bar u$ is a local minimizer of (P), there exists $\varepsilon > 0$ such that $J(\bar u) \le J(u)$ for all $u \in U_{ad} \cap B_\varepsilon(\bar u)$, where $B_\varepsilon(\bar u) = \{u \in L^2(Q_\omega) : \|u - \bar u\|_{L^2(Q_\omega)} < \varepsilon\}$. Given $p \in \big(\frac{4}{4-n}, \infty)$ we have for every $u \in U_{ad} \cap B_\varepsilon(\bar u)$

$$\|u - \bar u\|_{L^p(0,\infty;L^2(\omega))} \le (\beta - \alpha)^{1-\frac{2}{p}} \|u - \bar u\|^{\frac{2}{p}}_{L^2(Q_\omega)} < (\beta - \alpha)^{1-\frac{2}{p}} \varepsilon^{\frac{2}{p}}.$$

Therefore, we select $\varepsilon > 0$ small enough, such that $U_{ad} \cap B_\varepsilon(\bar u) \subset \mathcal{U}_p$ holds. Now, given $v \in C_{\bar u}$ we define for every integer $k \ge 1$ the function v_k by

$$v_k(x,t) = \begin{cases} 0 & \text{if } \alpha < \bar u(x,t) < \alpha + \frac{1}{k} \text{ or } \beta - \frac{1}{k} < \bar u(x,t) < \beta, \\ \mathrm{Proj}_{[-k,+k]}(v(x,t)) & \text{otherwise.} \end{cases}$$

It is obvious that $\{v_k\}_{k=1}^\infty \subset L^\infty(Q_\omega) \cap L^2(Q_\omega)$ and $v_k \to v$ in $L^2(Q_\omega)$ as $k \to \infty$. Further, if we set $\rho_k = \min\{\frac{1}{k^2}, \frac{\beta-\alpha}{k}, \frac{\varepsilon}{\|v\|_{L^2(Q_\omega)}}\}$, then $\bar u + \rho v_k \in U_{ad} \cap B_\varepsilon(\bar u)$ for every $\rho \in (0, \rho_k)$. In view of (2.13), it is straightforward to check that the condition $J'(\bar u)v = 0$ in the definition of $C_{\bar u}$ is equivalent to $(\bar\varphi + \nu\bar u)(x,t)v(x,t) = 0$ for almost all $(x,t) \in Q_\omega$. Using this fact, it is immediate that $J'(\bar u)v_k = 0$ for every k. Then, performing a Taylor expansion we get for every $\rho \in (0, \rho_k)$

$$\begin{aligned} 0 &\le J(\bar u + \rho v_k) - J(\bar u) = \rho J'(\bar u)v_k + \frac{\rho^2}{2} J''(\bar u + \theta_{\rho,k}\rho v_k)v_k^2 \\ &= \frac{\rho^2}{2} J''(\bar u + \theta_{\rho,k}\rho v_k)v_k^2. \end{aligned}$$

Dividing by $\frac{\rho^2}{2}$ we deduce $J''(\bar u + \theta_{\rho,k}\rho v_k)v_k^2 \ge 0$. Since $\bar u + \theta_{\rho,k}\rho v_k \to \bar u$ in $L^p(0,\infty; L^2(\omega))$ as $\rho \to 0$, we deduce $J''(\bar u)v_k^2 \ge 0$. Moreover, since $v_k \to v$ in $L^2(Q_\omega)$ we infer from Theorem A.3 that $z_{v_k} \to z_v$ in $L^2(Q_\omega)$. Hence, we can pass to the limit in the previous inequality and obtain $J''(\bar u)v^2 \ge 0$. □

Now, we establish the sufficient second order conditions for local optimality.

Theorem 3.5 *Let $\bar u \in U_{ad} \cap \mathcal{U}_\infty$ satisfy the first order optimality conditions (2.6)–(2.7) and the second order condition $J''(\bar u)v^2 > 0$ for every $v \in C_{\bar u}\setminus\{0\}$. Then, there exists $\kappa > 0$ and $\varepsilon > 0$ such that*

$$J(\bar{u}) + \frac{\kappa}{2}\|u - \bar{u}\|^2_{L^2(Q_\omega)} \leq J(u) \ \ \forall u \in U_{ad} \textit{ with } \|u - \bar{u}\|_{L^2(Q_\omega)} \leq \varepsilon. \quad (3.17)$$

The proof of this theorem follows by contradiction similarly to the proof of Theorem 3.3 with the obvious simplifications due to the constraints under consideration in this second case for U_{ad}. For the proof of these results for finite horizon control problems the reader is also referred to [5, 13]. The difficulties resulting from the infinite horizon can be overcome by following the arguments used in the proof of Theorem 3.3.

4 Approximation by finite horizon problems

In this section we consider the approximation of (P) by finite horizon optimal control problems and provide error estimates for these approximations. For every $0 < T < \infty$ we consider the control problem

$$(\mathrm{P}_T) \quad \min_{u \in U_{T,ad}} J_T(u),$$

where $U_{T,ad} = \{u \in L^2(Q_{T,\omega}) : u(t) \in K_{ad} \text{ for a.a. } t \in (0, T)\}$,

$$J_T(u) = \frac{1}{2}\int_{Q_T} (y_{T,u} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int_{Q_{T,\omega}} u^2 \,\mathrm{d}x\,\mathrm{d}t$$

with $Q_T = \Omega \times (0, T)$, $Q_{T,\omega} = \omega \times (0, T)$, and $y_{T,u}$ denotes the solution of the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x, t, y) = g + u\chi_\omega \text{ in } Q_T, \\ \partial_n y = 0 \text{ on } \Sigma_T = \Gamma \times (0, T), \ \ y(0) = y_0 \text{ in } \Omega. \end{cases} \quad (4.1)$$

For every control $u \in L^2(Q_{T,\omega})$ with associated state $y_{T,u}$ and adjoint state $\varphi_{T,u}$ we define extensions to Q_ω and Q, denoted by $\hat{u}$, $\hat{y}_{T,u}$, and $\hat{\varphi}_{T,u}$, by setting $(\hat{u}, \hat{\varphi}_{T,u})(x, t) = (0, 0)$ if $t > T$ and $\hat{y}_{T,u}$ is the solution of (1.1) associated with the extension $\hat{u}$. In this section, we assume that $0 \in K_{ad}$. Hence, if $u \in U_{T,ad}$, then $\hat{u} \in U_{ad}$ holds. Given a local minimizer u_T of (P_T), we denote by y_T and φ_T its associated state and adjoint state, respectively. Then, (u_T, y_T, φ_T) satisfies the optimality conditions established in Theorem 2.4 with Q and Q_ω replaced by Q_T and $Q_{T,\omega}$. As a consequence, Corollary 2.1 is also satisfied by (u_T, y_T, φ_T) with the same changes.

In case U_{ad} is given by (1.2), we define $\lambda_T(t) = \|\varphi_T(t) + \nu u_T(t)\|_{L^2(\omega)}$ for $t \in (0, T)$ and the Lagrange function

$$\mathcal{L}_T : L^p(0, T; L^2(\omega)) \times L^\infty(0, T) \longrightarrow \mathbb{R}$$
$$\mathcal{L}_T(u, \lambda) = J_T(u) + \frac{1}{2\gamma}\int_0^T \lambda(t)\|u(t)\|^2_{L^2(\omega)} \,\mathrm{d}t,$$

for every $p \in (\frac{4}{4-n}, \infty]$. Arguing as in Lemma 3.1 we also have

$$\frac{\partial \mathcal{L}_T}{\partial u}(u_T, \lambda_T)v = 0 \quad \forall v \in L^2(Q_{T,\omega}). \tag{4.2}$$

The next two theorems establish the convergence of the approximating problems $(\mathrm{P}_T)$to (P) as $T \to \infty$.

Theorem 4.1 *For every $T > 0$ the control problem (P_T)has at least one solution u_T. If (P) has a feasible control u_0, then the extensions $\{\hat{u}_T\}_{T>0}$ of any family of solutions are bounded in $L^2(Q_\omega)$. Every weak limit $\bar{u}$ in $L^2(Q_\omega)$ of a sequence $\{\hat{u}_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ as $k \to \infty$ is a solution of (P). Moreover, strong convergence $\hat{u}_{T_k} \to \bar{u}$ in $L^p(0, \infty; L^2(\omega))$ holds for every $p \in [2, \infty)$.*

Proof Since $U_{T,ad}$ is not empty, the existence of solution for $(\mathrm{P}_T)$is a classical result. Actually, one can easily adapt the existence proof of solution for (P) to (P_T). We denote by $\tilde{y}_T$ the extension of y_T by zero in $\Omega \times (T, \infty)$. We point out that $\tilde{y}_T \neq \hat{y}_T$. Let y^0 be the solution of (1.1) corresponding to u_0. By definition of feasible control we have that $J(u_0) < \infty$. Using the optimality of u_T we obtain

$$\begin{aligned}
&\frac{1}{2}\|\tilde{y}_T - y_d\|^2_{L^2(Q)} + \frac{\nu}{2}\|\hat{u}_T\|^2_{L^2(Q_\omega)} = J_T(u_T) + \frac{1}{2}\|y_d\|^2_{L^2(T,\infty;L^2(\Omega))}\\
&\le J_T(u_0) + \frac{1}{2}\|y_d\|^2_{L^2(Q)} \le J(u_0) + \frac{1}{2}\|y_d\|^2_{L^2(Q)} \ \forall T > 0.
\end{aligned}$$

This proves the boundedness of $\{\hat{u}_T\}_{T>0}$ and $\{\tilde{y}_T\}_{T>0}$ in $L^2(Q_\omega)$ and $L^2(Q)$, respectively. Let $\{(\hat{u}_{T_k}, \tilde{y}_{T_k})\}_{k=1}^\infty$ be a sequence with $T_k \to \infty$ as $k \to \infty$ converging weakly to $(\bar{u}, \bar{y})$ in $L^2(Q_\omega) \times L^2(Q)$. Since $\{\hat{u}_{T_k}\}_{k=1}^\infty \subset U_{ad}$ and U_{ad} is closed in $L^2(Q_\omega)$ and convex, we infer that $\bar{u} \in U_{ad}$. Moreover, we can apply Theorem A.2 to the Eq. (4.1) and deduce the existence of a constant M_1 independent of k such that for all $k \ge 1$

$$\begin{aligned}
&\|y_{T_k}\|_{L^2(0,T_k;H^1(\Omega))} + \|y_{T_k}\|_{L^\infty(Q_{T_k})} \le M_1 = C\Big(\|g + \hat{u}_{T_k}\chi_\omega\|_{L^2(Q)}\\
&+ \|g + \hat{u}_{T_k}\chi_\omega\|_{L^p(0,\infty;L^2(\Omega))} + \|y_0\|_{L^\infty(\Omega)} + \sup_{k\ge 1}\|\tilde{y}_{T_k}\|_{L^2(Q)} + M_f\Big).
\end{aligned}$$

From this estimate and (A.6) we get the existence of a constant M_2 such that

$$\|f(\cdot,\cdot,y_{T_k})\|_{L^2(Q_{T_k})} + \|f(\cdot,\cdot,y_{T_k})\|_{L^\infty(Q_{T_k})} \le M_2 \ \ \forall k \ge 1.$$

The two above estimates and (4.1) imply that

$$\|y_{T_k}\|_{W(0,T_k)} + \|y_{T_k}\|_{L^\infty(Q_{T_k})} \le M_3 \ \ \forall k \ge 1$$

for a constant independent of k. Using the convergence of $\tilde{y}_k \rightharpoonup \bar{y}$ in $L^2(Q)$, the compactness of the embedding $W(0, T) \subset L^2(Q_T)$ for every $T < \infty$, and the above estimate, it is obvious to pass to the limit in the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y_{T_k} + a y_{T_k} + f(x,t,y_{T_k}) = g + u_{T_k}\chi_\omega \text{ in } Q_T, \\ \partial_n y = 0 \text{ on } \Sigma_T = \Gamma \times (0,T),\ y_{T_k}(0) = y_0 \text{ in } \Omega \end{cases}$$

for each $T_k \geq T$, and to deduce that $\bar{y}$ is the solution of (4.1) associated to $\bar{u}$ for arbitrary $0 < T < \infty$. This proves that $\bar{y}$ is the solution of (1.1) corresponding to $\bar{u}$. Further, since $\bar{y} \in L^2(Q)$, we deduce that $\bar{u} \in \mathcal{U}_\infty$. Let us prove that $\bar{u}$ is a solution of (P). For every feasible control u of (P) we have

$$\begin{aligned} J(\bar{u}) &\leq \liminf_{k\to\infty} \Big(\frac{1}{2}\int_Q (\tilde{y}_{T_k} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int_{Q_\omega} \hat{u}_{T_k}^2 \,\mathrm{d}x\,\mathrm{d}t\Big) \\ &\leq \limsup_{k\to\infty} \Big(\frac{1}{2}\int_Q (\tilde{y}_{T_k} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int_{Q_\omega} \hat{u}_{T_k}^2 \,\mathrm{d}x\,\mathrm{d}t\Big) \\ &= \limsup_{k\to\infty} \Big(J_{T_k}(u_{T_k}) + \frac{1}{2}\|y_d\|^2_{L^2(T_k,\infty;L^2(\Omega))}\Big) \leq \limsup_{k\to\infty} J_{T_k}(u) = J(u). \end{aligned}$$

This proves that $\bar{u}$ is a solution of (P). Moreover, replacing u by $\bar{u}$ in the above inequalities we infer

$$\begin{aligned} \lim_{k\to\infty} \Big(\frac{1}{2}\int_Q (\tilde{y}_{T_k} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int_{Q_\omega} \hat{u}_{T_k}^2 \,\mathrm{d}x\,\mathrm{d}t\Big) &= \int_Q (\bar{y} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{\nu}{2}\int_{Q_\omega} \bar{u}^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

This convergence along with the weak convergence $(\hat{u}_{T_k}, \tilde{y}_{T_k}) \rightharpoonup (\bar{u}, \bar{y})$ in $L^2(Q_\omega) \times L^2(Q)$ implies the strong convergence. Finally, for any $p \in (2,\infty)$ we have

$$\|\hat{u}_{T_k} - \bar{u}\|_{L^p(0,\infty;L^2(\omega))} \leq \|\hat{u}_{T_k} - \bar{u}\|^{\frac{p-2}{p}}_{L^\infty(0,\infty;L^2(\omega))} \|\hat{u}_{T_k} - \bar{u}\|^{\frac{2}{p}}_{L^2(Q_\omega)} \to 0.$$

□

Theorem 4.2 *Let $\bar{u}$ be a strict local minimizer of (P). Then, there exist $T_0 \in (0,\infty)$ and a family $\{u_T\}_{T>T_0}$ of local minimizers to (P_T)such that the convergence $\hat{u}_T \to \bar{u}$ in $L^p(0,\infty;L^2(\omega))$ holds as $T \to \infty$ for every $p \in [2,\infty)$.*

Proof Since $\bar{u}$ is a strict local minimizer of (P), there exists $\rho > 0$ such that $J(\bar{u}) < J(u)$ for every $u \in U_{ad} \cap B_\rho(\bar{u})$ with $u \neq \bar{u}$, where $B_\rho(\bar{u})$ is the closed ball in $L^2(Q_\omega)$ centered at $\bar{u}$ and radius $\rho > 0$. We consider the control problems

$$(\mathrm{P}_\rho) \quad \min_{u\in B_\rho(\bar{u})\cap U_{ad}} J(u) \quad \text{and} \quad (\mathrm{P}_{T,\rho}) \quad \min_{u\in B_{T,\rho}(\bar{u})\cap U_{T,ad}} J_T(u),$$

where $B_{T,\rho}(\bar{u}) = \{u \in L^2(Q_{T,\omega}) : \|u - \bar{u}\|_{L^2(Q_{T,\omega})} \leq \rho\}$. Obviously $\bar{u}$ is the unique solution of (P_ρ). Existence of a solution u_T of $(\mathrm{P}_{T,\rho})$is straightforward. Then, arguing as in the proof of Theorem 4.1 and using the uniqueness of the solution of (P$_\rho$), we deduce the convergence $\hat{u}_T \to \bar{u}$ in $L^2(Q_\omega)$ as $T \to \infty$. This implies the existence of $T_0 > 0$ such that $\|u_T - \bar{u}\|_{L^2(Q_{T,\omega})} \le \|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)} < \rho$ for all $T > T_0$. Hence, u_T is also a local minimizer of (P$_T$)for $T > T_0$. The strong convergence $\hat{u}_T \to \bar{u}$ in $L^p(0, \infty; L^2(\omega))$ follows from the convergence in $L^2(Q_\omega)$ and the fact that $\|\hat{u}_T\|_{L^\infty(0,\infty;L^2(\omega))} \le \gamma$ for every $T > 0$. ◻

In the previous theorem we proved the existence of local minimizers $\{u_T\}_{T>T_0}$ of problems (P$_T$)converging to $\bar{u}$ assuming that $\bar{u}$ is a strict local minimizer of (P). Moreover, in the proof of the theorem, the existence of an $L^2(Q_\omega)$-closed ball $B_\rho(\bar{u})$ such that the minimum of J_T on the set $U_{ad} \cap B_\rho(\bar{u})$ is achieved at the local minimizer u_T was established. In particular, this implies that $J_T(u_T) \le J_T(\bar{u})$ for every $T > T_0$. In the next theorem the following question is addressed: if $\{u_T\}_{T>T_0}$ is a sequence of local minimizers of problems (P$_T$)converging to $\bar{u}$, does the inequality $J_T(u_T) \le J_T(\bar{u})$ hold for T large enough? The positive answer to this question is also important to establish the estimates in Theorem 4.4 below.

Theorem 4.3 *Suppose that U_{ad} is defined by* (1.2) *or* (1.3)*. Let $\bar{u}$ be a local minimizer of (P) satisfying the second order sufficient optimality condition given in Theorems* 3.3 *and* 3.5*, respectively. Let $\{u_T\}_{T>T_0}$ be a sequence of local minimizers of problems (P_T)such that $\hat{u}_T \to \bar{u}$ strongly in $L^2(Q_\omega)$. Then, there exists $T_0^* \in (T_0, \infty)$ such that $J_T(u_T) \le J_T(\bar{u})$ for every for every $T \ge T_0^*$.*

Proof The proof is carried out under the assumption that U_{ad} is given by (1.2). It is similar, even easier, if U_{ad} is given by (1.3). First, we observe that the convergence $\hat{u}_T \to \bar{u}$ in $L^2(Q_\omega)$ and the fact that $\|\hat{u}_T(t)\|_{L^2(\omega)} \le \gamma$ for almost every $t > 0$ implies that $\hat{u}_T \to \bar{u}$ strongly in $L^p(0, \infty; L^2(\omega))$ for every $p < \infty$. Then, for fixed $p > \frac{4}{4-n}$, there exists $\hat{T} \ge T_0$ such that $\hat{u}_T \in \mathcal{U}_p$ for every $T \ge \hat{T}$. This yields $\hat{y}_T = G_p(\hat{u}_T) \to G_p(\bar{u}) = \bar{y}$ in Y_p as $T \to \infty$. Given the adjoint state φ_T associated with u_T, we denote by $\hat{\varphi}_T$ its extension by 0 for $t > T$.

We proceed by contradiction. If the statement fails, then there exists a sequence $\{u_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ as $k \to \infty$ such that

$$\|\hat{u}_{T_k} - \bar{u}\|_{L^2(Q_\omega)} < \frac{1}{k} \text{ and } J_{T_k}(\bar{u}) < J_{T_k}(u_{T_k}). \tag{4.3}$$

Let us set $\rho_k = \|\hat{u}_{T_k} - \bar{u}\|_{L^2(Q_\omega)}$ and $v_{T_k} = \frac{1}{\rho_k}(\hat{u}_{T_k} - \bar{u})$. Taking a subsequence, denoted in the same way, we have $v_{T_k} \rightharpoonup v$ in $L^2(Q_\omega)$.

Now, we split the proof in three steps.

Step I $\hat{\varphi}_T \to \bar{\varphi}$ in $W(0, \infty) \cap L^\infty(Q)$ as $T \to \infty$. Let us set $\psi_T = \hat{\varphi}_T - \bar{\varphi}$ and denote by χ_T the real function taking the value 1 if $t \in [0, T]$ and 0 otherwise. Then, ψ_T satisfies the equation

$$\begin{cases} -\dfrac{\partial \psi_T}{\partial t} - \Delta \psi_T + a\psi_T + \dfrac{\partial f}{\partial y}(x, t, \bar{y})\psi_T \\ = \left[\frac{\partial f}{\partial y}(x, t, \bar{y}) - \frac{\partial f}{\partial y}(x, t, \hat{y}_T)\right]\hat{\varphi}_T + \chi_T(\hat{y}_T - \bar{y}) - (1 - \chi_T)(\bar{y} - y_d) \text{ in } Q, \\ \partial_n \psi_T = 0 \text{ on } \Sigma,\ \lim_{t\to\infty} \|\psi_T(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

Since $\hat{y}_T \to \bar{y}$ in Y_p, we deduce that $\hat{y}_T \to \bar{y}$ in $L^q(0,\infty; L^2(\Omega)) \cap L^\infty(Q)$ for every $q \geq 2$. Hence, with the mean value theorem and (3.2) we obtain that $\left[\frac{\partial f}{\partial y}(x,t,\bar{y}) - \frac{\partial f}{\partial y}(x,t,\hat{y}_T)\right] \to 0$ in $L^q(0,\infty; L^2(\Omega))$. Moreover, from Theorem A.4 and the fact that $y_d \in L^2(Q) \cap L^p(0,\infty; L^2(\Omega))$, we get that $\hat{\varphi}_T$ is bounded in $W(0,\infty) \cap L^\infty(Q)$. Therefore the first term of the right hand side in the above partial differential equation converges to 0 in $L^q(0,\infty; L^2(\Omega))$. The same convergence is true for the second term $\chi_T(\hat{y}_T - \bar{y})$. The third term $(1-\chi_T)(\bar{y} - y_d)$ converges to 0 in $L^q(0,\infty; L^2(\Omega))$ for $q = p$ if $p < \infty$ and $q < \infty$ arbitrary if $p = \infty$. Then, from Theorem A.4 the claimed convergence $\hat{\varphi}_T \to \bar{\varphi}$ in $W(0,\infty) \cap L^\infty(Q)$ follows.

Step II $v \in C_{\bar{u}}$. Using the local optimality of $\bar{u}$ we get

$$J'(\bar{u})v = \lim_{k\to\infty} J'(\bar{u})v_{T_k} = \lim_{k\to\infty} \frac{1}{\rho_k} J'(\bar{u})(\hat{u}_{T_k} - \bar{u}) \geq 0.$$

On the other side, using the convergence established in Step I and the convergence $\hat{u}_{T_k} \to \bar{u}$ in $L^2(Q_\omega)$ along with the local optimality of u_{T_k} we infer

$$\begin{aligned} J'(\bar{u})v &= \lim_{k\to\infty} \int_{Q_\omega} (\hat{\varphi}_{T_k} + \nu \hat{u}_{T_k}) v_{T_k} \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{k\to\infty} \frac{1}{\rho_k} \int_0^{T_k} \int_\omega (\varphi_{T_k} + \nu u_{T_k})(u_{T_k} - \bar{u}) \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{k\to\infty} \frac{1}{\rho_k} J'_{T_k}(u_{T_k})(u_{T_k} - \bar{u}) \leq 0. \end{aligned}$$

The last two inequalities imply that $J'(\bar{u})v = 0$. Now, the proof continues as in the Step I of the proof of Theorem 3.3.

Step III—Contradiction Since (u_{T_k}, φ_{T_k}) satisfies (2.10), we deduce the inequality $\mathcal{L}_{T_k}(\bar{u}, \lambda_{T_k}) < \mathcal{L}_{T_k}(u_{T_k}, \lambda_{T_k})$ with (4.3) and the fact that $\lambda_{T_k}(t)\|\bar{u}(t)\|_{L^2(\omega)} \leq \lambda_{T_k}(t)\gamma = \lambda_{T_k}(t)\|u_{T_k}(t)\|_{L^2(\omega)}$. Hence, performing a Taylor expansion and using (4.2) we infer

$$\begin{aligned} 0 > \mathcal{L}_{T_k}(\bar{u}, \lambda_{T_k}) - \mathcal{L}_{T_k}(u_{T_k}, \lambda_{T_k}) &= \frac{\partial \mathcal{L}_{T_k}}{\partial u}(u_{T_k}, \lambda_{T_k})(\bar{u} - u_{T_k}) \\ &\quad + \frac{1}{2}\frac{\partial^2 \mathcal{L}_{T_k}}{\partial u^2}(\bar{u} + \theta_k(u_{T_k} - \bar{u}), \lambda_{T_k})(\bar{u} - u_{T_k})^2 \\ &= \frac{1}{2}\frac{\partial^2 \mathcal{L}_{T_k}}{\partial u^2}(\bar{u} + \theta_k(u_{T_k} - \bar{u}), \lambda_{T_k})(\bar{u} - u_{T_k})^2. \end{aligned}$$

Dividing the above expression by $\rho_k^2/2$ we get

$$\frac{\partial^2 \mathcal{L}_{T_k}}{\partial u^2}(\bar{u} + \theta_k(u_{T_k} - \bar{u}), \lambda_{T_k})v_{T_k}^2 < 0.$$

We observe that for $k \to \infty$

$$\|\hat{\lambda}_{T_k} - \bar{\lambda}\|_{L^2(0,\infty)} \leq \|\hat{\varphi}_{T_k} - \bar{\varphi}\|_{L^2(0,\infty)} + \nu\|\hat{u}_{T_k} - \bar{u}\|_{L^2(0,\infty)} \to 0.$$

Setting $u_{\theta_k} = \bar{u} + \theta_k(u_{T_k} - \bar{u})$, we denote by y_{θ_k} the solution of (4.1) corresponding to the control u_{θ_k} and by φ_{θ_k} the corresponding adjoint state in $Q_{T_k,\omega}$. Then, putting $\psi_k = \hat{\varphi}_{\theta_k} - \bar{\varphi}$ we have

$$\begin{cases} -\dfrac{\partial \psi_k}{\partial t} - \Delta\psi_k + a\psi_k + \dfrac{\partial f}{\partial y}(x,t,\bar{y})\psi_k \\ = \left[\frac{\partial f}{\partial y}(x,t,\bar{y}) - \frac{\partial f}{\partial y}(x,t,\hat{y}_{\theta_k})\right]\hat{\varphi}_{\theta_k} + \chi_{T_k}(\hat{y}_{\theta_k} - \bar{y}) - (1-\chi_{T_k})(\bar{y} - y_d) \text{ in } Q, \\ \partial_n\psi_k = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\psi_k(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

Arguing as in Step I we obtain that $\psi_k \to 0$ in $W(0,\infty) \cap L^\infty(Q)$. Then, arguing as in Steps II and III of the proof of Theorem 3.3 and using the established convergences, we infer that $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})v^2 \leq 0$ and the contradiction follows. □

Under an extra assumption on f, the following theorem provides estimates for the difference $\hat{u}_T - \bar{u}$.

Theorem 4.4 *Suppose that U_{ad} is defined by (1.2) or (1.3) and that $\bar{u}$ is a local minimizer of (P) satisfying the second order sufficient optimality condition. We assume that $\frac{\partial f}{\partial y}(x,t,y) \geq 0$ holds for all $y \in \mathbb{R}$ and almost all $(x,t) \in Q$. Let $\{u_T\}_{T>T_0}$ be a sequence of local minimizers of problems (P_T)such that $\hat{u}_T \to \bar{u}$ in $L^2(Q_\omega)$. Then, there exist $T^* \in [T_0, \infty)$ and a constant C such that for every $T \geq T^*$*

$$\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)} + \|\hat{y}_T - \bar{y}\|_{W(0,\infty)} \leq \\ C\Big(\|y_T(T)\|_{L^2(\Omega)} + \|y_d\|_{L^2(T,\infty;L^2(\Omega))} + \|g\|_{L^2(T,\infty;L^2(\Omega))}\Big). \tag{4.4}$$

Proof We use the inequalities (3.9) or (3.17). For this purpose, we take $T^* \in [T_0^*, \infty)$ such that $\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)} < \varepsilon$ for all $T \geq T^*$, where T_0^* is introduced in Theorem 4.3. Then, given $T \geq T^*$, (3.9) or (3.17), and Theorem 4.3 yield

$$\begin{aligned} &\frac{\kappa}{2}\|\hat{u}_T - \bar{u}\|^2_{L^2(Q_\omega)} \leq J(\hat{u}_T) - J(\bar{u}) = J_T(u_T) - J_T(\bar{u}) \\ &\quad + \frac{1}{2}\int_T^\infty \|\hat{y}_T(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t - \frac{1}{2}\int_T^\infty \|\bar{y}(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\quad - \frac{\nu}{2}\int_T^\infty \|\bar{u}(t)\|^2_{L^2(\omega)}\,\mathrm{d}t \leq \frac{1}{2}\int_T^\infty \|\hat{y}_T(t) - y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t, \end{aligned}$$

which leads to

$$\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)} \leq \frac{1}{\sqrt{\kappa}}\|\hat{y}_T - y_d\|_{L^2(T,\infty;L^2(\Omega))}. \tag{4.5}$$

To prove the first estimate of (4.4) we observe that $\hat{y}_T$ satisfies the equation

$$\begin{cases} \dfrac{\partial \hat{y}_T}{\partial t} - \Delta \hat{y}_T + a\hat{y}_T + f(x,t,\hat{y}_T) = g \text{ in } \Omega \times (T,\infty), \\ \partial_n \hat{y}_T = 0 \text{ on } \Gamma \times (T,\infty),\ \ \hat{y}_T(T) = y_T(T) \text{ in } \Omega. \end{cases}$$

Testing this equation with $\hat{y}_T$, and using that $f(x,t,\hat{y}_T)\hat{y}_T \geq 0$ due to the monotonicity of f with respect to y and (1.4), it follows that

$$\begin{aligned} \frac{1}{2}\|\hat{y}_T(t)\|^2_{L^2(\Omega)} + \int_T^\infty \int_\Omega [|\nabla \hat{y}_T|^2 + a\hat{y}_T^2]\,\mathrm{d}x\,\mathrm{d}t \ &\leq \frac{1}{2}\|y_T(T)\|^2_{L^2(\Omega)} \\ &+ \int_T^\infty \int_\Omega g\hat{y}_T\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From this inequality we infer with (1.9) that

$$\|\hat{y}_T\|_{L^2(T,\infty;L^2(\Omega))} \leq C'\Big(\|y_T(T)\|_{L^2(\Omega)} + \|g\|_{L^2(T,\infty;L^2(\Omega))}\Big).$$

This inequality and (4.5) imply the estimate of the controls in (4.4). To get the estimate for the states we observe that $\phi_T = \hat{y}_T - \bar{y}$ satisfies the equation

$$\begin{cases} \dfrac{\partial \phi_T}{\partial t} - \Delta \phi_T + a\phi_T + \dfrac{\partial f}{\partial y}(x,t,y_{T,\theta})\phi_T = (\hat{u}_T - \bar{u})\chi_\omega \text{ in } Q, \\ \partial_n \phi_T = 0 \text{ on } \Sigma,\ \ \phi_T(0) = 0 \text{ in } \Omega, \end{cases}$$

where $y_{T,\theta} = \bar{y} + \theta_T(\hat{y}_T - \bar{y})$ with $\theta_T : Q \longrightarrow [0,1]$ measurable. Then, applying Theorem A.3 and Remark 5.2 we infer $\|\phi_T\|_{W(0,\infty)} \leq K_3\|\hat{u}_T - \bar{u}\|_{L^2(Q_\omega)}$. Combining this estimate with the one established for the controls we deduced (4.4). ◻

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Appendix

Here we prove $L^\infty(Q)$ estimates for the solution of the following equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g \text{ in } Q, \\ \partial_n y = 0 \text{ on } \Sigma,\ \ y(0) = y_0 \text{ in } \Omega, \end{cases} \tag{A.1}$$

assuming that $y_0 \in L^\infty(\Omega)$ and $g \in L^2(Q) \cap L^p(0,\infty; L^2(\Omega))$ with p satisfying (1.10).

Definition A.1 We call y a solution to (A.1) if $y \in L^2_{loc}(0,\infty; H^1(\Omega))$, and for every $T > 0$ the restriction of y to $Q_T = \Omega \times (0,T)$ belongs to $W(0,T) \cap L^\infty(Q_T)$ and satisfies the following equation in the variational sense

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay + f(x,t,y) = g \text{ in } Q_T, \\ \partial_n y = 0 \text{ on } \Sigma_T,\ \ y(0) = y_0 \text{ in } \Omega. \end{cases} \tag{A.2}$$

Since $y \in L^\infty(Q_T)$, we observe that (1.8) implies for $M_T = \|y\|_{L^\infty(Q_T)}$ and for almost all $(x,t) \in Q_T$

$$|f(x,t,y(x,t))| \le C_{M_T} M_T. \tag{A.3}$$

Theorem A.2 *Under the assumption (1.4)–(1.6), equation (A.1) has a unique solution y. In addition, if $y \in L^2(Q)$, then $y \in W(0,\infty) \cap L^\infty(Q)$ and $f(\cdot,\cdot,y) \in L^2(Q) \cap L^\infty(Q)$ holds. Moreover, the following estimates are satisfied*

$$\|y\|_Q \le K_1\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|y\|_{L^2(Q)}\Big), \tag{A.4}$$

$$\|y\|_{L^\infty(Q)} \le K_2\Big(\|y\|_{L^2(Q)} + \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(Q)} + \|g\|_{L^p(0,\infty;L^2(\Omega))} + M_f\Big), \tag{A.5}$$

$$\|f(\cdot,\cdot,y)\|_{L^\infty(Q)} \le C_{K_\infty}\|y\|_{L^\infty(Q)},\ \ \|f(\cdot,\cdot,y)\|_{L^2(Q)} \le C_{K_\infty}\|y\|_{L^2(Q)}, \tag{A.6}$$

$$\lim_{t\to\infty} \|y(t)\|_{L^2(\Omega)} = 0, \tag{A.7}$$

where M_f is given by (1.5), $K_\infty = \|y\|_{L^\infty(Q)}$, C_{K_∞} as in (1.6) with $M = K_\infty$, and

$$\|y\|_Q = \Big(\|y\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \|y\|^2_{L^2(0,\infty;H^1(\Omega))}\Big)^{\frac{1}{2}}.$$

Proof The existence and uniqueness of a solution y of (A.1) is a consequence of [10, Theorem 2.1]. Now, we assume that $y \in L^2(Q)$. The proof is split into several steps.

Step I $y \in L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$. Testing equation (A.2) with y, integrating in $(0,t)$ with $t \in (0,T)$, using (1.5), and arguing as in (A.3) we get

$$\begin{aligned}
&\frac{1}{2}\|y(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega [|\nabla y|^2 + ay^2]\,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega g\, y\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_f(s)} |f(x,s,y)||y|\,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac{1}{2}\|y_0\|^2_{L^2(\Omega)} + \|g\|_{L^2(Q)}\|y\|_{L^2(Q)} + C_{M_f}\|y\|^2_{L^2(Q)},
\end{aligned}$$

where $\Omega_f(t) = \{x \in \Omega : |y(x,t)| < M_f\}$. This inequality along with (1.9) proves that $y \in L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ and (A.4) holds.

Step II $y \in L^\infty(Q)$. Let us first observe that without loss of generality we may suppose that $y_0 \in H^1(\Omega)$. Indeed, if this is not the case we use the fact that $f(\cdot,\cdot,y) \in L^2(Q_T)$ by (A.3) and $g \in L^2(Q_T)$ to deduce that $y \in C([T_0,T]; H^1(\Omega))$ for each $0 < T_0 < T < \infty$; see, for instance, [17, Corollary III.2.4]. Since $y \in L^\infty(Q_{T_0})$ for each $0 < T_0 < \infty$, it is enough to prove that $y \in L^\infty(\Omega \times (T_0,\infty))$. Then there is no loss of generality if we assume that $y_0 \in H^1(\Omega)$ and, consequently, $y \in H^1(Q_T)$ for every $T < \infty$; see [17, Proposition III.2.5].

For every real number $\rho \ge \max\{\|y_0\|_{L^\infty(\Omega)}, M_f\}$, M_f given by (1.5), we introduce the function $y_\rho(x,t) = y(x,t) - \mathrm{Proj}_{[-\rho,+\rho]}(y(x,t))$. Then, we still have that $y_\rho \in H^1(Q_T)$ for all $T < \infty$. We set $A_\rho(t) = \{x \in \Omega : |y(x,t)| > \rho\}$ for every $t \in (0,\infty)$.

First we prove the result for $n = 2$ or 3. Let us choose a number $\alpha \in \left(\frac{pn}{2p-4}, \frac{n}{n-2}\right)$. Observe that $\frac{pn}{2p-4} < \frac{n}{n-2}$ obviously holds if $n = 2$ and it is also true for $n = 3$ due to the assumption $p > \frac{4}{4-n}$. Note that α satisfies $\alpha \in (1,\infty)$ in case $n = 2$, and $\alpha \in (\frac{3}{2}, 3)$ in case $n = 3$. We will denote $\alpha' = \frac{\alpha}{\alpha-1}$.

Testing equation (A.2) with y_ρ, integrating in $(0,t)$ with $t \in (0,T)$, and using that $\frac{\partial y}{\partial t} y_\rho = \frac{\partial y_\rho}{\partial t} y_\rho$, $\nabla y \cdot \nabla y_\rho = |\nabla y_\rho|^2$, and $f(x,t,y)y_\rho \ge 0$ due to (1.5) and $\rho \ge M_f$, we infer

$$\begin{aligned}
&\frac{1}{2}\|y_\rho(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega [|\nabla y_\rho|^2 + ay_\rho^2]\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega g\, y_\rho\,\mathrm{d}x\,\mathrm{d}s \\
&\le \|g\|_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^t \|y_\rho(s)\|^{p'}_{L^2(A_\rho(s))}\,\mathrm{d}s\Big)^{\frac{1}{p'}} \\
&\le \|g\|_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^t \|y_\rho(s)\|^{p'}_{L^{2\alpha}(A_\rho(s))}|A_\rho(s)|^{\frac{p'}{2\alpha'}}\,\mathrm{d}s\Big)^{\frac{1}{p'}} \\
&\le C_1\|g\|_{L^p(0,\infty;L^2(\Omega))}\left(\int_0^t \|y_\rho(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t |A_\rho(s)|^{\frac{p'}{\alpha'(2-p')}}\,\mathrm{d}s\right)^{\frac{2-p'}{2p'}} \\
&\le \frac{C_a^2}{2}\int_0^t \|y_\rho(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s + \frac{C_1^2}{2C_a^2}\|g\|^2_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^t |A_\rho(s)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}s\Big)^{\frac{p-2}{p}},
\end{aligned}$$

where C_a is given by (1.9) and C_1 is the embedding constant of $H^1(\Omega) \subset L^{2\alpha}(\Omega)$. The above estimate leads to

$$\begin{aligned}\|y_\rho(t)\|^2_{L^2(\Omega)} &+ C_a^2 \int_0^t \|y_\rho(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s \\ &\leq \frac{C_1^2}{C_a^2}\|g\|^2_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^t |A_\rho(s)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}s\Big)^{\frac{p-2}{p}}.\end{aligned}$$

From the above inequalities we get

$$\|y_\rho\|_{Q_T} \leq C_2\|g\|_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^T |A_\rho(s)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}s\Big)^{\frac{p-2}{2p}} \tag{A.8}$$

with C_2 independent of T and ρ.

On the other hand, taking $\kappa = 1 + \frac{2\alpha'(p-2)}{pn}$, $r = \frac{2p\kappa}{\alpha'(p-2)}$, and $q = 2\kappa$ we get $\frac{1}{r} + \frac{n}{2q} = \frac{n}{4}$. Then, using [16, Formula II-(3.4)] we obtain the existence of a constant C_3 independent of T such that

$$\|y\|_{L^r(0,T;L^q(\Omega))} \leq C_3\|y\|_{Q_T}. \tag{A.9}$$

For every $j = 0, 1, 2, \dots$ we set $k_j = \rho(2 - 2^{-j})$. We observe that $\rho \leq k_j \leq 2\rho$, $A_\rho(t) = A_{k_0}(t) \supset A_{k_1}(t) \supset A_{k_2}(t) \supset \dots$, and $A_{k_j}(t) \supset A_{2\rho}(t)$ for every $j \geq 0$. Then, we have

$$\begin{aligned}\|y_{k_j}\|_{L^r(0,T;L^q(\Omega))} &= \Big(\int_0^T \|y_{k_j}\|^r_{L^q(A_{k_j}(t))}\,\mathrm{d}t\Big)^{\frac{1}{r}} \\ &\geq \Big(\int_0^T \|y_{k_j}\|^r_{L^q(A_{k_{j+1}}(t))}\,\mathrm{d}t\Big)^{\frac{1}{r}} \geq (k_{j+1} - k_j) \\ &\quad\times \Big(\int_0^T |A_{k_{j+1}}(t)|^{\frac{r}{q}}\,\mathrm{d}t\Big)^{\frac{1}{r}} \\ &= (k_{j+1} - k_j)\Big(\int_0^T |A_{k_{j+1}}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\Big)^{\frac{\alpha'(p-2)}{2p\kappa}}.\end{aligned}$$

Combining this inequality with (A.8) and (A.9) we deduce

$$\begin{aligned}&\Big(\int_0^T |A_{k_{j+1}}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\Big)^{\frac{\alpha'(p-2)}{2p\kappa}} \\ &\leq \frac{C_2C_3}{k_{j+1} - k_j}\|g\|_{L^p(0,\infty;L^2(\Omega))}\Big(\int_0^T |A_{k_j}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\Big)^{\frac{p-2}{2p}}.\end{aligned}$$

Using that $k_{j+1} - k_j = \rho 2^{-(j+1)}$ we infer

$$\left(\int_0^T |A_{k_{j+1}}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\right)^{\frac{p-2}{2p}}$$
$$\leq \left[\frac{2C_2C_3}{\rho}\|g\|_{L^p(0,\infty;L^2(\Omega))}\right]^{\frac{\kappa}{\alpha'}}(2^{\frac{\kappa}{\alpha'}})^j\left[\left(\int_0^T |A_{k_j}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\right)^{\frac{p-2}{2p}}\right]^{\frac{\kappa}{\alpha'}}.$$

Setting

$$c = \left[\frac{2C_2C_3}{\rho}\|g\|_{L^p(0,\infty;L^2(\Omega))}\right]^{\frac{\kappa}{\alpha'}},\ b = 2^{\frac{\kappa}{\alpha'}},\ \xi_j = \left(\int_0^T |A_{k_j}(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\right)^{\frac{p-2}{2p}}$$

for $j \geq 0$, we have $\xi_{j+1} \leq cb^j\xi_j^{\frac{\kappa}{\alpha'}}$ for $j \geq 0$. Moreover, since $\alpha > \frac{pn}{2p-4}$ we get that $\beta = \frac{\kappa}{\alpha'} > 1$. Then, from [16, LemmaII−5.6] we obtain

$$\xi_j \leq c^{\frac{\beta^j-1}{\beta-1}}b^{\frac{\beta^j-1}{(\beta-1)^2}-\frac{j}{\beta-1}}\xi_0^{\beta^j}. \tag{A.10}$$

Let us estimate ξ_0. For this purpose we distinguish two cases. First, we assume that $n = 2$ and $p \in (2, 4]$. Using that $|A_\rho(t)| \leq \frac{1}{\rho^2}\|y(t)\|^2_{L^2(A_\rho(t))}$, we get

$$\xi_0 = \left(\int_0^T |A_\rho(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\right)^{\frac{p-2}{2p}} \leq \frac{1}{\rho^{\frac{1}{\alpha'}}}\left(\int_0^T \|y(t)\|^{\frac{2p}{\alpha'(p-2)}}_{L^2(\Omega)}\,\mathrm{d}t\right)^{\frac{p-2}{2p}} \leq \left(\frac{C_4}{\rho}\|y\|_Q\right)^{\frac{1}{\alpha'}}.$$

The last inequality follows from the fact $y \in L^2(Q) \cap L^\infty(0,\infty; L^2(\Omega))$ and $\frac{2p}{\alpha'(p-2)} \geq 2$ because $\alpha' \in (1, \frac{p}{2})$ and $\frac{p}{2} \leq \frac{p}{p-2}$ for $p \leq 4$. For the remaining cases we observe that $\frac{2\alpha'(p-2)}{p} > 1$ and additionally $\frac{2\alpha'(p-2)}{p} < 6$ if $n = 3$. Now, we argue as follows

$$\xi_0 = \left(\int_0^T |A_\rho(t)|^{\frac{p}{\alpha'(p-2)}}\,\mathrm{d}t\right)^{\frac{p-2}{2p}} \leq \frac{1}{\rho^{\frac{p-2}{p}}}\left(\int_0^T \|y(t)\|^2_{L^{\frac{2\alpha'(p-2)}{p}}(A_\rho(t))}\,\mathrm{d}t\right)^{\frac{p-2}{2p}}$$
$$\leq \left(\frac{C_4}{\rho}\|y\|_{L^2(0,\infty;H^1(\Omega))}\right)^{\frac{p-2}{p}} \leq \left(\frac{C_4}{\rho}\|y\|_Q\right)^{\frac{p-2}{p}}.$$

Selecting

$$\rho = C_4 b^{\frac{1}{(\beta-1)^2}}\|y\|_Q + 2C_2C_3\|g\|_{L^p(0,\infty;L^2(\Omega))} + \|y_0\|_{L^\infty(\Omega)} + M_f$$

we get with (A.10)

$$\xi_j \leq [c^{\frac{1}{\beta-1}}b^{\frac{1}{(\beta-1)^2}}\xi_0]^{\beta^j}c^{-\frac{1}{\beta-1}}b^{-\frac{1}{(\beta-1)^2}}b^{-\frac{j}{\beta-1}} \leq c^{-\frac{1}{\beta-1}}b^{-\frac{1}{(\beta-1)^2}}b^{-\frac{j}{\beta-1}} \to 0$$

as $j \to \infty$. Finally, we get

$$\left(\int_0^T |A_{2\rho}(t)|^{\frac{p}{\alpha'(p-2)}} \, \mathrm{d}t \right)^{\frac{p-2}{2p}} \leq \lim_{j\to\infty} \xi_j = 0.$$

Hence, $|A_{2\rho}(t)| = 0$ for almost every $t \in (0, T)$ holds. Since $T > 0$ was arbitrarily selected and all the constants above are independent of T, we deduce that $|y(x,t)| \leq 2\rho$ for almost all $(x,t) \in Q$ and (A.5) follows with (A.4).

Now, we explain the changes in the proof for the case $n = 1$. To get an analogous inequality to (A.8), we use the following Gagliardo-Nirenberg inequality

$$\|y\|_{L^\infty(\Omega)} \leq C\|y\|^{\frac{1}{2}}_{H^1(\Omega)}\|y\|^{\frac{1}{2}}_{L^2(\Omega)};$$

see, for instance, [6, P. 233]. Then, we have with Hölder and Young inequalities

$$\begin{aligned}
&\frac{1}{2}\|y_\rho(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega [|\nabla y_\rho|^2 + a y_\rho^2] \, \mathrm{d}x \, \mathrm{d}s \leq \int_0^t \int_\Omega g \, y_\rho \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \|g\|_{L^2(0,\infty;L^2(\Omega))} \left(\int_0^t \|y_\rho(s)\|^2_{L^2(A_\rho(s))} \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq \|g\|_{L^2(0,\infty;L^2(\Omega))} \left(\int_0^t \|y_\rho(s)\|^2_{L^\infty(\Omega)} |A_\rho(s)| \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq C\|g\|_{L^2(0,\infty;L^2(\Omega))} \left(\int_0^t \|y_\rho(s)\|_{L^2(\Omega)} \|y_\rho(s)\|_{H^1(\Omega)} |A_\rho(s)| \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq C\|g\|_{L^2(0,\infty;L^2(\Omega))} \|y_\rho\|^{\frac{1}{2}}_{L^\infty(0,\infty;L^2(\Omega))} \|y_\rho\|^{\frac{1}{2}}_{L^2(0,\infty;H^1(\Omega))} \left(\int_0^t |A_\rho(s)|^2 \, \mathrm{d}s \right)^{\frac{1}{4}} \\
&\leq \varepsilon\big(\|y_\rho\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \|y_\rho\|^2_{L^2(0,\infty;H^1(\Omega))} \big) \\
&\quad + C_\varepsilon \|g\|^2_{L^2(0,\infty;L^2(\Omega))} \left(\int_0^t |A_\rho(s)|^2 \, \mathrm{d}s \right)^{\frac{1}{4}}.
\end{aligned}$$

From here we infer

$$\|y_\rho\|_{Q_T} \leq C_2\|g\|_{L^2(0,\infty;L^2(\Omega))} \left(\int_0^T |A_\rho(s)|^2 \, \mathrm{d}s \right)^{\frac{1}{4}}.$$

On the other side, we apply (A.9) with $r = 8$ and $q = 4$ and arguing as for the cases $n = 2$ or 3 we obtain

$$\begin{aligned}
\|y_{k_j}\|_{Q_T} &\geq \frac{1}{C_3}(k_{j+1} - k_j) \left(\int_0^T |A_{k_{j+1}}(t)|^{\frac{r}{q}} \, \mathrm{d}t \right)^{\frac{1}{r}} \\
&= \frac{1}{C_3}(k_{j+1} - k_j) \left(\int_0^T |A_{k_{j+1}}(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{8}}.
\end{aligned}$$

Thus we get

$$\left(\int_0^T |A_{k_{j+1}}(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \leq \frac{C_4\|g\|^4_{L^2(0,\infty;L^2(\Omega))}}{(k_{j+1}-k_j)^4}\left[\left(\int_0^T |A_{k_j}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}\right]^2.$$

To estimate ξ_0 we use again that $y \in L^2(Q) \cap L^\infty(0,\infty; L^2(\Omega))$ and proceed as follows

$$\xi_0 = \left(\int_0^T |A_\rho(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \leq \frac{1}{\rho^2}\left(\int_0^T \|y_\rho(t)\|^4_{L^2(A_\rho(t))} \,\mathrm{d}t\right)^{\frac{1}{2}} \leq \left(\frac{C}{\rho}\|y\|_Q\right)^2.$$

The rest of the proof follows as for the cases $n = 2$ or 3.

Step III—Proof of (A.6) *and* (A.7). The inequalities of (A.6) are an immediate consequence of (1.4), (1.6), and the mean value theorem:

$$|f(x,t,y(x,t)| = \left|\frac{\partial f}{\partial y}(x,t,\theta(x,t)y(x,t))\right| |y(x,t)|, \tag{A.11}$$

with $0 \leq \theta(x,t) \leq 1$.

Since $y \in L^2(0,\infty; H^1(\Omega))$, we have that $\Delta y \in L^2(0,\infty; H^1(\Omega)^*)$. From the state equation and $g, f(\cdot,\cdot,y) \in L^2(Q)$ we infer that $\frac{\partial y}{\partial t} \in L^2(0,\infty; H^1(\Omega)^*)$ and, hence, $y \in W(0,\infty)$. Finally, the fact that $y \in W(0,\infty)$ implies (A.7); see [9, Theorem 2.4] for details. □

Remark 5.1 The proof of the boundedness of y in Q follows some ideas of the proof of [16, Theorem III−7.1]. In that theorem, the boundedness is established for finite time horizon and the $L^\infty(Q_T)$ estimates depend on time T. In our theorem, we have avoided the dependence with respect to time exploiting the fact that $y \in L^2(Q)$, which was used to estimate ξ_0. By a simple modification of our proof, the $L^\infty(Q)$ estimate of y can be also obtained in terms of $\|g\|_{L^r(0,\infty;L^q(\Omega))}$ if $\frac{1}{r} + \frac{n}{2q} < 1$. We observe that the assumption $y \in L^2(Q)$ is natural in the context of our optimal control problem due to the structure of its cost functional. Another difference of our estimates with respect to [16, Theorem III−7.1] concerns the choice of the boundary condition. Here we have treated the Neumann case while the Dirichlet case was considered in the mentioned reference. The only difference in our proof for the Dirichlet case consists in the definition of ρ that should include the $L^\infty(\Sigma)$ norm of the Dirichlet datum, if it is not zero.

Now, we analyze the following linear equation

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az + d(x,t,y)z = h \text{ in } Q, \\ \partial_n z = 0 \text{ on } \Sigma,\ z(0) = z_0 \text{ in } \Omega. \end{cases} \tag{A.12}$$

We assume that

$$y \in L^\infty(Q) \text{ and } \lim_{t\to\infty} \|y(t)\|_{L^2(\Omega)} = 0, \tag{A.13}$$

and that $d : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying

$$\forall M > 0\, \exists C_M \text{ such that } |d(x,t,s)| \le C_M \ \forall |s| \le M, \tag{A.14}$$

$$\begin{cases} \exists m_d > 0,\ \exists \delta_d \in [0,1), \text{ and } \exists C_d > 0 \text{ such that} \\ d(x,t,s) \ge -C_d|s| - \delta_d a(x,t)\ \forall |s| \le m_d, \end{cases} \tag{A.15}$$

for almost all $(x,t) \in Q$.

Theorem A.3 *Under assumptions* (A.13)–(A.15)*, equation* (A.12) *has a unique solution* $z \in W(0,\infty)$ *for every* $h \in L^2(0,\infty; H^1(\Omega)^*)$ *and* $z_0 \in L^2(\Omega)$*, which satisfies*

$$\|z\|_{W(0,\infty)} \le K_{3,y}\Big(\|h\|_{L^2(0,\infty;H^1(\Omega)^*)} + \|z_0\|_{L^2(\Omega)}\Big), \tag{A.16}$$

where $K_{3,y}$ *depends on* $\|y\|_{L^\infty(Q)}$*. In addition, if* $h \in L^2(Q) \cap L^p(0,\infty; L^2(\Omega))$ *with* $p \in (\frac{4}{4-n}, \infty]$ *and* $z_0 \in L^\infty(\Omega)$*, then* $z \in L^\infty(Q)$ *and the estimate*

$$\|z\|_{L^\infty(Q)} \le K_{4,y}\Big(\|h\|_{L^2(Q)} + \|h\|_{L^p(0,\infty;L^2(\Omega))} + \|z_0\|_{L^\infty(\Omega)}\Big) \tag{A.17}$$

holds for a constant $K_{4,y}$ *also depending on* $\|y\|_{L^\infty(Q)}$.

Proof Due to the fact that $y \in L^\infty(Q)$ and (A.14) we have that $d(\cdot,\cdot,y) \in L^\infty(Q)$, and hence the existence and uniqueness of $z \in W(0,T) \cap L^\infty(Q_T)$ holds for every $T < \infty$. Let us prove that $z \in W(0,\infty)$. We put $K = \|y\|_{L^\infty(Q)}$ and $C_K = \|d(\cdot,\cdot,y)\|_{L^\infty(Q)}$. Given $\delta_d \in [0,1)$ we know that there exists a constant C_{a,δ_d} such that

$$C_{a,\delta_d}\|w\|_{H^1(\Omega)} \le \left(\int_\Omega [|\nabla w|^2 + (1-\delta_d)aw^2]\,\mathrm{d}x\right)^{\frac{1}{2}} \quad \forall w \in H^1(\Omega). \tag{A.18}$$

We select $\varepsilon > 0$ such that $\max\{C_d, \frac{C_K}{m_d}\}C_1^2\varepsilon \le \frac{C^2_{a,\delta_d}}{4}$, where m_d and C_d are given in (A.15) and C_1 is the embedding constant for $H^1(\Omega) \subset L^4(\Omega)$. Using (A.13) we deduce the existence of $T_\varepsilon > 0$ such that

$$\|y(t)\|_{L^2(\Omega)} \le \varepsilon \quad \forall t \ge T_\varepsilon. \tag{A.19}$$

For $t > 0$ we set $\Omega_{m_d}(t) = \{x \in \Omega : |y(x,t)| \le m_d\}$. Now, we test (A.12) with z and integrate over $\Omega \times (T_\varepsilon, t)$ for every $t > T_\varepsilon$, use assumption (A.15), and (A.19)

$$\begin{aligned} &\frac{1}{2}\|z(t)\|^2_{L^2(\Omega)} + C^2_{a,\delta_d}\int_{T_\varepsilon}^t \|z(t)\|^2_{H^1(\Omega)}\,\mathrm{d}s \\ &\le \frac{1}{2}\|z(t)\|^2_{L^2(\Omega)} + \int_{T_\varepsilon}^t\int_\Omega [|\nabla z|^2 + (1-\delta_d)az^2]\,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} \\ &\quad + \int_{T_\varepsilon}^t \langle h(s), z(s)\rangle\,\mathrm{d}s + C_d\int_{T_\varepsilon}^t\int_{\Omega_{m_d}(t)} |y|z^2\,\mathrm{d}x\,\mathrm{d}s + C_K\int_{T_\varepsilon}^t\int_{\Omega\setminus\Omega_{m_d}(t)} z^2\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \|h\|_{L^2(0,\infty;H^1(\Omega)^*)}\|z\|_{L^2(T_\varepsilon,t;H^1(\Omega))} \\
&\quad + \max\{C_d, \frac{C_K}{m_d}\} \int_{T_\varepsilon}^t \int_\Omega |y|z^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{C^2_{a,\delta_d}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)} + \frac{C^2_{a,\delta_d}}{4}\|z\|^2_{L^2(T_\varepsilon,t;H^1(\Omega))} \\
&\quad + \max\{C_d, \frac{C_K}{m_d}\} \int_{T_\varepsilon}^t \|y(s)\|_{L^2(\Omega)}\|z(s)\|^2_{L^4(\Omega)}\,\mathrm{d}s \\
&\leq \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{C^2_{a,\delta_d}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)} \\
&\quad + \frac{C^2_{a,\delta_d}}{4}\int_{T_\varepsilon}^t \|z\|^2_{H^1(\Omega)}\,\mathrm{d}s + C_1^2\max\{C_d, \frac{C_K}{m_d}\}\varepsilon \int_{T_\varepsilon}^t \|z(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s \\
&\leq \frac{1}{2}\|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{C^2_{a,\delta_d}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)} + \frac{C^2_{a,\delta_d}}{2}\int_{T_\varepsilon}^t \|z\|^2_{H^1(\Omega)}\,\mathrm{d}s.
\end{aligned}
$$

This implies

$$
\|z(t)\|^2_{L^2(\Omega)} + C^2_{a,\delta_d}\int_{T_\varepsilon}^t \|z(t)\|^2_{H^1(\Omega)}\,\mathrm{d}s \leq \|z(T_\varepsilon)\|^2_{L^2(\Omega)} + \frac{2}{C^2_{a,\delta_d}}\|h\|^2_{L^2(0,\infty;H^1(\Omega)^*)}.
$$

Since z solves (A.12) in $(0, T_\varepsilon)$, we have $z \in W(0, T_\varepsilon)$ and $\|z\|_{W(0,T_\varepsilon)}$ can be estimated by $\|h\|_{L^2(0,\infty;H^1(\Omega)^*)} + \|z_0\|_{L^2(\Omega)}$. This along with the above estimate implies the desired estimate of z in $L^2(0, \infty; H^1(\Omega)) \cap L^\infty(0, \infty; L^2(\Omega))$. From the equation (A.12) we infer that $\frac{\partial z}{\partial t} \in L^2(0, \infty; H^1(\Omega)^*)$ and estimate (A.16) follows.

Finally, under the additional regularity of h and z_0, applying Theorem A.2 to the equation

$$
\frac{\partial z}{\partial t} - \Delta z + az = g = h - d(x, t, y)z \in L^p(0, \infty; L^2(\Omega)) \cap L^2(Q)
$$

with $f = 0$ and $M_f = 0$ there, we infer that $z \in L^\infty(Q)$ and (A.17) holds. Here we have used that $L^\infty(0, \infty; L^2(\Omega)) \cap L^2(Q) \subset L^p(0, \infty; L^2(\Omega))$ for every $p \geq 2$. □

We finish this appendix by analyzing the following adjoint equation

$$
\begin{cases}
-\dfrac{\partial\varphi}{\partial t} - \Delta\varphi + a\varphi + d(x, t, y)\varphi = h \text{ in } Q, \\
\partial_n\varphi = 0 \text{ on } \Sigma,\ \lim_{t\to\infty}\|\varphi(t)\|_{L^2(\Omega)} = 0.
\end{cases}
\tag{A.20}
$$

Theorem A.4 *Under assumptions (A.13)–(A.15), equation (A.20) has a unique solution $\varphi \in W(0, \infty)$ for all $h \in L^2(0, \infty; H^1(\Omega)^*)$ which satisfies*

$$
\|\varphi\|_{W(0,\infty)} \leq K_{5,y}\|h\|_{L^2(0,\infty;H^1(\Omega)^*)},
\tag{A.21}
$$

where $K_{5,y}$ depends on $\|y\|_{L^\infty(Q)}$. In addition, if $h \in L^2(Q) \cap L^p(0,\infty; L^2(\Omega))$ with $p \in (\frac{4}{4-n}, \infty]$, then $\varphi \in L^\infty(Q)$ and the estimate

$$\|\varphi\|_{L^\infty(Q)} \leq K_{6,y}\Big(\|h\|_{L^2(Q)} + \|h\|_{L^p(0,\infty;L^2(\Omega))}\Big) \tag{A.22}$$

holds for a constant $K_{6,y}$ also depending on $\|y\|_{L^\infty(Q)}$.

Proof First, we prove uniqueness. For this purpose we establish that the unique solution to (A.20) with $h = 0$ is $\varphi = 0$. Indeed, in this case, we take $z \in W(0,\infty)$ solution to (A.12) with $h = \varphi$ and $z_0 = 0$. Then we have,

$$\begin{aligned}\int_Q |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_Q [\frac{\partial z}{\partial t} - \Delta z + az + d(x,t,y)z]\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q z[-\frac{\partial \varphi}{\partial t} - \Delta\varphi + a\varphi + d(x,t,y)\varphi]\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

To prove the existence of a solution we denote by $\varphi_T \in W(0,T)$ the solutions of

$$\begin{cases} -\dfrac{\partial \varphi_T}{\partial t} - \Delta\varphi_T + a\varphi_T + d(x,t,y)\varphi_T = h \text{ in } Q_T, \\ \partial_n\varphi_T = 0 \text{ on } \Sigma_T,\ \varphi_T(T) = 0 \text{ in } \Omega. \end{cases} \tag{A.23}$$

The existence and uniqueness of φ_T is known because the function $d(x,t,y(x,t))$ is bounded. We extend φ_T by 0 to $(0,\infty)$ and estimate this extension in $L^2(0,\infty; H^1(\Omega))$ independently of T. For this purpose we take $\phi \in L^2(0,\infty; H^1(\Omega)^*)$ arbitrary and denote by z the solution of (A.12) with $h = \phi$ and $z_0 = 0$. Then, we have with (A.16)

$$\begin{aligned}\int_0^\infty \langle\phi(t), \varphi_T(t)\rangle \,\mathrm{d}t &= \int_0^T \langle\phi(t), \varphi_T(t)\rangle \,\mathrm{d}t = \int_0^T \langle\frac{\partial z}{\partial t} - \Delta z + az + d(x,t,y)z, \varphi_T\rangle \,\mathrm{d}t \\ &= \int_0^T \langle -\frac{\partial \varphi_T}{\partial t} - \Delta\varphi_T + a\varphi_T + d(x,t,y)\varphi_T, z\rangle \,\mathrm{d}t = \int_0^T \langle h, z\rangle \,\mathrm{d}t \\ &= \int_0^\infty \langle h, z\rangle \,\mathrm{d}t \leq \|h\|_{L^2(0,\infty;H^1(\Omega)^*)}\|z\|_{L^2(0,\infty;H^1(\Omega))} \\ &\leq K_{3,y}\|h\|_{L^2(0,\infty;H^1(\Omega)^*)}\|\phi\|_{L^2(0,\infty;H^1(\Omega)^*)}.\end{aligned}$$

This implies that

$$\|\varphi_T\|_{L^2(0,\infty;H^1(\Omega))} \leq K_{3,y}\|h\|_{L^2(0,\infty;H^1(\Omega)^*)} \quad \forall T > 0.$$

From (A.23) and the above estimate we deduce the boundedness of $\{\varphi_T\}_{T>0}$ in $W(0,\infty)$. Then, there exists a sequence $\{T_k\}_{k=1}^\infty$ with $T_k \to \infty$ and a function $\varphi \in W(0,\infty)$ such that $\varphi_{T_k} \rightharpoonup \varphi$ in $W(0,\infty)$ as $k \to \infty$. It is obvious that we can pass to the limit in (A.23) and deduce that φ satisfies (A.20) and estimate (A.21) holds.

To prove that $\varphi \in L^\infty(Q)$ under the additional regularity assumption on h we introduce the functions $z_T(x,t) = \varphi_T(x, T-t)$ for every $T > 0$. Then $z_T \in W(0,T)$ and it satisfies (A.12) in Q_T with $z_0 = 0$ and $h_T(x,t) = h(x, T-t)$. Since

$$\|z_T\|_{L^2(Q_T)} = \|\varphi_T\|_{L^2(Q_T)} \le \|\varphi_T\|_{L^2(0,\infty;H^1(\Omega))} \le K_{3,y}\|h\|_{L^2(0,\infty;H^1(\Omega)^*)},$$

$\|h_T\|_{L^2(Q_T)} = \|h\|_{L^2(Q_T)} \le \|h\|_{L^2(Q)}$ and $\|h_T\|_{L^p(0,\infty;L^2(\Omega))} \le \|h\|_{L^p(0,\infty;L^2(\Omega))}$, we infer from Theorem A.2 that $\{\varphi_T\}_{T>0}$ is uniformly bounded in $L^\infty(Q)$ and, consequently, estimate (A.22) holds. ◻

Remark 5.2 If the function f in (A.1) satisfies $\frac{\partial f}{\partial y}(x,t,y) \ge 0$ for every $y \in \mathbb{R}$ and almost all $(x,t) \in Q$, then the term $\|y\|_{L^2(Q)}$ in the estimates (A.4) and (A.5) can be removed. Under this assumption on f, the constants M_f and C_{M_f} in (1.5) and (1.7) are zero. Then, it is enough to use this in the proof of Theorem A.2 to get the independence of the estimates with respect to y.

Moreover, by an analogous argument, if the assumptions (A.13)–(A.15) are replaced by

$$y \in L^\infty(Q) \text{ and } \forall M > 0\, \exists C_M \text{ such that } 0 \le d(x,t,s) \le C_M \ \forall |s| \le M$$

and almost all $(x,t) \in Q$, then the constants $K_{3,y}$ until $K_{6,y}$ in the estimates (A.16), (A.17), (A.21), and (A.22) can be chosen independently of y.

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