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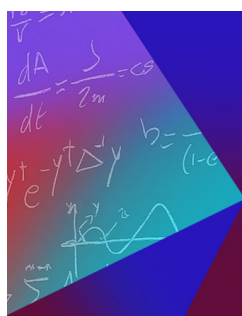**Articles You May Be Interested In**

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The relativistic spherical δ-shell interaction in $\mathbb{R}^3$: Spectrum and approximation

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This note revolves on the free Dirac operator in $\mathbb{R}^3$ and its δ-shell interaction with electrostatic potentials supported on a sphere. On one hand, we characterize the eigenstates of those couplings by finding sharp constants and minimizers of some precise inequalities related to an uncertainty principle. On the other hand, we prove that the domains given by Dittrich *et al.* [J. Math. Phys. **30**(12), 2875–2882 (1989)] and by Arrizabalaga *et al.* [J. Math. Pures Appl. **102**(4), 617–639 (2014)] for the realization of an electrostatic spherical shell interaction coincide. Finally, we explore the spectral relation between the shell interaction and its approximation by short range potentials with shrinking support, improving previous results in the spherical case. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.5000381]

I. INTRODUCTION

Since 1929, when Paul Dirac formulated it, the "Dirac equation" has played a fundamental role in quantum mechanics attracting the attention of both mathematicians and physicists. The newness of Dirac's approach comes from the fact that he could describe the behaviour of the particles by using a 4-component spinor. With this approach, one can observe the existence of negative-energy particles, see Ref. 17 and the references therein. By the way, it is not clear whether one should interpret the Dirac equation as a quantum mechanical evolution equation, like the Schrödinger equation for a single particle. An example that highlights this difference is, for instance, the comparison between the Schrödinger operator and the Dirac operator coupled with a singular potential supported on a hyper-surface: the so-called δ interaction.

The idea of coupling Hamiltonians with singular potentials supported on subsets of lower dimension with respect to the ambient space is quite classic in quantum mechanics. For the Schrödinger operator, this problem is described in the monograph[1] for finite and infinite δ-point interactions and in Ref. 9 for singular potentials supported on hyper-surfaces. Regarding the Dirac operator, in the 1-dimensional case, the problem is well-understood. Thanks to Refs. 1, 10, and 11, we get the description of the domain, some properties of the spectrum, and a resolvent formula. In three dimensions, the first result is Ref. 8. By using the decomposition into spherical harmonics, they can reduce their analysis to a 1-dimensional question and they can construct the domain of the Dirac operator coupled with a singular potential supported on the sphere. In the case of a general surface Σ, the first work is Ref. 2. In this work, they characterize the domain of the δ-shell Dirac operator with coupling constant $\lambda \neq \pm 2$, by the interactions between certain functions $u \in H^1(\mathbb{R}^3)$ and $g \in L^2(\Sigma)$. Comparing this work with the general abstract theory given in Ref. 15, one could suppose that this kind of interaction is forcing g to be in $H^{1/2}(\Sigma)$. Indeed, recently, in Ref. 14, they proved that this conjecture is true. Moreover, they also define the domain of δ-shell Dirac operator when the coupling constant $\lambda = \pm 2$. Finally, in Refs. 5 and 7, they can define the domain of the δ-shell Dirac operator by using the abstract theory of boundary triples.

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Nevertheless, one has to keep in mind that, even if this kind of model is easier to be mathematically understood, it is an ideal model that cannot be physically reproduced. This is the reason why it is interesting to approximate this kind of operators by more regular ones. And the results we obtain show the difference between the Schrödinger operator and the Dirac operator. For instance, in one dimension, if $V \in C_c^\infty(\mathbb{R})$, then

$$V_\epsilon(t) := \frac{1}{\epsilon} V\left(\frac{t}{\epsilon}\right) \to (\textstyle\int V)\delta_0 \quad \text{when } \epsilon \to 0$$

in the sense of distributions, where δ_0 denotes the Dirac measure at the origin. In Ref. 1, it is proved that $\Delta + V_\epsilon \to \Delta + (\int V)\delta_0$ in the norm resolvent sense when $\epsilon \to 0$, and in Ref. 6, this result is generalized to higher dimensions for singular perturbations on general smooth hyper-surfaces.

These kinds of results do not hold for the Dirac operator, and it is physically known as "Klein's paradox." In fact, in Ref. 16, it is proved that, in the 1-dimensional case, the convergence holds in the norm resolvent sense but the coupling constant does depend non-linearly on the potential V, unlike in the case of Schrödinger operators. Though, in the 3-dimensional case, the convergence holds in the strong resolvent sense for bounded smooth hyper-surfaces under certain hypothesis over the potential V, see Ref. 12.

This note revolves on the free Dirac operator in $\mathbb{R}^3$ and its δ-shell interactions with singular electrostatic potentials supported on a sphere. On one hand, we answer an open question posed in Ref. 3 which provides eigenstates of those couplings by finding sharp constants and minimizers of some precise inequalities related to an uncertainty principle (see Question III.4, Theorem III.6, and Corollary III.5). On the other hand, we prove that the domains given in Refs. 8 and 2 coincide in the spherical case and that the conjecture that comes from the comparison to Ref. 15 holds (see Theorem II.2 and Remark II.3). Moreover, we explore the spectral relation between the electrostatic δ-shell interaction and its approximation by the coupling of the free Dirac operator with shrinking short range potentials, improving the results in Ref. 12 in the case of a spherical shell interaction (see Theorem IV.2).

II. PRELIMINARIES

In this section, we review some notation and basic rudiments on the construction of the electrostatic δ-shell interactions mentioned in the Introduction. We first recall the approach presented in Ref. 2 for interactions on general smooth bounded domains and then we review the one from Ref. 8 in the case of a spherical interaction. At the end of the section, we prove that both self-adjoint realizations coincide in the spherical case whenever the strength of the interaction differs from a concrete value.

Given $m \geq 0$, the free Dirac operator in $\mathbb{R}^3$ is defined by

$$H = -i\alpha \cdot \nabla + m\beta,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{for } i = 1, 2, 3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is the family of *Pauli's matrices*. Thus H acts on spinors of the form $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$. Despite the massless case has its own interest, throughout this article we will assume that $m > 0$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. We set $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. Besides, let σ and ν denote the surface measure and unit outward (with respect to Ω) normal vector field on $\partial\Omega$, respectively. A fundamental solution of H is given by

$$\phi(x) = \frac{e^{-m|x|}}{4\pi|x|} \left(m\beta + (1 + m|x|) \, i\alpha \cdot \frac{x}{|x|^2} \right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

see Ref. 2, Lemma 3.1. Given $g \in L^2(\partial\Omega)^4$, we introduce the operator

$$\Phi(g)(x) = \int_{\partial\Omega} \phi(x-y)g(y) \, d\sigma(y) \quad \text{for } x \in \mathbb{R}^3 \setminus \partial\Omega. \tag{1}$$

By Ref. 2, Corollary 2.3, $\Phi : L^2(\partial\Omega)^4 \to L^2(\mathbb{R}^3)^4$ is linear and bounded. For $x \in \partial\Omega$, we also define

$$C_\sigma g(x) = \lim_{\epsilon \searrow 0} \int_{\partial\Omega \cap \{|x-y|>\epsilon\}} \phi(x-y)g(y) \, d\sigma(y) \quad \text{and} \quad C_\pm g(x) = \lim_{\Omega_\pm \ni y \xrightarrow{nt} x} \Phi(g)(y), \tag{2}$$

where $\Omega_\pm \ni y \xrightarrow{nt} x$ means that y tends to x non-tangentially from $\Omega_\pm$, respectively. From Ref. 2, Lemma 3.3, we know that both C_σ and $C_\pm$ are linear and bounded in $L^2(\partial\Omega)^4$, and the following Plemelj-Sokhotski jump formulae hold

$$C_\pm = \mp \frac{i}{2}(\alpha \cdot \nu) + C_\sigma. \tag{3}$$

Let $H^1(\mathbb{R}^3)^4$ denote the Sobolev space of $\mathbb{C}^4$-valued functions such that all its components have all its zero and first order derivatives in $L^2(\mathbb{R}^3)$. It is well known that the trace operator on $\partial\Omega$, initially defined on smooth functions, extends to a bounded operator

$$\text{tr}_{\partial\Omega} : H^1(\mathbb{R}^3)^4 \to H^{1/2}(\partial\Omega)^4 \subset L^2(\partial\Omega)^4,$$

where $H^{1/2}(\partial\Omega)$ denotes the Sobolev-Slobodeckij space on $\partial\Omega$ of order 1/2.

We are ready to introduce the electrostatic δ-shell interaction $H + \lambda\delta_{\partial\Omega}$ studied in Ref. 2. Here, $\lambda \in \mathbb{R}$ is a parameter that represents the strength of the interaction. Following Ref. 2, Theorem 3.8, we define

$$D(H + \lambda\delta_{\partial\Omega}) = \{u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4, \ g \in L^2(\partial\Omega)^4, \ \lambda \, \text{tr}_{\partial\Omega} \, u = -(1 + \lambda C_\sigma)g\},$$
$$(H + \lambda\delta_{\partial\Omega})\varphi = H\varphi + \lambda \frac{\varphi_+ + \varphi_-}{2} \sigma \quad \text{for } \varphi \in D(H + \lambda\delta_{\partial\Omega}), \tag{4}$$

where $H\varphi$ in the right hand side of the second statement in (4) is understood in the sense of distributions and $\varphi_\pm$ denotes the boundary traces of φ when one approaches to $\partial\Omega$ from $\Omega_\pm$. In particular, one has $(H + \lambda\delta_{\partial\Omega})\varphi = Hu \in L^2(\mathbb{R}^3)^4$ for all $\varphi = u + \Phi(g) \in D(H + \lambda\delta_{\partial\Omega})$. Furthermore, $H + \lambda\delta_{\partial\Omega}$ is self-adjoint for all $\lambda \neq \pm 2$. For shortness sake, we put

$$H_\lambda = H + \lambda\delta_{\partial\Omega}.$$

Roughly speaking, to construct the interaction H_λ presented in Ref. 2, one finds the domain of definition of the operator looking in the space of distributions and imposing some restrictions to get a big enough domain for the adjoint. However, in Ref. 8, the electrostatic δ-shell interactions are obtained using self-adjoint extensions of the restricted operator $H|_{C_c^\infty(\mathbb{R}^3\setminus\partial\Omega)^4}$, as we will see below.

Unless we say the contrary, from now on, we restrict our study to the case

$$\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}.$$

For clarity, let us denote $B_\pm = \Omega_\pm$ and $\mathbb{S}^2 = \partial\Omega$. We now review the approach from Ref. 8, where the authors construct self-adjoint and rotationally invariant extensions of $H|_{C_c^\infty(\mathbb{R}^3\setminus\mathbb{S}^2)^4}$ by using the decomposition in the classical spherical harmonics Y_n^l. The indices refer to $n = 0, 1, 2, \ldots$ and $l = -n, -n+1, \ldots, n,$, and the functions satisfy $\Delta_{\mathbb{S}^2} Y_n^l = n(n+1)Y_n^l$, where $\Delta_{\mathbb{S}^2}$ denotes the usual spherical Laplacian. Moreover, $\{Y_n^l\}_{l,n}$ is a complete orthonormal set in $L^2(\mathbb{S}^2)$. For $j = 1/2, 3/2, 5/2, \ldots$ and $m_j = -j, -j+1, \ldots, j$, set

$$\psi_{j-1/2}^{m_j} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j} \, Y_{j-1/2}^{m_j-1/2} \\ \sqrt{j-m_j} \, Y_{j-1/2}^{m_j+1/2} \end{pmatrix},$$

$$\psi_{j+1/2}^{m_j} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j} \, Y_{j+1/2}^{m_j-1/2} \\ -\sqrt{j+1+m_j} \, Y_{j+1/2}^{m_j+1/2} \end{pmatrix}.$$

Then $\{\psi_{j\pm 1/2}^{m_j}\}_{j,m_j}$ is a complete orthonormal set in $L^2(\mathbb{S}^2)^2$. Moreover, if we set

$$r = |x|, \quad \hat{x} = x/|x| \quad \text{and} \quad L = -ix \times \nabla \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

then

$$(\sigma \cdot \hat{x})\psi_{j\pm 1/2}^{m_j} = \psi_{j\mp 1/2}^{m_j} \quad \text{and} \quad (1 + \sigma \cdot L)\psi_{j\pm 1/2}^{m_j} = \pm(j + 1/2)\psi_{j\pm 1/2}^{m_j},$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of *Pauli's matrices*. For $k_j = \pm(j + 1/2)$, we define

$$\Phi_{m_j,k_j}^+ = \begin{pmatrix} i\,\psi_{j\pm 1/2}^{m_j} \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_{m_j,k_j}^- = \begin{pmatrix} 0 \\ \psi_{j\mp 1/2}^{m_j} \end{pmatrix}.$$

Then, the set $\mathcal{B} = \{\Phi_{m_j,k_j}^+, \Phi_{m_j,k_j}^-\}_{j,k_j,m_j}$ is a complete orthonormal base of $L^2(\mathbb{S}^2)^4$. We refer to Ref. 17, Sec. 4.6 for the details.

We now introduce the subspaces

$$\mathcal{C}_{m_j,k_j} = \left\{ \frac{1}{r}\left(f_{m_j,k_j}^+(r)\Phi_{m_j,k_j}^+(\hat{x}) + f_{m_j,k_j}^-(r)\Phi_{m_j,k_j}^-(\hat{x}) \right) \in L^2(\mathbb{R}^3)^4 : f_{m_j,k_j}^\pm \in C_c^\infty(0, +\infty) \right\}.$$

From Ref. 17, Theorems 1.1 and 4.14, we know that the operator

$$\mathring{H} = H|_{C_c^\infty(\mathbb{R}^3 \setminus \{0\})^4}$$

is essentially self-adjoint and leaves the partial wave subspace $\mathcal{C}_{m_j,k_j}$ invariant. Its action on each subspace is represented, in terms of the base $\mathcal{B}$, by the operator

$$D(\mathring{h}_{m_j,k_j}) = C_c^\infty(0, +\infty)^2, \quad \mathring{h}_{m_j,k_j}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} m & -\frac{d}{dr} + \frac{k_j}{r} \\ \frac{d}{dr} + \frac{k_j}{r} & -m \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{5}$$

By Ref. 17, Lemma 4.15, $\mathring{h}_{m_j,k_j}$ is essentially self-adjoint, and if we set $h_{m_j,k_j} = \overline{\mathring{h}_{m_j,k_j}}$, we get that

$$H \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} h_{m_j,k_j},$$

where "$\cong$" means that the operators are unitarily equivalent and H is the free Dirac operator defined in $H^1(\mathbb{R}^3)$.

For m_j and k_j as above, let us define

$$D(\hat{h}_{m_j,k_j}) = C_c^\infty((0, 1) \cup (1, +\infty))^2 \subset D(\mathring{h}_{m_j,k_j}), \quad \hat{h}_{m_j,k_j}\varphi := \mathring{h}_{m_j,k_j}\varphi, \text{ for all } \varphi \in D(\hat{h}_{m_j,k_j}).$$

Assume that $\lambda \in \mathbb{R} \setminus \{\pm 2\}$ and set

$$M_\lambda^\pm = \begin{pmatrix} \lambda/2 & \pm 1 \\ \mp 1 & \lambda/2 \end{pmatrix}.$$

In Ref. 8, it is proved that the operator $h(\lambda)_{m_j,k_j}$ defined by

$$\begin{aligned} D(h(\lambda)_{m_j,k_j}) = \Big\{ \varphi \in L^2(0, +\infty)^2 : h_{m_j,k_j}\varphi \in L^2(0, +\infty)^2, \; \varphi \in AC((0, 1) \cup (1, +\infty))^2, \\ M_\lambda^-\varphi(1^+) + M_\lambda^+\varphi(1^-) = 0 \Big\}, \\ h(\lambda)_{m_j,k_j}\varphi = h_{m_j,k_j}\varphi \quad \text{for all } \varphi \in D(h(\lambda)_{m_j,k_j}) \end{aligned} \tag{6}$$

is a self-adjoint extension of $\hat{h}_{m_j,k_j}$. Here, $AC((0, 1) \cup (1, +\infty))$ denotes the space of absolutely continuous functions on the open set $(0, 1) \cup (1, +\infty)$. Furthermore, if one sets

$$\delta_1(\varphi) = \frac{\varphi(1^+) + \varphi(1^-)}{2},$$

then $h(\lambda)_{m_j,k_j} = \mathring{h}_{m_j,k_j} + \lambda\delta_1$ on $D(h(\lambda)_{m_j,k_j})$, with the understanding that here $\mathring{h}_{m_j,k_j}$ just means the differential operator given by the matrix on the right hand side of (5) acting in the sense of distributions. Let us finally introduce the subspaces

$$\mathcal{H}(\lambda)_{m_j,k_j} = \left\{ \frac{1}{r}\left(f_{m_j,k_j}^+(r)\Phi_{m_j,k_j}^+(\hat{x}) + f_{m_j,k_j}^-(r)\Phi_{m_j,k_j}^-(\hat{x}) \right) \in L^2(\mathbb{R}^3)^4 : f_{m_j,k_j}^\pm \in D(h(\lambda)_{m_j,k_j}) \right\}.$$

The electrostatic δ-shell interaction with strength λ studied in Ref. 8 is given by

$$
\begin{aligned}
D(\widehat{H}(\lambda)) &= \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} \mathcal{H}(\lambda)_{m_j,k_j}, \\
\widehat{H}(\lambda) &\cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} h(\lambda)_{m_j,k_j},
\end{aligned}
\tag{7}
$$

which is a self-adjoint operator.

In order to compare the notions of a δ-shell interaction given by (4) in the spherical case and (7), let us first prove an auxiliary result.

Lemma II.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2. Then

$$
H^1(\mathbb{R}^3 \setminus \partial\Omega)^4 = \left\{ u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4, \ g \in H^{1/2}(\partial\Omega)^4 \right\}.
$$

Proof. If $f = u + \Phi(g)$ for some $u \in H^1(\mathbb{R}^3)^4$ and $g \in H^{1/2}(\partial\Omega)^4$, by Ref. 11, Lemma 3.1, we have that $f \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$.

Let us consider now $f \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$. Since $f \in H^1(\Omega_\pm)^4$, by the trace theorem, we also have $f_\pm \in H^{1/2}(\partial\Omega)^4$. Set

$$
g := i\,(\alpha \cdot \nu)(f_+ - f_-) \in H^{1/2}(\partial\Omega)^4
$$

and $u = f - \Phi(g)$. Once again, Ref. 11, Lemma 3.1 shows that $u \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$. Moreover, by (3),

$$
u_+ - u_- = f_+ - f_- - C_+ g + C_- g = f_+ - f_- + i\,(\alpha \cdot \nu)g = 0,
$$

thus $u_+ = u_-$ and u has a well defined boundary trace in $H^{1/2}(\partial\Omega)^4$. This implies that actually $u \in H^1(\mathbb{R}^3)^4$, and we are done since $f = u + \Phi(g)$. $\square$

Theorem II.2. *Assume that* $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. *For any* $\lambda \in \mathbb{R} \setminus \{\pm 2\}$, *the self-adjoint realizations* H_λ *and* $\widehat{H}(\lambda)$ *defined by* (4) *and* (7), *respectively, coincide.*

Proof. Consider the operator

$$
\begin{aligned}
D(\widetilde{H}_\lambda) &= \left\{ u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4, \ g \in H^{1/2}(\mathbb{S}^2)^4, \ \lambda\,\mathrm{tr}_{\mathbb{S}^2}\, u = -(1 + \lambda C_\sigma)g \right\}, \\
\widetilde{H}_\lambda &= H_\lambda|_{D(\widetilde{H}_\lambda)}.
\end{aligned}
$$

Since $H^{1/2}(\mathbb{S}^2)^4 \subset L^2(\mathbb{S}^2)^4$, by construction, we get $\widetilde{H}_\lambda \subset H_\lambda$. We are going to prove that

$$
\widehat{H}(\lambda) \subset \widetilde{H}_\lambda.
\tag{8}
$$

With this at hand, we deduce that $\widehat{H}(\lambda) \subset \widetilde{H}_\lambda \subset H_\lambda$, and since both $\widehat{H}(\lambda)$ and H_λ are self-adjoint operators for $\lambda \neq \pm 2$, we finally conclude that $\hat{H}(\lambda) = H_\lambda$ and the theorem follows. Let us focus on (8). Fixed m_j and k_j as in (7), for simplicity of notation, we put

$$
\begin{aligned}
f^\pm(r) &= f^\pm_{j,m_j}(r), \\
\Phi^\pm(\hat{x}) &= \Phi^\pm_{j,m_j}(\hat{x}), \\
\mathcal{H}(\lambda) &= \mathcal{H}(\lambda)_{m_j,k_j}, \\
h(\lambda) &= h(\lambda)_{m_j,k_j}.
\end{aligned}
$$

Thus, any $\varphi \in \mathcal{H}(\lambda)$ can be written as

$$
\varphi(x) = \frac{1}{r}\,(f^+(r)\Phi^+(\hat{x}) + f^-(r)\Phi^-(\hat{x})) = \frac{1}{r} \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix}.
$$

In the last expression above, "·" just means the "scalar product." As before, we denote by $\varphi_\pm$ the boundary values of φ when we approach $\mathbb{S}^2$ from $\Omega_\pm$. Let $\mathcal{M}_\lambda^\pm$ be the operator defined in $\mathcal{H}(\lambda)$ by the action of the matrix $M_\lambda^\pm$ on the basis $\{\Phi^+, \Phi^-\}$, that is, for any $\hat{x} \in \mathbb{S}^2$,

$$\mathcal{M}_\lambda^+ \varphi_+(\hat{x}) = \frac{1}{r} \left(M_\lambda^+ \begin{pmatrix} f^+(1^+) \\ f^-(1^+) \end{pmatrix} \right) \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix},$$

$$\mathcal{M}_\lambda^- \varphi_-(\hat{x}) = \frac{1}{r} \left(M_\lambda^- \begin{pmatrix} f^+(1^-) \\ f^-(1^-) \end{pmatrix} \right) \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix}.$$

So, in particular, we have that

$$\mathcal{M}_\lambda^+ \varphi_-(\hat{x}) + \mathcal{M}_\lambda^- \varphi_+(\hat{x}) = 0 \quad \text{for all } \hat{x} \in \mathbb{S}^2. \tag{9}$$

Moreover, since $\varphi \in H^1(\mathbb{R}^3 \setminus \mathbb{S}^2)^4$, using Lemma II.1, we can write $\varphi = u + \Phi(g)$ for some $u \in H^1(\mathbb{R}^3)^4$ and $g \in L^2(\mathbb{S}^2)^4$. Then, since $\nu(\hat{x}) = \hat{x}$ for all $\hat{x} \in \mathbb{S}^2$, using (3), we observe that (9) is equivalent to

$$\begin{aligned} 0 &= (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-) \mathrm{tr}_{\mathbb{S}^2} u(\hat{x}) + (\mathcal{M}_\lambda^+ C_+ + \mathcal{M}_\lambda^- C_-) g(\hat{x}) \\ &= (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-) \mathrm{tr}_{\mathbb{S}^2} u(\hat{x}) + \frac{1}{2} \left(\mathcal{M}_\lambda^- - \mathcal{M}_\lambda^+ \right) i(\alpha \cdot \hat{x}) g(\hat{x}) + (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-) C_\sigma g(\hat{x}). \end{aligned} \tag{10}$$

Since $M_\lambda^+ + M_\lambda^- = \lambda \mathbb{I}_2$, where $\mathbb{I}_2$ denotes the 2×2 identity matrix, we get that, for $\hat{x} \in \mathbb{S}^2$,

$$(\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-) u(\hat{x}) = \lambda u(\hat{x}), \tag{11}$$
$$(\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-) C_\sigma g(\hat{x}) = \lambda C_\sigma g(\hat{x}). \tag{12}$$

Note also that

$$\frac{1}{2} \left(M_\lambda^- - M_\lambda^+ \right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

that is, the matrix that represents the operator $-i(\alpha \cdot \hat{x})$ on the basis $\{\Phi^+, \Phi^-\}$ (see Ref. 17, Eq. 4.123). So

$$\frac{1}{2} \left(\mathcal{M}_\lambda^- - \mathcal{M}_\lambda^+ \right) (i\alpha \cdot \hat{x}) g(\hat{x}) = g(\hat{x}) \tag{13}$$

for $\hat{x} \in \mathbb{S}^2$. Combining (11)–(13) and (10) becomes

$$0 = \lambda \, \mathrm{tr}_{\mathbb{S}^2} u + (1 + \lambda C_\sigma) g.$$

In conclusion, we have seen that if $\varphi \in \mathcal{H}(\lambda)$ then $\varphi \in H_\lambda$. Since these arguments are valid for any m_j and k_j, (8) follows. □

Remark II.3. From the proof of Theorem II.2, we also observe that if $\lambda \neq \pm 2$ then $\widetilde{H}_\lambda = H_\lambda$, which means that the condition $\lambda \, \mathrm{tr}_{\mathbb{S}^2} u = -(1 + \lambda C_\sigma) g$ in (4) forces g to belong to $H^{1/2}(\mathbb{S}^2)^4$.

III. ON THE SPECTRUM FOR THE SPHERICAL δ-SHELL INTERACTION

In this section, we answer affirmatively a question posed in Ref. 3, Sec. 4.2.3. As commented there, this yields a relation between the eigenvalues in the gap $(-m, m)$ for the electrostatic spherical δ-shell interaction and the minimizers of some precise quadratic form inequality. Before going further, we must recall some rudiments from Ref. 3, Sec. 4. Throughout this section, Ω denotes the unit ball and $\partial\Omega = \mathbb{S}^2$. Given $a \in [-m, m]$, set

$$k^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|} \mathbb{I}_2 \quad \text{and} \quad w^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|^3} \left(1 + \sqrt{m^2-a^2}|x| \right) i\sigma \cdot x$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. Given $f \in L^2(\sigma)^2$ and $x \in \mathbb{S}^2$, set

$$K^a f(x) = \int_{\mathbb{S}^2} k^a(x-z) f(z) \, d\sigma(z) \quad \text{and} \quad W^a f(x) = \lim_{\epsilon \searrow 0} \int_{\{|x-z| > \epsilon\} \cap \mathbb{S}^2} w^a(x-z) f(z) \, d\sigma(z).$$

Then

$$C_\sigma^a = \begin{pmatrix} (a+m)K^a & W^a \\ W^a & (a-m)K^a \end{pmatrix},$$

where C_σ^a is defined as C_σ in (2) replacing the kernel ϕ by

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a + m\beta + \left(1 + \sqrt{m^2-a^2}|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}.$$

The following corresponds to Ref. 3, Lemma 4.3.

Lemma III.1. Given $a \in (-m, m)$, there exist positive numbers $d_{j\pm1/2}$ and purely imaginary numbers $p_{j\pm1/2}$ for all $j = 1/2, 3/2, 5/2, \ldots$ and $m_j = -j, -j+1, \ldots, j$, such that

(i). $K^a \psi_{j\pm1/2}^{m_j} = d_{j\pm1/2}\psi_{j\pm1/2}^{m_j}$ and $\lim_{j\to\infty} d_{j\pm1/2} = 0$. Moreover,

$$0 \le d_{j\pm1/2} \le d_0 = \frac{1 - e^{-2\sqrt{m^2-a^2}}}{2\sqrt{m^2-a^2}}.$$

(ii). $W^a \psi_{j\pm1/2}^{m_j} = p_{j\pm1/2}\psi_{j\mp1/2}^{m_j}$ and $p_{j+1/2} = -p_{j-1/2}$. Moreover,

$$|p_{j\pm1/2}|^2 = \frac{1}{4} - (m^2-a^2)d_{j+1/2}\,d_{j-1/2} \ge \frac{1}{4}e^{-2\sqrt{m^2-a^2}}\left(2 - e^{-2\sqrt{m^2-a^2}}\right).$$

The following result allows us to construct eigenstates for H_λ from the eigenfunctions of K^a; it corresponds to Ref. 3, Lemma 4.6.

Lemma III.2. Let H_λ be as in (4). If $\lambda > 0$ and $a \in (-m, m)$ satisfy

$$\frac{\lambda^2}{4} - ((m+a)d_{j\mp1/2} - (m-a)d_{j\pm1/2})\lambda = 1 \quad \text{for some } j, \tag{14}$$

then, for any m_j, $\psi_{j\pm1/2}^{m_j}$ gives rise to an eigenfunction for H_λ with eigenvalue a.

Remark III.3. In Lemma III.2, the expression "gives rise to an eigenfunction" means that if one defines

$$g = \begin{pmatrix} f \\ h \end{pmatrix} \in L^2(\mathbb{S}^2)^4, \quad \text{where } h = \psi_{j\pm1/2}^{m_j} \quad \text{and} \quad f = -(1/\lambda + (a+m)K^a)^{-1}W^a h,$$

setting $\varphi = \phi * (a\Phi^a(g)) + \Phi(g)$, one gets that $H_\lambda\varphi = a\varphi$. Here, Φ^a is defined as Φ in (1) replacing ϕ by ϕ^a.

In Ref. 3, Question 4.7, the following question was raised:

Question III.4. Let $d_{j\pm1/2}$ be the coefficients given by Lemma III.1. *Is it true that $d_{j+1/2}d_{j-1/2} < d_1d_0$ for all $j = 3/2, 5/2, 7/2 \ldots$?*

Theorem III.6 answers it in the affirmative and, as commented at the end of [Ref. 3, Sec. 4.2.3], it yields the following result related to Lemma III.2. We first recall the values of d_0 and d_1 from Lemma III.1 (computed in Ref. 3) and a precise constant d_* that will appear below, see Ref. 3, Eqs. (4.31), (4.32), and (4.39), respectively,

$$d_0 = \frac{1 - e^{-2\sqrt{m^2-a^2}}}{2\sqrt{m^2-a^2}},$$

$$d_1 = \frac{1}{2\sqrt{m^2-a^2}}\left(1 - \frac{1}{m^2-a^2} + \left(1 + \frac{1}{\sqrt{m^2-a^2}}\right)^2 e^{-2\sqrt{m^2-a^2}}\right),$$

$$d_* = \frac{1}{2\sqrt{m^2-a^2}} - \frac{1}{2}\left(1 + \frac{1}{\sqrt{m^2-a^2}}\right)e^{-2\sqrt{m^2-a^2}}.$$

Corollary III.5. Let $a \in (-m, m)$ and $\lambda > 0$. Then, for any $f \in L^2(\sigma)^2$,

$$\int_{\mathbb{S}^2} |f|^2 \, d\sigma \leq \frac{1/\lambda + (m + a)d_0}{2d_*^2} \int_{\mathbb{S}^2} (1/\lambda + (m + a)K^a)^{-1} W^a f \cdot \overline{W^a f} \, d\sigma$$

$$+ \frac{1}{2(1/\lambda + (m + a)d_0)} \int_{\mathbb{S}^2} (1/\lambda + (m + a)K^a)(\sigma \cdot \nu)f \cdot \overline{(\sigma \cdot \nu)f} \, d\sigma. \tag{15}$$

The equality in (15) *is only attained at linear combinations of ψ_1^l for $l \in \{-1/2, 1/2\}$. If*

$$\frac{\lambda^2}{4} - ((m + a)d_0 - (m - a)d_1)\lambda = 1, \tag{16}$$

then the minimizers of (15) *give rise to eigenfunctions of H_λ. Besides, these conclusions also hold if we exchange the roles of d_0 and d_1 in* (15) *and* (16) *and we replace ψ_1^l by ψ_0^l (that is, we exchange the roles of $j + 1/2$ and $j - 1/2$ for $j = 1/2$).*

Theorem III.6. *Let $d_{j\pm1/2}$ be the coefficients given by* Lemma III.1. *Then,*

$$d_{j\pm1/2} = I_{(j+1/2)\pm1/2}(\sqrt{m^2 - a^2})K_{(j+1/2)\pm1/2}(\sqrt{m^2 - a^2}), \tag{17}$$

where I and K denote the standard second order Bessel's functions. Moreover,

$$d_{j+1/2}d_{j-1/2} < d_0d_1 \quad \text{for all } j = 3/2, 5/2, 7/2 \ldots. \tag{18}$$

Proof. Let us first compute $d_{j\pm1/2}$ in terms of Bessel's functions. Fixed m_j and k_j, due to Ref. 17, Lemma 4.15 and Theorem II.2, it is enough to find some $a \in (-m, m)$ which is an eigenvalue for the operator $h(\lambda)_{m_j,k_j}$. We want to find some

$$\begin{pmatrix} f \\ g \end{pmatrix} \in D(h(\lambda)_{m_j,k_j})$$

verifying the following system of differential equations:

$$\begin{cases} (m - a)f + (-\frac{d}{dr} + \frac{k_j}{r})g = 0, \\ (\frac{d}{dr} + \frac{k_j}{r})f - (m + a)g = 0. \end{cases} \tag{19}$$

Set $M = \sqrt{m^2 - a^2}$. Since $k_j = \pm(j + 1/2)$, we set

$$f(r) = \begin{cases} A\sqrt{r} \, I_{(j+1/2)\pm1/2}(Mr) & \text{if } r < 1 \\ B\sqrt{r} \, K_{(j+1/2)\pm1/2}(Mr) & \text{if } r > 1 \end{cases}, \quad g(r) = \begin{cases} \frac{AM}{m+a}\sqrt{r} \, I_{(j+1/2)\mp1/2}(Mr) & \text{if } r < 1 \\ -\frac{BM}{m+a}\sqrt{r} \, K_{(j+1/2)\mp1/2}(Mr) & \text{if } r > 1 \end{cases}, \tag{20}$$

for some $(A, B) \neq (0, 0)$. If we put

$$\varphi = \begin{pmatrix} f \\ g \end{pmatrix},$$

then $\varphi \in L^2(0, +\infty)^2$, $h_{m_j,k_j}\varphi \in L^2(0, +\infty)^2$, $\varphi \in AC((0, 1) \cup (1, +\infty))^2$, and φ satisfies (19). Thus, to get that φ is an eigenvector for the operator $h(\lambda)_{m_j,k_j}$, it remains to prove that $\varphi \in D(h(\lambda)_{m_j,k_j})$, that is, we have to show that $M_\lambda^- \varphi(1^+) + M_\lambda^+ \varphi(1^-) = 0$. In other words, the following linear system must hold:

$$\begin{cases} A \left(\frac{M}{a+m} I_{(j+1/2)\mp1/2}(M) + \frac{\lambda}{2} I_{(j+1/2)\pm1/2}(M) \right) \\ \quad + B \left(\frac{M}{a+m} K_{(j+1/2)\mp1/2}(M) + \frac{\lambda}{2} K_{(j+1/2)\pm1/2}(M) \right) = 0, \\[2mm] A \left(\frac{\lambda M}{2(a+m)} I_{(j+1/2)\mp1/2}(M) - I_{(j+1/2)\pm1/2}(M) \right) \\ \quad + B \left(K_{(j+1/2)\pm1/2}(M) - \frac{\lambda M}{2(a+m)} K_{(j+1/2)\mp1/2}(M) \right) = 0. \end{cases}$$

Since this is a 2×2 homogeneous linear system in A and B and we are supposing that $(A, B) \neq (0, 0)$, we deduce that the associated matrix has a null determinant. This means that

$$0 = -\frac{\lambda^2 M}{4(a+m)} \left(I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\mp1/2}(M) + I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\pm1/2}(M) \right)$$

$$+ \frac{\lambda}{m+a} \left((m+a)I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\pm1/2}(M) - (m-a)I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\mp1/2}(M) \right)$$

$$+ \frac{M}{m+a} \left(I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\mp1/2}(M) + I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\pm1/2}(M) \right). \tag{21}$$

By Ref. 13, Eq. 10.20.2, we get that

$$I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\mp1/2}(M) + I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\pm1/2}(M) = \frac{1}{M}. \tag{22}$$

Finally, combining (21) and (22), we see that the following must hold:

$$\frac{\lambda^2}{4} - \left((m+a)\, I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\pm1/2}(M) \right.$$
$$\left. - (m-a)I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\mp1/2}(M) \right)\lambda - 1 = 0. \tag{23}$$

In conclusion, if we define

$$D_{j\pm1/2}(a,\lambda) = \frac{\lambda^2}{4} - \left((m+a)I_{(j+1/2)\pm1/2}(M)K_{(j+1/2)\pm1/2}(M) \right.$$
$$\left. - (m-a)I_{(j+1/2)\mp1/2}(M)K_{(j+1/2)\mp1/2}(M) \right)\lambda - 1 \tag{24}$$

and we take $\varphi = \begin{pmatrix} f \\ g \end{pmatrix}$ with f and g given by (20), then φ is an eigenfunction for $h(\lambda)_{m_j,k_j}$ with eigenvalue a if and only if $D_{j\pm1/2}(a,\lambda) = 0$. In this case, the function

$$\psi(x) = \frac{1}{r} \left(f(r)\Phi^+_{m_j,k_j}(\hat{x}) + g(r)\Phi^-_{m_j,k_j}(\hat{x}) \right)$$

is an eigenfunction for H_λ with eigenvalue a. For this reason, a comparison of (23) and (14) yields (17), as desired.

Let us finally prove (18). We put $n = j + 1/2 \in \mathbb{N}$. Since $j > 1/2$, we have $n > 1$. Then (18) is equivalent to

$$d_n d_{n-1} < d_0 d_1, \quad \text{for all } n \geq 2. \tag{25}$$

We are going to show (25) by induction. For $n = 2$, we have to check that $d_1 d_2 < d_1 d_0$, which is equivalent to $d_1(d_2 - d_0) < 0$. Since $d_n \geq 0$ for all $n > 1$, it is enough to show that $d_2 - d_0 < 0$. But, from (17), we easily get that

$$d_2 - d_0 = \frac{3\left(M^3 + 2M^2 + 3M + 3\right)\sinh(M) - 3M\left(M^2 + 3M + 3\right)\cosh(M)}{e^M M^5} < 0.$$

Let us now suppose that (25) holds for $n - 1$. Then, we can split

$$d_{n-1}d_n - d_0 d_1 = d_{n-1}(d_n - d_{n-2}) + d_{n-1}d_{n-2} - d_0 d_1.$$

On one hand, $d_{n-1}d_{n-2} - d_0 d_1 < 0$ by (25). On the other hand, $d_n - d_{n-2} \leq 0$ by Ref. 4, Theorem 2 and $d_{n-1} \geq 0$. Thus (25) holds for all $n \geq 2$.   □

IV. APPROXIMATION BY SHORT-RANGE POTENTIALS

In this section, we investigate the spectral relation between the electrostatic δ-shell interaction on the boundary of a smooth domain and its approximation by the coupling of the Dirac operator with a short-range potential which depends on a parameter $\epsilon > 0$ in such a way that it shrinks to the boundary of the domain as $\epsilon \to 0$; see the definition of $H_{\lambda,\epsilon}$ below. From Ref. 12, Theorem 1.2, we know that if $a \in \sigma(H_\lambda)$, where $\sigma(\cdot)$ denotes the spectrum, then there exists a sequence $\{a_\epsilon\}$ such that $a_\epsilon \in \sigma(H_{\mu,\epsilon})$ and $a_\epsilon \to a$ for $\epsilon \to 0$, where $\lambda = 2\tan\left(\frac{\mu}{2}\right)$. However, the vice-versa spectral implication may not hold in general. In this section, we are going to show that the reverse does hold in the spherical case, that is, if $a_\epsilon \to a$ with $a_\epsilon \in \sigma(H_{\mu,\epsilon})$, then $a \in \sigma(H_\lambda)$ (see Theorem IV.2). In particular, this means that, when passing to the limit, we do not lose any element of the spectrum for electrostatic interactions

with potentials shrinking on $\mathbb{S}^2$.

Given $\epsilon > 0$ and $x \in \mathbb{R}^3$, we define

$$V_\epsilon(x) = \frac{1}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)}(|x|) \quad \text{and} \quad \mathbf{V}_\epsilon = V_\epsilon\mathbb{I}_4,$$

where $\mathbb{I}_4$ denotes the 4×4 identity matrix. For $\lambda \in \mathbb{R}$, we also introduce the operators

$$D(\mathring{H}_{\lambda,\epsilon}) = C_c^\infty(\mathbb{R}^3 \setminus \{0\})^4 \quad \text{and} \quad \mathring{H}_{\lambda,\epsilon} = H + \lambda\mathbf{V}_\epsilon,$$

$$D(H_{\lambda,\epsilon}) = H^1(\mathbb{R}^3)^4 \quad \text{and} \quad H_{\lambda,\epsilon} = H + \lambda\mathbf{V}_\epsilon.$$

Since $|V_\epsilon| \le \frac{1}{2\epsilon}$, $\mathring{H}_{\lambda,\epsilon}$ is essentially self-adjoint and $H_{\lambda,\epsilon}$ is self-adjoint by Ref. 17, Theorem 4.2. Moreover, $\sigma_{ess}(H_{\lambda,\epsilon}) = \sigma_{ess}(H) = \sigma(H) = (-\infty, -m] \cup [m, +\infty)$. For this reason, we are looking for some $a \in (-m, m)$ eigenvalue of $H_{\lambda,\epsilon}$.

Our aim is to find a precise relation between a, λ, and ϵ, say $R_\epsilon(a, \lambda)$, which must hold in order to get an eigenfunction for $H + \lambda V_\epsilon$ with eigenvector a. Then, we will take the limit of $R_\epsilon(a, \lambda)$ for $\epsilon \to 0$ and we will compare the result to (14). To do so, we use the same approach developed in Sec. III.

Clearly, if $\lambda = 0$, we get that $H_{\lambda,\epsilon} = H$, i.e., we are not perturbing the free Hamiltonian H, thus we can exclude this case in our study. Assuming that $\lambda \ne 0$, we note that if a is an eigenvalue of $H_{\lambda,\epsilon}$ with eigenfunction $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ then $-a$ is an eigenvalue of $H_{-\lambda,\epsilon}$ with eigenfunction $\tilde{\psi} = \begin{pmatrix} -\chi \\ \phi \end{pmatrix}$. For this reason, from now on, we will further assume that $\lambda > 0$.

Observe that $\mathring{H}_{\lambda,\epsilon}$ leaves the partial wave subspace $\mathcal{C}_{m_j,k_j}$ invariant. Its action on each subspace is represented with respect to the basis $\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_J}\}$ by the operator

$$D(\mathring{h}(\lambda, \epsilon)_{m_j,k_j}) = C_c^\infty(0, +\infty)^2,$$
$$\mathring{h}(\lambda, \epsilon)_{m_j,k_j}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} m + \frac{\lambda}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)} & -\frac{d}{dr} + \frac{k_j}{r} \\ \frac{d}{dr} + \frac{k_j}{r} & -m + \frac{\lambda}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{26}$$

Since $\mathring{H}_{\lambda,\epsilon}$ is essentially self-adjoint $\mathring{h}(\lambda, \epsilon)_{m_j,k_j}$ too, thus setting $h(\lambda, \epsilon)_{m_j,k_j} = \overline{\mathring{h}(\lambda, \epsilon)_{m_j,k_j}}$, we get

$$D\left(h(\lambda, \epsilon)_{m_j,k_j}\right) = \{\varphi \in L^2(0, +\infty)^2 : h(\lambda, \epsilon)_{m_j,k_j}\varphi \in L^2(0, +\infty)^2 \text{ and } \varphi \in AC(0, +\infty)^2\}, \tag{27}$$

and the action of $h(\lambda, \epsilon)_{m_j,k_j}$ on its domain of definition is formally given by the right hand side of the second equation in (26). Moreover, $a \in (-m, m)$ is an eigenvalue for $H_{\lambda,\epsilon}$ if and only if a is an eigenvalue for $h(\lambda, \epsilon)_{m_j,k_j}$ for some $\{m_j, k_j\}$. For this reason, we want to solve

$$\begin{cases} (m - a)f + (-\frac{d}{dr} + \frac{k}{r})g = 0 \\ (\frac{d}{dr} + \frac{k}{r})f - (m + a)g = 0 \end{cases} \quad \text{if } 0 < r < 1 - \epsilon \text{ or } r > 1 + \epsilon,$$

$$\begin{cases} (m - a + \frac{\lambda}{2\epsilon})f + (-\frac{d}{dr} + \frac{k}{r})g = 0 \\ (\frac{d}{dr} + \frac{k}{r})f - (m + a - \frac{\lambda}{2\epsilon})g = 0 \end{cases} \quad \text{if } 1 - \epsilon < r < 1 + \epsilon$$

for $\begin{pmatrix} f \\ g \end{pmatrix} \in D\left(h(\lambda, \epsilon)_{m_j,k_j}\right)$.

Since $k_j = \pm(j + 1/2)$, a non-trivial solution is given by

$$f(r) = \begin{cases} A\sqrt{r}\, I_{(j+\frac{1}{2})\pm\frac{1}{2}}(Mr) & r < 1 - \epsilon \\ B_1\sqrt{r}\, J_{(j+\frac{1}{2})\pm\frac{1}{2}}(Lr) + B_2\sqrt{r}\, Y_{(j+\frac{1}{2})\pm\frac{1}{2}}(Lr) & 1 - \epsilon < r < \epsilon + 1 \\ C\sqrt{r}\, K_{(j+\frac{1}{2})\pm\frac{1}{2}}(Mr) & r > 1 + \epsilon \end{cases}$$

$$g(r) = \begin{cases} \frac{AM}{a+m}\sqrt{r}\, I_{(j+1/2)\mp 1/2}(Mr) & 0 < r < 1 - \epsilon, \\ \frac{L\sqrt{r}}{a-\frac{\lambda}{2\epsilon}+m}\left(B_1\, J_{(j+\frac{1}{2})\mp\frac{1}{2}}(Lr) + B_2\, Y_{(j+\frac{1}{2})\mp\frac{1}{2}}(Lr)\right) & 1 - \epsilon < r < 1 + \epsilon, \\ -\frac{CM}{a+m}\sqrt{r}\, K_{(j+1/2)\mp 1/2}(Mr) & r > 1 + \epsilon, \end{cases} \tag{28}$$

where J and Y denote the first order Bessel's functions and I and K denote the second order Bessel's functions,

$$M = \sqrt{m^2 - a^2}, \qquad L = \sqrt{\left(\frac{\lambda}{2\epsilon} - a\right)^2 - m^2}$$

and $(A, B_1, B_2, C) \neq 0$ are some constants. Note that $M \in \mathbb{R}$ by the assumptions on a, but L could be complex. Note also that $f, g \in H^1((0, +\infty) \setminus \{1 - \epsilon, 1 + \epsilon\}, r\, dr)$. To ensure that they belong to $D(h(\lambda, \epsilon)_{m_{j,k}})$, we have to verify that both f and g are continuous in $1 - \epsilon$ and $1 + \epsilon$, which means that the following linear system must hold:

$$
\begin{cases}
0 = A\sqrt{1 - \epsilon}\, \mathrm{I}_{(j+1/2)\pm 1/2}(M(1 - \epsilon)) - B_1\sqrt{1 - \epsilon}\, \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 - \epsilon)) \\
\qquad - B_2\sqrt{1 - \epsilon}\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 - \epsilon)), \\[2mm]
0 = A\dfrac{\sqrt{1 - \epsilon}\, M\, \mathrm{I}_{(j+1/2)\mp 1/2}(M(1 - \epsilon))}{a + m} - B_1\dfrac{2\epsilon L\sqrt{1 - \epsilon}\, \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 - \epsilon))}{2a\epsilon - \lambda + 2m\epsilon} \\
\qquad - B_2\dfrac{2\epsilon L\sqrt{1 - \epsilon}\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 - \epsilon))}{2a\epsilon - \lambda + 2m\epsilon}, \\[2mm]
0 = B_1\sqrt{1 + \epsilon}\, \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 + \epsilon)) + B_2\sqrt{1 + \epsilon}\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 + \epsilon)) \\
\qquad - C\sqrt{1 + \epsilon}\, \mathrm{K}_{(j+1/2)\pm 1/2}(M(1 + \epsilon)), \\[2mm]
0 = B_1\dfrac{2\epsilon L\sqrt{1 + \epsilon}\, \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 + \epsilon))}{2a\epsilon - \lambda + 2m\epsilon} + B_2\dfrac{2\epsilon L\sqrt{1 + \epsilon}\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 + \epsilon))}{2a\epsilon - \lambda + 2m\epsilon} \\
\qquad + C\dfrac{\sqrt{1 + \epsilon}\, M\, \mathrm{K}_{(j+1/2)\mp 1/2}(M(1 + \epsilon))}{a + m}.
\end{cases}
\tag{29}
$$

Since this is a 4×4 homogeneous linear system on A, B_1, B_2, and C and we are assuming that $(A, B_1, B_2, C) \neq 0$, we deduce that the associated matrix has a null determinant. So, if we set

$$
\begin{aligned}
D_{j\pm 1/2}^{\epsilon}(a, \lambda) :=& \\
&\frac{2(a + m)\, \mathrm{K}_{(j+1/2)\pm 1/2}\left(\sqrt{m^2 - a^2}(1 + \epsilon)\right)}{\epsilon(-2a\epsilon + \lambda - 2m\epsilon)^2} \\
&\quad \times \Big\{-2L\epsilon(a + m)\, \mathrm{I}_{(j+1/2)\pm 1/2}\left(\sqrt{m^2 - a^2}(1 - \epsilon)\right) \\
&\qquad \times \Big[\, \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 + \epsilon))\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 - \epsilon)) \\
&\qquad\quad - \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 - \epsilon))\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 + \epsilon))\Big] \\
&\qquad - \sqrt{m^2 - a^2}(2a\epsilon - \lambda + 2m\epsilon)\, \mathrm{I}_{(j+1/2)\mp 1/2}\left(\sqrt{m^2 - a^2}(1 - \epsilon)\right) \\
&\qquad \times \Big[\, \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 - \epsilon))\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 + \epsilon)) \\
&\qquad\quad - \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 + \epsilon))\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 - \epsilon))\Big]\Big\} \\
&+ \frac{\sqrt{m^2 - a^2}\, \mathrm{K}_{(j+1/2)\mp 1/2}\left(\sqrt{m^2 - a^2}(1 + \epsilon)\right)}{\epsilon^2 L(2a\epsilon - \lambda + 2m\epsilon)} \\
&\quad \times \Big\{-2L\epsilon(a + m)\, \mathrm{I}_{(j+1/2)\pm 1/2}\left(\sqrt{m^2 - a^2}(1 - \epsilon)\right) \\
&\qquad \times \Big[\, \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 + \epsilon))\, \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1 - \epsilon)) \\
&\qquad\quad - \mathrm{J}_{(j+1/2)\mp 1/2}(L(1 - \epsilon))\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 + \epsilon))\Big] \\
&\qquad + \sqrt{m^2 - a^2}(2a\epsilon - \lambda + 2m\epsilon)\, \mathrm{I}_{(j+1/2)\mp 1/2}\left(\sqrt{m^2 - a^2}(1 - \epsilon)\right) \\
&\qquad \times \Big[\, \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 + \epsilon))\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 - \epsilon)) \\
&\qquad\quad - \mathrm{J}_{(j+1/2)\pm 1/2}(L(1 - \epsilon))\, \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1 + \epsilon))\Big]\Big\},
\end{aligned}
\tag{30}
$$

then the determinant of the matrix associated with the linear system (29) is $\frac{\epsilon(\epsilon^2-1)}{(a+m)^2}D_{j\pm1/2}^\epsilon(a,\lambda)$, which vanishes if and only if

$$D_{j\pm1/2}^\epsilon(a,\lambda)=0. \tag{31}$$

We can conclude that, given

$$\varphi=\begin{pmatrix}f\\g\end{pmatrix} \quad \text{with } f \text{ and } g \text{ as in (29)},$$

φ is an eigenfunction for $h(\lambda,\epsilon)_{m_j,k_j}$ with eigenvalue a if and only if $D_{j\pm1/2}^\epsilon(a,\lambda)=0$. This means that the function

$$\psi(x)=\frac{1}{r}\left(f(r)\Phi_{m_j,k_j}^+(\hat{x})+g(r)\Phi_{m_j,k_j}^-(\hat{x})\right)$$

is an eigenfunction for $H_{\lambda,\epsilon}$ with eigenvalue a.

In order to compare (30) and (23), let us draw some pictures of these relations for some concrete values of the underlying parameters, say $m=1$, $k=1$, and $\epsilon=2^{-10}$. Figures 1 and 2 describe the set of $(a,\lambda)\in(-1,1)\times(0,10)$ that verify (31) and (23), respectively.

Looking at Figs. 1 and 2, we note that there is no apparent relation between (31) and (23). However, the next result proves that there is indeed a precise connection between both equations when one takes the limit $\epsilon\to0$ in $D_{j\pm1/2}^\epsilon(a,\lambda)$.

Lemma IV.1. Let $j=1/2$, $3/2,\dots$ and $D_{j\pm1/2}^\epsilon$ and $D_{j\pm1/2}$ be defined by (30) and (24), respectively. Then, for any $\lambda>0$,

$$\lim_{\epsilon\to0}D_{j\pm1/2}^\epsilon(a,\lambda)=\frac{4(a+m)}{\lambda\pi\left(1+\tan\left(\frac{\lambda}{2}\right)^2\right)}D_{j\pm1/2}\left(a,2\tan\left(\frac{\lambda}{2}\right)\right) \quad \text{uniformly on } a\in(-m,m).$$
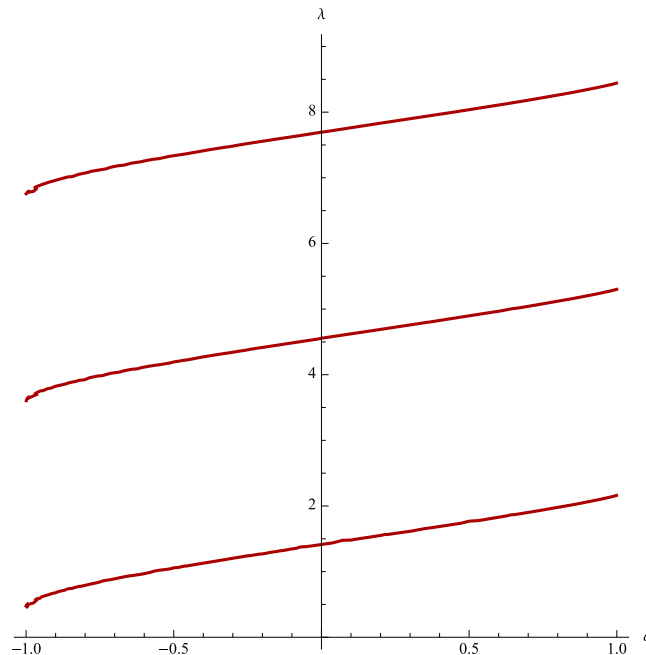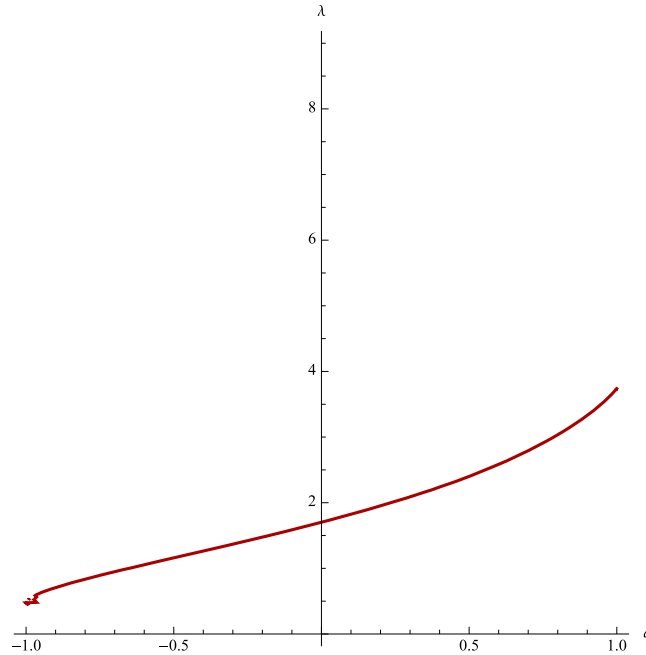

FIG. 1. The set of points (a,λ) satisfying (31).

FIG. 2. The set of points (a, λ) satisfying (23).

Proof. Note that $L \to +\infty$ uniformly in $a \in (-m, m)$ when $\epsilon \to 0$, thus we can use the asymptotics

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \left(\cos\left(z - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) + e^{|\Im(z)|} o(1) \right) \quad \text{for } |z| \to +\infty,$$

$$Y_n(z) = \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) + e^{|\Im(z)|} o(1) \right) \quad \text{for } |z| \to +\infty,$$

see Ref. 13, Eq. 10.7.8. Inserting these two relations in (30) and taking $\epsilon \to 0$, we get that, uniformly on $a \in (-m, m)$,

$$\lim_{\epsilon \to 0} D^\epsilon_{j\pm1/2}(a, \lambda)$$
$$= \frac{4}{\lambda \pi} \Big(M \, \mathrm{I}_{(j+1/2)\mp1/2}(M) \left((a+m)\cos(\lambda) \, \mathrm{K}_{(j+1/2)\pm1/2}(M) - M \sin(\lambda) \, \mathrm{K}_{(j+1/2)\mp1/2}(M) \right)$$
$$+ (a+m) \, \mathrm{I}_{(j+1/2)\pm1/2}(M) \left((a+m)\sin(\lambda) \, \mathrm{K}_{(j+1/2)\pm1/2}(M) + M \cos(\lambda) \, \mathrm{K}_{(j+1/2)\mp1/2}(M) \right) \Big).$$

Setting $t = 2\tan\left(\frac{\lambda}{2}\right)$, we know that $\sin(\lambda) = \frac{t}{1+\frac{t^2}{4}}$ and $\cos(\lambda) = \frac{1-\frac{t^2}{4}}{1+\frac{t^2}{4}}$. Using (22), hence

$$\lim_{\epsilon \to 0} D^\epsilon_{j\pm1/2}(a, \lambda) = \frac{16(a+m)}{\lambda \pi (4+t^2)} \left(\frac{t^2}{4} - \left((m+a) \, \mathrm{I}_{(j+1/2)\pm1/2}(M) \, \mathrm{K}_{(j+1/2)\pm1/2}(M) \right. \right.$$
$$\left. \left. - (m-a) \, \mathrm{I}_{(j+1/2)\mp1/2}(M) \, \mathrm{K}_{(j+1/2)\mp1/2}(M) \right) t - 1 \right),$$

which coincides with (24) if one replaces λ by $t = 2\tan\left(\frac{\lambda}{2}\right)$ in there. $\square$

The following result resumes what we have proven so far with the aid of Lemma IV.1.

Theorem IV.2. *Let* $\mu \in \mathbb{R} \setminus \{0\}$ *and*

$$\lambda = 2\tan\left(\frac{\mu}{2}\right).$$

Let $h(\lambda)_{m_j,k_j}$ be as in (6) and, for $\epsilon > 0$, let $h(\mu,\epsilon)_{m_j,k_j}$ be as in (27). If $a_\epsilon \in \sigma_p(h(\mu,\epsilon)_{m_j,k_j})$ and $\lim_{\epsilon \to 0} a_\epsilon = a$ for some $a \in (-m, m)$, then $a \in \sigma_p(h(\lambda)_{m_j,k_j})$.

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