Informal strategies of students with autism spectrum disorder in solving Cartesian product problems

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The present work focuses on arithmetic word problem solving and explores the strategies used by 26 students diagnosed with autism spectrum disorder when solving multiplicative Cartesian product problems. The students solved two outfit problems involving small and large numbers, respectively. The success in both problems was low. We found a variety of correct strategies, predominantly operation strategies. Most incorrect strategies were based on additive relations with modelling. We detail the difficulties observed during the problem-solving process, and implications for teaching students diagnosed with the disorder are drawn.

Keywords: Primary education, combinatorial thinking, cartesian product problems, solution strategies, autism spectrum disorder.

Introduction

Combinatorics constitutes a significant component of the mathematics curriculum, building on a rich structure of principles that underlie several other areas, such as counting, numeration, computation and probability. While developing their combinatorial thinking skills, children learn key mathematical skills such as constructing meaningful representations, reasoning mathematically, and generalizing mathematical concepts (English, 1991, 2005).

An important process in the development of combinatorial thinking skills is the acquisition of combinatorial strategies. A standard task to help children acquire these strategies is the Cartesian product problem (English, 1991), which consists in finding all possible combinations of two items, taken out of two different sets of items. Mulligan and Mitchelmore (1997) found that children in grades 2 and 3 used three main intuitive strategies for solving different types of problems with multiplicative structure, and all three are encountered among the correct resolutions of Cartesian product problems. These strategies are: i) direct modeling with counting strategies (when concrete manipulatives or drawings are used to model the problem situation, and objects are counted with no obvious reference to the multiplicative structure); ii) counting strategies (when the same actions are performed as in the previous level, but without the use of manipulatives); and iii) operation strategies (when multiplications are used). Several studies have shown that children develop these strategies intuitively, and that they acquire increasingly more sophisticated strategies for this type of problem, depending on age and experience (English, 1991; Maher & Martino, 1996). Mulligan and Mitchelmore (1997) note, though, that the Cartesian product problems are considered very difficult by the children, and most of the responses obtained in their study were incorrect. Furthermore, the majority of these incorrect responses were based on applying an inappropriate additive strategy, in which the numbers were added instead of multiplied. The prevalence of this incorrect strategy in the resolution of Cartesian product problems is confirmed by Nesher (1992), for students in grades 3 to

6, and by Ivars and Fernández (2016), for students in grades 1 to 6. The latter study additionally performs a more detailed analysis of the incorrect responses, identifying strategies such as one-to-one combinations (when elements are combined one to one without repetition) and nonsensical strategies (which include blank responses).

The focus of our research is on students with autism spectrum disorder (ASD). This disorder is characterized by deficits in social development, communication, and restrictive and repetitive behaviors or interests (American Psychiatric Association, 2013). These characteristics may lead to poor problem-solving capabilities, in particular since they often result in low reading comprehension and difficulties in thinking ahead or planning tasks. In order to improve problem-solving capabilities in ASD students, adapted instruction is required. To that end, there has been a growing interest in researching mathematical learning in this group (Bullen et al., 2020; Polo-Blanco et al., in press a; Polo-Blanco et al., in press b) and, in particular, in the strategies they employ when solving mathematical problems (Polo-Blanco et al., 2019, 2021). This research is especially relevant since students with ASD are increasingly incorporated into mainstream educational settings at all levels of education (Roberts & Webster, 2020).

The literature on probabilistic thinking, in particular combinatorial thinking, in students with ASD is very scarce. To our knowledge, only the work by López-Mojica, e.g. (2013), analyzes the resolution of combinatorial tasks in one student with ASD. The author highlights the need to explore combinatorial activities in order to introduce the idea of probability (López Mojica, 2013). At the same time, the importance of combinatorial thinking in students in general is clear, as emphasized by several authors (e.g., Eizenberg & Zaslavsky, 2004; English, 1991, 2005): First, as mentioned before, it allows them to acquire the mathematical skills that are present in the educational curricula. Second, people use basic principles of combinatorics in many everyday situations, for instance by enumerating all possible ways an event can occur, which is key to making informed decisions (Yee, 2009). Combinatorics therefore develops skills needed in daily life, and we consider this aspect to be especially relevant for ASD students, whom it helps to be more autonomous in their adult life.

For these reasons, in this paper we set out to investigate the strategies used by students with an ASD diagnosis when solving Cartesian product problems. In particular, we study the strategies they use when solving two "outfit problems", which require a multiplication to obtain all possible combinations. Based on the results in previous studies with students of typical development, we anticipate that the students with ASD will also experience difficulties in the task, and that they will use basic strategies in their resolution.

Our research questions are:

- What strategies do students with ASD employ to solve multiplicative Cartesian product problems?
- What are the main difficulties they encounter during the process of solving Cartesian product problems?

Methodology

We conducted an exploratory and descriptive investigation (Yin, 2017) in which we detailed the solving strategies of 26 students with ASD, as well as the main difficulties identified, when solving combination problems with multiplicative structure.

Participants

The participants were 26 students aged 6 to 12 years (23 males and 3 females), diagnosed with ASD according to DSM-5 (American Psychiatric Association, 2013), with minimum IQ of 70 on the WISC-V (Wechsler, 2014), and minimum equivalent mathematical age of 5.5 years. All of them were attending primary education in 19 ordinary schools in Cantabria (Spain). The mean chronological age of participants at the time was 9.35 years, with a standard deviation of 2.06. The mean IQ of the participants was 89.88, with a standard deviation of 11.77.

Data collection instrument

Based on Mulligan and Mitchelmore (1997), we designed a questionnaire with 16 multiplication and division problems of the types: equal groups, multiplicative comparison and Cartesian product. Of these 16 problems, students first solved 8 problems involving small numbers. Then, the students who had provided the correct solution for a problem were asked to solve the corresponding large-number problem. In this study we analyze the two Cartesian product problems that required a multiplication for their resolution, one with small numbers and one with large numbers. These problems are:

- Outfits Problem, Small (OPS): I have 3 shirts of different colors and 4 different pairs of pants. If I wear one shirt and one pair of pants each time, in how many ways can I dress?
- Outfits Problem, Large (OPL): I have 8 shirts of different colors and 3 different pairs of pants. If I wear one shirt and one pair of pants each time, in how many ways can I dress?

The students solved these problems individually, in one session of approximately 25 minutes and in a classroom free of distractions, with only the interviewer and the student present. Before starting to solve the problems, the interviewer explained what the test consisted of, and made sure that he or she understood the statements, reading them with him or her in cases where the student was confused. The student was told that he or she could write, use manipulatives (interlocking blocks) or answer orally. All sessions were videotaped, and the solutions were transcribed for later analysis. The students' strategies were coded by the fourth author. An experienced mathematics education teacher, who was blind to the hypotheses of the study, recoded 30% of the students' strategies. The mean interobserver reliability for strategy categorization was 94%, calculated as the number of agreements divided by the number of agreements plus disagreements and multiplied by 100.

Analysis categories

We adhered to the following system for classifying the strategies used to solve multiplicative structure problems (Ivars & Fernández, 2016; Mulligan & Mitchelmore, 1997): incorrect strategies (level 0), direct modeling with counting (level 1), counting (level 2) and operation strategies (level 3). The incorrect strategies (level 0) considered were inappropriate additive relationships, one-to-one combinations, and given number (when one of the numbers in the problem is given as the answer).

Results

Table 1 shows the strategies followed by the students on the Cartesian product problems with small numbers, OPS. Eight out of the 26 students followed correct resolution strategies, the most frequent one being operations based (six students). Two students (S7 and S15) represented this strategy symbolically in the form of a horizontal algorithm, while another two (S13 and S26) expressed the multiplication verbally ("Three times four"). The last two students (S32 and S35) started by manipulating cubes and then gave the answer, one verbally and the other symbolically. In Figure 1, we can see that S32 used the orange and purple blocks to create structures of different heights, representing respectively the three T-shirts and the four pants. He then selected an orange structure and hit it against each of the purple ones, saying aloud the numbers "*one*" till "*four*". Finally, he said "*four times three*" and wrote the number 12 as the solution. S35 joined four blocks and then another three, and wrote the number "7", as we can see in Figure 1. He then corrected "*ah, but it asks you how many ways... Seven is the total*". He wrote the multiplication in the form of a vertical algorithm as the result and said, "*Twelve ways. I think 12 ways*", and he crossed out the number seven he wrote earlier.

Table 1: Strategies followed for the OPS problem

Correct strategies			Incorrect strategies (level 0)			
Direct modeling with counting (level 1)	Counting (level 2)	Operation strategies (level 3)	Inappropriate additive relationships	One-to-one combinations	Given number	Other
S10	S19	S7, S13, S15, S26, S32, S35	S3, S4, S11, S12, S16, S17, S20, S21, S24, S25, S27, S31	S8, S29	S28, S34	S14, S30

One student (S10) demonstrated a matching strategy that he expressed through drawings of all possible combinations of shirts with pants. As shown in Figure 1, he assigned a number to each shirt and pair of pants, and used the symbol "+" to express the pairing. After finishing the drawing, S10 counted the pairs obtained and provided the answer.

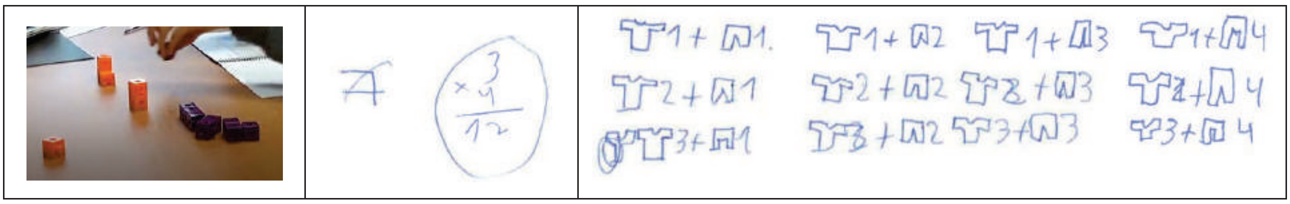

Figure 1: Examples of correct solution strategies, by S32 (left), S35 (middle), and S10 (right)

Another student (S19) used a correct counting strategy, although he made a calculation error when executing it. In particular, his strategy consisted in the repeated addition of the same number ("*four*"). He performed mental calculation to keep track of the running total, while using his fingers to represent the amount of times he had added this number. Eventually, however, he got confused and raised an additional, fourth finger, answering: "*I would say sixteen*".

The most often encountered incorrect solution strategy for the OPS problem was the application of inappropriate additive relations (12 students). In this case, the students added the quantities given in the statement instead of multiplying them, obtaining "7" as a result. Out of them, five students responded verbally: three stated this orally and two (S17 and S31) simply wrote the result without further explanation. In Figure 2, we can see that two students responded by expressing the sum symbolically, one in the form of a vertical algorithm (S16) and one in the form of a horizontal algorithm (S20). One student (S3) used drawings to help him perform the additive strategy. After reading the problem, he drew a boy wearing pants and a T-shirt, and an additional two T-shirts and three pairs of pants around it. Interestingly, one of the T-shirts resembled very much the one he was wearing at the moment, and from his remarks he was imagining these were his clothes: "*Okay, I always wear this one... ah, no, only on one day I wear this one*". Finally, he said "*It would be one plus one equals two*", and concluded "*seven*", which he wrote down as the result. Another student (S4) used cubes to calculate the result, as we can see in Figure 2. He picked up three cubes with one hand and placed them on the problem sheet, and then put four more cubes, concluding that there were "*seven*" different shapes.

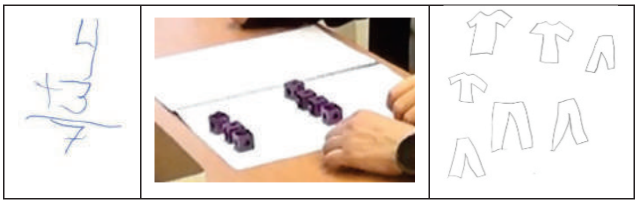

Figure 2: Examples of incorrect solution strategies, by S16 (left), S4 (middle), and S12 (right)

Three students (S11, S12 and S27) tried multiple representations to solve the problem. S11 initially took three red marbles with his left hand and four orange blocks with his right hand, said "*seven",* and wrote the number "7". He argued to the interviewer that this was the result by saying, "*Because I added the t-shirts [shows his right hand with four orange blocks] and also the pants and in total it would give... [starts singing, playing with the chips]*." S12 initially answered, "*Three and four, seven."* After the interviewer asked him what that "seven" was, S12 began to draw the 3 shirts and the 4 pants, as we can see in Figure 2. After the interviewer insisted "*how many ways can I dress?"* S12 repeated, "*Seven"*. S27 made arguments apparently unrelated to the task and first said that the answer was "14", writing down "14 and 30" and finally ended up saying that it was "7":

S27: I got it, seven.
Interviewer: And how do you know it's seven?
S27: The first one you put the shirt on, then socks and pants and lastly combing our hair and brushing our teeth. Okay? That's it.
Interviewer: So you... you count seven things that you do. But why do you know it's seven?
S27: Because I do, because three plus four is seven.

Two students (S8 and S29) performed an incorrect one-to-one combination strategy, by matching each garment from one set with one from the other set, without repetition. Both students expressed this verbally. For instance, S29 wrote "*three ways"* and argued *"because there are four pants, I can only use three because... [he thinks] because I have one pair of pants left over*".

Two of the students (S28 and S34) responded a number already given in the statement. S28 verbally expressed that the solution was "*three*", and argued that "*because he had heard it*". S34 answered several times as a result "*many*", and, after the interviewer requested that he specify how many, he said "*three or four*", which are the number of shirts and pants given in the statement, respectively.

One student (S14) performed a strategy that could not be identified as any of the previous. After reading the problem, S14 said "*Three, four... Ouch! Let's see...*" and wrote the number "5", and argued: "*Three, four, five*". We interpret that he provided "5" as the answer because it was the next number in the numerical sequence. A final case of a strategy classified as "other" is that of S30, who drew a picture of a boy wearing a tracksuit, copying some letters from his own jacket. Although the interviewer insisted that he continue, S30 was tired and distracted and did not answer anything else.

All students who obtained a correct answer in the OPS problem went on to solve the large-number multiplication problem, OPL. These students, seven in total, again used a correct strategy to solve this second problem, as summarized in Table 2. Specifically, most of them used operation strategies (5 students), which three of them (S7, S15 and S32) represented symbolically in the form of a horizontal algorithm, and the other two (S13 and S26) expressed verbally. For instance, S26 read the problem and said "*I think I am going to multiply eight by three*", and then wrote "24" as a result.

Table 2: Strategies followed for the OPL problem

Correct strategies		
Direct modeling with counting (level 1)	Counting (level 2)	Operation strategies (level 3)
S10, S35		S7, S13, S15, S26, S32

Student S10 repeated the matching strategy he had applied successfully in the OPS problem, drawing all possible combinations of shirts and pants. This time, he represented them by the letters "C" (from "*camiseta*", in Spanish) and "P" (from "*pantalones*") accompanied by numbers, as shown in Figure 3. When finished drawing, he counted the pairs obtained and provided the answer.

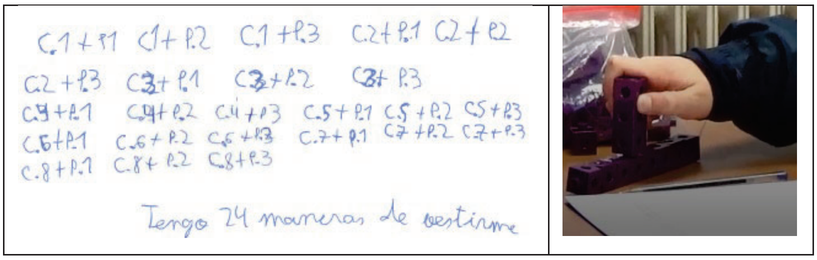

Figure 3: Examples of correct solution strategies for OPL, by S10 (left) and S35 (right)

Finally, student S35 used a modeling strategy with a manipulative type of representation making use of blocks. He first joined eight blocks, and then another three blocks, after which he combined both groups forming an inverted "T", as we can see in Figure 3. Following this, he touched each of the blocks in the row of eight and repeated this step three times. He then said "Twenty-four".

Interviewer: Okay, how did you know?
S35: By counting per pair of pants how many shirts there are.
Interviewer: And what did you count?
S35: Well I counted [touching the blocks in the row where there are eight]: one, two, three, four, five, six, seven, eight [counts the row again] nine, ten, eleven,...
Interviewer: Okay okay.

Discussion and conclusion

This work contributes to the area of problem solving in students with ASD. Specifically, we have analyzed the strategies used by students with ASD when solving Cartesian product problems that involve multiplications. Most of the students failed to solve the problems correctly, and a variety of strategies were found in the analysis of their solutions. The most frequently used correct approach consisted of operation strategies based on internalized calculations. In line with the results found in the literature for students of typical development (e.g., Ivars & Fernández, 2016), the most frequent incorrect strategy was the use of additive relations, carried out on many occasions through modeling.

The results show significant difficulties in understanding the problems, confirming previous studies on problem solving in ASD students (Polo-Blanco et al., 2019), which could be related to the language difficulties characteristic of the disorder. In order to facilitate the understanding of Cartesian product problem solving, the problems could be contextualized to topics familiar to the student, in line with previous work (Polo-Blanco et al., 2021). In addition, basic modeling strategies could help the student understand the situation and the combinations posed in the problem. In order to move from modeling and counting strategies to operation strategies, it is advisable to adapt the instruction to the needs observed, and to start from the strategy used by the student. For instance, if the student uses a table to list the combinations, it may be useful to help them see that the number of combinations coincides with the result of the multiplication. In general, teaching methodologies adapted to the characteristics of ASD students should be designed for the resolution of these problems (Polo-Blanco et al., in press a), for instance, by including self-instruction lists with the support of visual guides.

The results of this work allow us to further explore the elements that hinder the learning of students with ASD, in order to offer effective instructions to achieve an improvement in academic performance and, ultimately, a greater autonomy and quality of life in adulthood.

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