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ABSTRACT

We investigate the properties of self-adjointness of a two-dimensional Dirac operator on an infinite sector with infinite mass boundary conditions and in the presence of a Coulomb-type potential with the singularity placed on the vertex. In the general case, we prove the appropriate Dirac–Hardy inequality and exploit the Kato–Rellich theory. In the explicit case of a Coulomb potential, we describe the self-adjoint extensions for all the intensities of the potential relying on a radial decomposition in partial wave subspaces adapted to the infinite-mass boundary conditions. Finally, we integrate our results, giving a description of the spectrum of these operators.

I. INTRODUCTION

In this paper, we are interested in the two-dimensional Dirac operator on an infinite sector, subject to infinite mass boundary conditions, in the presence of a singular potential of Coulomb type, centered in the corner of the sector. The descriptions of the self-adjointness and spectral properties of the Dirac operator in a sector with infinite mass boundary conditions and of the Dirac operator with a Coulomb-type perturbation, respectively, are well understood, but a detailed analysis of the coupling of the two features is missing. It is interesting to describe their interaction since the two share the same singular nature: this is particularly evident in the case of an explicit Coulomb perturbation; see Remark II.2.

The Dirac operator was introduced in Ref. 1 as the Hamiltonian generating the evolution of a relativistic particle with spin $\frac{1}{2}$ on the whole three-dimensional space, and its analysis has been the subject of many investigations (see the monography by Thaller[2]). Parallel to that, it has found many other applications for both quark models in the atomic nucleus or, in its two-dimensional version, in the analysis of materials with Dirac fermion low-energy excitations, the most famous being certainly graphene; see, e.g., Ref. 3 for a review. For these models, it is physically meaningful to consider the operator on some domain with boundaries to model either the confining property of quarks or the edge of a material. From a mathematical point of view, the introduction of boundaries requires that appropriate boundary conditions have to be considered in order to preserve self-adjointness.

The free Dirac operator in two spatial dimensions is given by the following formal expression:

$$D_0 := -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla + m\sigma_3 = \begin{pmatrix} m & -\mathrm{i}(\partial_{x_1} - \mathrm{i}\partial_{x_2}) \\ -\mathrm{i}(\partial_{x_1} - \mathrm{i}\partial_{x_2}) & -m \end{pmatrix},$$

where $m \geq 0$ is the mass of the particle and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2)$, $\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The free Dirac operator in $\mathbb{R}^2$ is realized as a self-adjoint operator with domain $H^1(\mathbb{R}^2;\mathbb{C}^2)$. Its spectrum is purely essential and $\sigma(D_0) = \sigma_{\text{ess}}(D_0) = (-\infty, -m] \cup [m, +\infty)$. In fact, $-\mathrm{i}\boldsymbol{\sigma} \cdot \nabla$ is equivalent to the multiplication operator $\boldsymbol{\sigma} \cdot \boldsymbol{k}$ through a Fourier transform; see Ref. 2, Chap. 1 for details.

In the analysis of boundary value problems on connected domains, one of the most interesting examples of boundary conditions for the applications is the one known as *infinite mass boundary condition*. As its name suggests, it is given by considering the limit case of infinite mass outside the domain; see Refs. 4 and 5. In detail, let $\Omega \subset \mathbb{R}^2$ be a connected domain such that its boundary $\partial\Omega$ is regular enough: we denote by $\mathbf{n}$ the outward normal and by $\mathbf{t}$ the tangent vector to $\partial\Omega$ chosen in such a way that $(\mathbf{n}, \mathbf{t})$ is positively oriented.

The infinite mass boundary condition is defined as

$$\mathcal{B}_{\mathbf{n}}\psi = \psi \quad \text{on } \partial\Omega, \tag{1.1}$$

where the matrix $\mathcal{B}_{\mathbf{n}}$ is given by

$$\mathcal{B}_{\mathbf{n}} = -\mathrm{i}\sigma_3\, \boldsymbol{\sigma} \cdot \mathbf{n}.$$

The regularity of Ω plays a fundamental role in this sort of problem. When Ω is C^2-regular, the Dirac operator D_0 acting on the set of functions in $H^1(\Omega;\mathbb{C}^2)$ and that verifies (1.1) is self-adjoint; see Ref. 6.

Such a result is not anymore valid if we relax the regularity hypothesis of Ω and consider, for instance, domains with corners. Let $\omega \in (0, 2\pi]$, and let S_ω be the two-dimensional open sector of aperture ω,

$$S_\omega := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : r > 0,\ 0 < \theta < \omega\}. \tag{1.2}$$

The problem of self-adjointness for the Dirac operator on S_ω with infinite mass boundary conditions is well understood: we resume some of the results from Refs. 7 and 8 in the following theorem.

Theorem I.1 *(Refs. 7 and 8). Let $\omega \in (0, 2\pi]$ and S_ω be defined as in (1.2). Let H_ω be the operator*

$$\begin{aligned} &H_\omega\psi := D_0\psi, \\ &\mathcal{D}(H_\omega) := \{\psi \in H^1(S_\omega;\mathbb{C}^2) : \mathcal{B}_{\mathbf{n}}\psi = \psi \quad \text{on } \partial S_\omega\}. \end{aligned} \tag{1.3}$$

Then, the following holds:

(i) *if $0 < \omega \leq \pi$, H_ω is self-adjoint;*
(ii) *if $\pi < \omega \leq 2\pi$, H_ω admits infinite self-adjoint extensions, and among them, there exists a unique distinguished one whose domain is included in the Sobolev space $H^{1/2}(S_\omega;\mathbb{C}^2)$.*

Remark I.2. It is well known that when Ω is a bounded connected Lipschitz domain, the boundary trace operator $\mathrm{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is well defined and bounded. However, for bounded domains, H^1 is not the maximal domain for the differential expression $\boldsymbol{\sigma} \cdot \nabla$. For this reason, it is convenient to introduce

$$\mathcal{K}(\Omega) := \{u \in L^2(\Omega;\mathbb{C}^2) : \boldsymbol{\sigma} \cdot \nabla u \in L^2(\Omega;\mathbb{C}^2)\}.$$

For this space, a weaker notion of boundary trace can be given. Indeed, when Ω is a *curvilinear polygon*, then the operator $\boldsymbol{\sigma} \cdot \mathbf{n}tr : H^1(\Omega;\mathbb{C}^2) \to L^2(\partial\Omega,\mathbb{C}^2)$ extends to a bounded operator $T : \mathcal{K}(\Omega) \to H^{-1/2}(\partial\Omega;\mathbb{C}^2)$; see Ref. 8, Lemma 2.3. Then, for $0 < \omega \leq \pi$, the boundary condition $\mathcal{B}_{\mathbf{n}}\psi = \psi$ on ∂S_ω in (1.3) is intended in the sense of $H^{1/2}(\partial S_\omega;\mathbb{C}^2)$, while for $\pi < \omega \leq 2\pi$, it has to be intended in the weaker sense of $H^{-1/2}(\partial\Omega;\mathbb{C}^2)$.

We refer to Ref. 8 for the description of the more general quantum dot boundary conditions. Moreover, we refer to Ref. 9 for the analysis of the self-adjointness in the case of discontinuous infinite mass boundary conditions. Finally, the analogous problem in the three-dimensional setting, namely, the self-adjointness of the Dirac operator on a three-dimensional cone with MIT bag boundary conditions, can be found in Ref. 10.

A strictly related topic is the description of Dirac operators with δ-shell interactions (see, e.g., Refs. 11–17, the review papers,[18,19] and the references therein) in both the two- and three-dimensional settings. In fact, it is possible to describe Dirac operators on domains as Dirac operators coupled with δ-shell interactions generating confinement; see Ref. 20, Sec. 2.3. In this field of research, there has been a big effort to lower the regularity assumptions for the boundary of the considered domain: we refer to Refs. 14 and 21 and references therein for the general case of domains with Lipshitz boundary. In particular, in Ref. 14, it is shown that the Dirac operator on a compact region with a corner admits a unique self-adjoint realization whose domain is included in $H^{1/2}(S_\omega;\mathbb{C}^2)$, but in the particular case of the sector, the authors of Refs. 7 and 8 provided a more precise description of the domain. In addition, we refer to Ref. 22 for the description of the Dirac operator with Lorentz-scalar δ-shell interactions supported on star-graphs.

As mentioned before, the analysis of the two-dimensional Dirac equation has attained a certain amount of interest from low-energy condensed matter physics. The successful experimental isolation of a single plane of graphene provides an interesting test for non-perturbative quantum electrodynamics.[23] In fact, depending on the material graphene is deposited on, electronic excitations can be well described in terms of a massive or a massless Dirac equation.[24] These substrates interact with graphene, resulting in effective potentials that may break symmetries of the lattice or generate gaps in the electronic spectrum. The analysis of charged impurities is of particular importance as they play an important role in the transport properties of graphene. In this context, parameters entering the Dirac equation translate in a small mass and strong interaction characterized by a large value of the effective fine structure constant, and one consequently expects that charge impurities may lead to phenomena beyond the perturbative description of quantum electrodynamics, such as the "vacuum polarization."[25] These problems are treated in the literature by adding a (critical or, even, supercritical) Coulomb potential to the Dirac equation. Therefore, we can think of an excitation of graphene to be modeled by a Dirac equation in two spatial dimension with a $\nu/|x|$ potential centered in the position of an impurity.[26]

Naturally, the history of the Dirac–Coulomb operator begins in the three-dimensional setting as the very first motivation for its introduction was the analysis of the relativistic correction to the spectral lines of the hydrogen atom. We summarize its very interesting and rich history; see Refs. 28 and 27 or Ref. 29, Sec. 1.3 for more details. Rigorous analysis of the Dirac–Coulomb Hamiltonian devoted to establishing its self-adjointness dates back to the early 1950s in the works of Rellich[30] and Kato;[31] only in the early 1970s, it was recognized by several authors that the operator with purely Coulomb potential was essentially self-adjoint if and only if $|\nu| \le \frac{\sqrt{3}}{2}$. In the same years, three (in principle) *distinguished* self-adjoint extensions were built by Schmincke,[45] Nenciu,[33] and Wüst[34] in the regime of higher nuclear charge[32] ($\frac{\sqrt{3}}{2} < |\nu| < 1$), and just before the end of the decade, it was recognized that the three extensions were, in fact, the *same*.[35] It took several years to develop powerful Hardy–Dirac inequalities to push the definition of *the* distinguished extension up to the value $|\nu| = 1$ in Refs. 29 and 36.

In the regime $\frac{\sqrt{3}}{2} < |\nu|$, the Dirac–Coulomb operator in three spatial dimension is *not* essentially self-adjoint, so the research focused on the classification of *all* the self-adjoint realizations of the formal operator in this regime. This result was achieved correctly in Ref. 37 for $\frac{\sqrt{3}}{2} < |\nu| < 1$, in Ref. 38 for $|\nu| > 1$, and in Refs. 27 and 39 for any $\nu \in \mathbb{R}$ with different techniques: the adaptation of Kreĭn–Višik–Birman–Grubb extension scheme, von Neumann extension theory, and the restriction of the domain of the adjoint and boundary triplets, respectively. More recently, Dereziński and Ruba[40] classified and carefully analyzed closed extensions with complex-valued potentials. Leaving the realm of electrostatic fields generated by one point charge, we mention Ref. 44, where the authors proved the existence of a distinguished self-adjoint extension for a generic (in a certain sense "subcritical") charge distribution and Ref. 43 where the authors construct a self-adjoint realisation of the two-body Dirac-Coulomb operator.

Let us emphasize that the analyses of Refs. 27 and 37–39 rely on the angular decomposition of the Dirac–Coulomb operator; therefore, the results can be translated directly to the two-dimensional case modifying only the eigenvalues of the angular momentum appearing in the radial operator. In particular, the two-dimensional Dirac–Coulomb operator ceases to be essentially self-adjoint when $\nu \neq 0$ and the distinguished extension exists for $|\nu| < \frac{1}{2}$; see Ref. 41.

A. Main results

In this paper, we are interested in perturbing the Dirac operator H_ω on a sector with a potential of Coulomb-type. To study self-adjointness, we use two different approaches: the Kato–Rellich theory and the explicit radial decomposition of the operator. In the following, we assume $m = 0$ without loss of generality since a bounded perturbation does not influence such a property.

The first and crucial tool in the analysis of perturbations of self-adjoint operators is the Kato–Rellich theorem. Its use has a deep impact in physical applications being, for example, the key ingredient to prove self-adjointness of atomic Hamiltonians in non-relativistic quantum mechanics: in this setting, the inter-particle interaction is "small" in a certain sense with respect to the graph norm of an (essential) self-adjoint Hamiltonian. From an analytical point of view, the smallness of the inter-particle potential requires the validity of a Hardy inequality. Mimicking this approach for the Dirac operator on a sector, we present, as the first result of this paper, a Hardy inequality for the Dirac operator on sectors with infinite mass boundary conditions.

Theorem I.3 *(Dirac–Hardy inequality). Let $\omega \in (0, 2\pi]$ and S_ω be as in (1.2), and let H_ω be defined as in (1.3). For any $\psi \in \mathcal{D}(H_\omega)$, we have that*

$$\int_{S_\omega} |\boldsymbol{\sigma} \cdot \nabla \psi|^2 \, dx \ge \frac{(\pi - \omega)^2}{4\omega^2} \int_{S_\omega} \frac{|\psi|^2}{|x|^2} \, dx. \tag{1.4}$$

Remark I.4. We underline that $\| -i\boldsymbol{\sigma}\cdot\nabla\psi\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)} = \|\nabla\psi\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)}$. Consequently, a non-trivial Hardy inequality as in (1.4) does not hold if we replace S_ω with the whole $\mathbb{R}^2$ since it does not hold for the gradient. Theorem I.3 shows that the Hardy-type estimate (1.4) holds when we restrict to the domain S_ω: such a phenomenon is also known to happen for the Hardy inequality for the gradient.

Thanks to the Kato–Rellich and Wüst theorems (Ref. 45, Theorems X.12 and X.14), Theorem I.3 immediately implies the following stability result for the self-adjointness of H_ω under unbounded perturbations of Coulomb-type.

Corollary I.5. Let $\omega\in(0,\pi)$, S_ω be as in (1.2), and H_ω be defined as in (1.3). Let $V : S_\omega \to \mathbb{C}^{2\times 2}$ such that $V(x)$ is Hermitian for a.a. $x\in S_\omega$ and such that for some $\nu>0$,

$$|V(x)| \le \frac{\nu}{|x|} \quad \text{for a.a. } x\in S_\omega,$$

with $|V(x)|$ being the operator norm of the matrix $V(x)\in\mathbb{C}^{2\times 2}$. Then, the following holds:

(i) *if $\nu < \frac{\pi-\omega}{2\omega}$, $H_\omega + V$ is self-adjoint with $\mathcal{D}(H_\omega+V) = \mathcal{D}(H_\omega)$;*
(ii) *if $\nu = \frac{\pi-\omega}{2\omega}$, $H_\omega + V$ is essentially self-adjoint on $\mathcal{D}(H_\omega+V) = \mathcal{D}(H_\omega)$.*

Remark I.6. Hypotheses of Corollary I.5 are satisfied for potentials that *locally* diverge logarithmically. This is important because the divergence of the two-dimensional electrostatic field in dimension 2 close to the charge is logarithmic. However, as discussed above, the interest for potentials of the type $1/|x|^\alpha$ arises when restricting a three-dimensional model to a two-dimensional effective one.

In the particular case of the *Coulomb potential,*

$$V(x) := \frac{\nu}{|x|}\mathbb{1}_2 \qquad \text{for all } x\in S_\omega\setminus\{0\},$$

we can provide a much more detailed description of the self-adjoint realizations of D_0+V exploiting the radial symmetry: in the following, we extend (and improve) the results in Corollary I.5 to any angle $\omega\in(0,2\pi]$ and $\nu\in\mathbb{R}$. We define the *minimal* operator $H_{\min}$ as follows:

$$\begin{aligned}\mathcal{D}(H_{\min}) &:= \left\{u\in C_c^\infty\left(\overline{S_\omega}\setminus\{0\};\mathbb{C}^2\right) : \mathcal{B}_{\mathbf{n}}u = u \quad \text{on } \partial S_\omega\right\},\\ H_{\min}u &:= (D_0+V)u.\end{aligned} \tag{1.5}$$

The operator $H_{\min}$ is symmetric, as can be seen with an explicit integration by parts.

Our next result is the classification of the self-adjoint extensions of the minimal operator $H_{\min}$. For this purpose, we denote by $\mathbb{N}$ the set of all natural numbers, including zero.

Theorem I.7. *Let $\omega\in(0,2\pi]$, S_ω be as in (1.2), and $H_{\min}$ be as in (1.5). Then, the following holds:*

(i) *If $\nu^2 \le \frac{\pi^2-\omega^2}{4\omega^2}$, the operator $H_{\min}$ is essentially self-adjoint and*

$$\mathcal{D}(\overline{H_{\min}}) = \mathcal{D}(H_\omega) = \left\{\psi\in H^1(S_\omega;\mathbb{C}^2) : \mathcal{B}_{\mathbf{n}}\psi = \psi \quad \text{on } \partial S_\omega\right\}.$$

(ii) *If $\nu^2 > \frac{\pi^2-\omega^2}{4\omega^2}$, the operator $H_{\min}$ has infinitely many self-adjoint extensions and there exists a one-to-one correspondence between the self-adjoint extensions of $H_{\min}$ and the space $\mathcal{U}(d+1)$ of the unitary matrices on $\mathbb{C}^{d+1}$, being*

$$d := \max\left\{k\in\mathbb{N} : k < \frac{\omega}{\pi}\sqrt{\nu^2+\frac{1}{4}} - \frac{1}{2}\right\}. \tag{1.6}$$

Remark I.8. For $\omega\in(0,\pi)$, $\frac{(\pi-\omega)^2}{4\omega^2} < \frac{\pi^2-\omega^2}{4\omega^2}$, so Theorem I.7 (i) gives a better result than Corollary I.5. This is not surprising: already in the whole space, a similar phenomenon occurs. In fact, the Dirac–Hardy inequality (1.4) does not allow us to exploit the peculiar matricial form of the Coulomb potential and provides a more general (and weaker) result. For a discussion on this feature in the three-dimensional setting, we refer to the introduction of Ref. 46.

When $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2 \le \frac{\pi^2}{4\omega^2}$, it is possible to select a *distinguished* self-adjoint extension among all the self-adjoint extensions given in Theorem I.7 (ii), requiring that the functions in its domain have the best possible behavior in the origin. For this purpose, for $w\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, set

$$\mathcal{D}(w,B_1) := \{u\in L^2(\mathbb{R}^2) : wu\in L^2(B_1)\},$$

where B_1 denotes the ball of radius 1 centered at the origin.

Theorem I.9. *Under the assumptions of Theorem I.7, assume, moreover, that* $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2$. *Then, the following holds:*

(i) *If* $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2 < \frac{\pi^2}{4\omega^2}$, *there exists a unique self-adjoint extension* $T^{(D)}$ *of* $H_{\min}$ *such that*

$$\mathcal{D}(T^{(D)}) \subset \mathcal{D}(|x|^{-a}, B_1) \qquad \textit{for all } 0 \le a < \frac{1}{2} + \sqrt{\frac{\pi^2}{4\omega^2} - \nu^2}.$$

Thus, $T^{(D)}$ *is the distinguished extension.*

(ii) *If* $\frac{\pi^2}{4\omega^2} = \nu^2$, *there exists a unique self-adjoint extension* $T^{(D)}$ *of* $H_{\min}$ *such that*

$$\mathcal{D}(T^{(D)}) \subset \mathcal{D}((|x|^a \log^2|x|)^{-1}, B_1) \qquad \textit{for all } 0 \le a \le \frac{1}{2}. \tag{1.7}$$

Thus, $T^{(D)}$ *is the distinguished extension.*

(iii) *If* $\frac{\pi^2}{4\omega^2} < \nu^2$, *there exist infinite extensions T of* $H_{\min}$ *such that*

$$\mathcal{D}(T^{(D)}) \subset \mathcal{D}((|x|^a \log^2|x|)^{-1}, B_1) \qquad \textit{for all } 0 \le a \le \frac{1}{2}. \tag{1.8}$$

Thus, $H_{\min}$ *does not have any distinguished extension.*

Remark I.10. If $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2 < \frac{\pi^2}{4\omega^2}$, the distinguished extension $T^{(D)}$ can be characterized in terms of Sobolev regularity. Indeed, combining Theorem I.9 with Theorem 1.4.5.3 in Ref. 47, $T^{(D)}$ is the unique extension of $H_{\min}$ that verifies

$$\mathcal{D}(T^{(D)}) \subset H^s(S_\omega; \mathbb{C}^2) \qquad \textit{for } s < \frac{1}{2} + \sqrt{\frac{\pi^2}{4\omega^2} - \nu^2}.$$

Nevertheless, this characterization fails in the case $\nu^2 \ge \frac{\pi^2}{4\omega^2}$, where one can see that there exists infinite self-adjoint extension verifying the following property:

$$\mathcal{D}(T) \subset H^s(S_\omega; \mathbb{C}^2) \qquad \textit{for } s < \frac{1}{2}.$$

Having established the self-adjointness of Coulomb-type perturbations of H_ω, we turn our analysis to a description of their spectrum. In the following part of the Introduction, we consider, in general, $m \ge 0$ in the definition of H_ω. Our first result in this direction complements Corollary I.5 and Theorem I.7, investigating the stability of the essential spectrum of H_ω under general Coulomb-type perturbations.

Theorem I.11. *Let* $H_{\min}$ *be the operator defined in (1.5), and let T be any self-adjoint extension of* $H_{\min}$. *Then,*

$$\sigma_{\text{ess}}(T) = (-\infty, -m] \cup [m, +\infty).$$

Moreover, when $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2$, *that is, when* $H_{\min}$ *is not essentially self-adjoint, for any* $\lambda \in (-m, m)$, *there exists T a self-adjoint extension of* $H_{\min}$ *for which* λ *is an eigenvalue.*

Remark I.12. The result of Theorem I.11 can be translated immediately to the case of singular potentials as V verifying the hypothesis of Corollary I.5. In this case, the self-adjoint realization $H_\omega + V$ has $\sigma_{\text{ess}}(H_\omega + V) = (-\infty, -m] \cup [m, +\infty)$.

The infinite mass boundary conditions prevent the massive operator H_ω to be diagonalized by the unitary transformation of Proposition A.3. This makes difficult to mimic the computation of eigenvalues or the characterization of the discrete spectrum of Dirac operators with explicit Coulomb potentials as in Ref. 48. Hence, we cannot provide further details in the case of an explicit Coulomb potential using the radial decomposition.

II. THE RADIAL OPERATOR AND PROOFS OF THEOREM I.7 AND THEOREM I.9

To prove Theorems I.7 and I.9, we decompose the Dirac operators $H_{\min}$ for $m = 0$ in the direct sum of one-dimensional Dirac operators on the half-line. We introduce some notations: for $k \in \mathbb{N}$, set

$$\lambda_k := \frac{(2k+1)\pi}{2\omega}, \tag{2.1}$$

and let $d_{\nu,k}$ be the differential expression

$$d_{\nu,k} := \begin{pmatrix} \frac{\nu}{r} & -\partial_r - \frac{\lambda_k}{r} \\ \partial_r - \frac{\lambda_k}{r} & \frac{\nu}{r} \end{pmatrix}. \tag{2.2}$$

We define the following Dirac operators on the half-line:

$$\begin{aligned} &\mathcal{D}(h_{\nu,k}) := C_c^\infty\big((0,+\infty);\mathbb{C}^2\big), \\ &h_{\nu,k}u := d_{\nu,k}u. \end{aligned} \tag{2.3}$$

Proposition II.1. Let $\nu \in \mathbb{R}$, $\omega \in (0,2\pi]$, S_ω be defined as in (1.2), and $H_{\min}$ be defined as in (1.5), and for all $k \in \mathbb{N}$, let $h_{\nu,k}$ be defined as in (2.3). Then,

$$H_{\min} \cong \bigoplus_{k\in\mathbb{N}} h_{\nu,k},$$

where "≅" means that the operators are unitarily equivalent.

Remark II.2. The *partial wave subspace* decomposition of H_ω (given by Proposition II.1 for $\nu = 0$) leads us to the analysis of an orthogonal sum of half-line Dirac operators perturbed by off-diagonal Coulomb potentials, expressed via the eigenvalues of the *spin–orbit operator*. This is the reason why we perturb the operator H_ω with an external Coulomb-potential. Roughly speaking, the presence of a corner in the origin and the presence of an external Coulomb perturbation have the same singular nature: they both imply the presence of a singular term of order $\sim 1/r$ in the radial operators $h_{\nu,k}$.

The Proof of Proposition II.1 exploits the radial symmetry of the problem and takes advantage of the decomposition of the Hilbert space $L^2(S_\omega;\mathbb{C}^2)$ in the *partial wave subspaces*. We omit these details here, and we leave the Proof of Proposition II.1 to the Appendix.

Thanks to Proposition II.1, the Proof of Theorem I.7 follows from the analysis of the same properties on the reduced operators $h_{\nu,k}$. For any $k \in \mathbb{N}$, the operator $h_{\nu,k}$ is a *radial* Dirac operator and it has been studied in several works. Indeed, the operator $h_{\nu,k}$ is precisely in the form of the one defined in Ref. 27, Eq. (2.19) with $m = \lambda = \mu = 0$ and $k_j = -\lambda_k$.

In the following proposition, we study its self-adjointness.

Proposition II.3. Let $k \in \mathbb{N}$, and let $h_{\nu,k}$ be defined as in (2.3). Let, moreover,

$$\chi \in C_c^\infty(\mathbb{R};[0,1]) \text{ such that } \chi'(r) \le 0 \text{ and } \chi(r) = \begin{cases} 1 & \text{for } r \le 1, \\ 0 & \text{for } r \ge 2. \end{cases} \tag{2.4}$$

Thus, the following holds:

(i) *If $\lambda_k^2 - \nu^2 \ge 1/4$, then $h_{\nu,k}$ is essentially self-adjoint. Moreover,*

$$\begin{aligned} &\mathcal{D}(\overline{h_{\nu,k}}) = H_0^1((0,+\infty);\mathbb{C}^2) \qquad \text{if } \lambda_k^2 - \nu^2 > \frac{1}{4}, \\ &\mathcal{D}(\overline{h_{\nu,k}}) \supsetneq H_0^1((0,+\infty);\mathbb{C}^2) \qquad \text{if } \lambda_k^2 - \nu^2 = \frac{1}{4}. \end{aligned}$$

(ii) *If $0 < \lambda_k^2 - \nu^2 < 1/4$, then $h_{\nu,k}$ is not essentially self-adjoint and it admits a one parameter family of self-adjoint extensions $\{t_{\nu,k}^{(\alpha)}\}_{\alpha\in[0,\pi)}$ such that*

$$\mathcal{D}(t_{\nu,k}^{(\alpha)}) = \operatorname{span}\Big\{u_{\nu,k}^{(\alpha)}\Big\} + H_0^1((0,+\infty);\mathbb{C}^2),$$

being

$$u_{\nu,k}^{(\alpha)}(r) = P_{\nu,k}\cdot \begin{pmatrix} \cos(\alpha) r^{\sqrt{\lambda_k^2-\nu^2}} \\ \sin(\alpha) r^{-\sqrt{\lambda_k^2-\nu^2}} \end{pmatrix} \chi(r) \qquad \text{for } r > 0,$$

where $P_{\nu,k} \in \mathbb{R}^{2\times 2}$ *is the invertible matrix,*

$$P_{\nu,k} := \frac{1}{2\sqrt{\lambda_k^2 - \nu^2}\,(-\lambda_k - \sqrt{\lambda_k^2 - \nu^2})}\begin{pmatrix} -\lambda_k - \sqrt{\lambda_k^2 - \nu^2} & \nu \\ -\nu & \lambda_k + \sqrt{\lambda_k^2 - \nu^2} \end{pmatrix}.$$

(iii) *If* $\lambda_k^2 - \nu^2 = 0$, *then* $h_{\nu,k}$ *is not essentially self-adjoint and it admits a one parameter family of self-adjoint extensions* $\{t_{\nu,k}^{(\alpha)}\}_{\alpha\in[0,\pi)}$ *such that*

$$\mathcal{D}(t_{\nu,k}^{(\alpha)}) = \operatorname{span}\Big\{u_{\nu,k}^{(\alpha)}\Big\} + H_0^1((0,+\infty);\mathbb{C}^2),$$

being

$$u_{\nu,k}^{(\alpha)}(r) = (Q_{\nu,k}\log(r) + \mathbb{1}_2)\cdot\begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix}\chi(r) \qquad \text{for } r>0,$$

where $Q_{\nu,k} \in \mathbb{R}^{2\times 2}$ *is the rank 1 matrix and defined as follows:*

$$Q_{\nu,k} := \begin{pmatrix} \lambda_k & -\nu \\ \nu & -\lambda_k \end{pmatrix}.$$

(iv) *If* $\lambda_k^2 - \nu^2 < 0$, *then* $h_{\nu,k}$ *is not essentially self-adjoint and it admits a one parameter family of self-adjoint extensions* $\{t_{\nu,k}^{(\alpha)}\}_{\alpha\in[0,\pi)}$ *such that*

$$\mathcal{D}(t_{\nu,k}^{(\alpha)}) = \operatorname{span}\Big\{u_{\nu,k}^{(\alpha)}\Big\} + H_0^1((0,+\infty);\mathbb{C}^2),$$

being

$$u_{\nu,k}^{(\alpha)}(r) = R_{\nu,k}\cdot\begin{pmatrix}\cos(\alpha) r^{\mathrm{i}\sqrt{\nu^2-\lambda_k^2}}\\ \sin(\alpha) r^{-\mathrm{i}\sqrt{\nu^2-\lambda_k^2}}\end{pmatrix}\chi(r) \qquad \text{for } r>0$$

where $R_{\nu,k} \in \mathbb{C}^{2\times 2}$ *is the invertible matrix,*

$$R_{\nu,k} := \frac{1}{2\mathrm{i}\sqrt{\nu^2 - \lambda_k^2}\,(-\lambda_k - \mathrm{i}\sqrt{\nu^2 - \lambda_k^2})}\begin{pmatrix} -\lambda_k - \mathrm{i}\sqrt{\nu^2 - \lambda_k^2} & \nu \\ -\nu & \lambda_k + \mathrm{i}\sqrt{\nu^2 - \lambda_k^2} \end{pmatrix}.$$

Before giving the Proof of Proposition II.3, let us now first characterize the closure of $h_{\nu,k}$.

Proposition II.4. Let $h_{\nu,k}$ *be defined as in (2.2). Then,*

$$\mathcal{D}(\overline{h_{\nu,k}}) = \begin{cases} H_0^1\big((0,+\infty);\mathbb{C}^2\big) & \text{if } \lambda_k^2 - \nu^2 \neq \frac{1}{4}, \\ \left\{\begin{array}{l} u \in L^2\big((0,+\infty);\mathbb{C}^2\big): \\ \left(\lambda_k + \frac{1}{2}\right)\left(u_1' - \frac{u_1}{r}\right) - \nu\left(u_2' - \frac{u_2}{r}\right) \in L^2(0,+\infty), \\ \nu u_1 - \left(\lambda_k + \frac{1}{2}\right)u_2 \in H_0^1(0,+\infty) \end{array}\right\} & \text{if } \lambda_k^2 - \nu^2 = \frac{1}{4}. \end{cases}$$

Proof. We denote by C any positive constant. Applying the one-dimensional Hardy inequality (see Ref. 27, Proposition 2.4),

$$\int_0^\infty |f'(r)|^2\,dr \geq \frac{1}{4}\int_0^\infty \frac{|f(r)|^2}{r^2}\,dr \qquad \text{for } f \in C_c^\infty(0,+\infty),$$

we have that

$$\|h_{\nu,k}u\|_{L^2} \le C\|\partial_r u\|_{L^2} \qquad \textit{for } u \in \mathcal{D}(h_{\nu,k}).$$

This implies that

$$H_0^1((0,+\infty);\mathbb{C}^2) \subset \mathcal{D}(\overline{h_{\nu,k}}).$$

Set

$$\tilde{\lambda}_k := \begin{cases} \sqrt{\lambda_k^2 - \nu^2} & \textit{if } \lambda_k^2 - \nu^2 \ge 0, \\ \mathrm{i}\sqrt{\nu^2 - \lambda_k} & \text{otherwise,} \end{cases}$$

and let M_1, M_2 be the matrices defined as

$$M_1 := \begin{pmatrix} -\tilde{\lambda}_k - \lambda_k & -\nu \\ \nu & \tilde{\lambda}_k + \lambda_k \end{pmatrix} \quad M_2 := \begin{pmatrix} \tilde{\lambda}_k + \lambda_k & -\nu \\ \nu & -\tilde{\lambda}_k - \lambda_k \end{pmatrix}.$$

We get with an easy computation that

$$M_1 \cdot \begin{pmatrix} \frac{\nu}{r} & -\partial_r - \frac{\lambda_k}{r} \\ \partial_r - \frac{\lambda_k}{r} & \frac{\nu}{r} \end{pmatrix} = \begin{pmatrix} 0 & -\partial_r - \frac{\tilde{\lambda}_k}{r} \\ \partial_r - \frac{\tilde{\lambda}_k}{r} & 0 \end{pmatrix} \cdot M_2. \tag{2.5}$$

Let $\tilde{h}_{\nu,k}$ be the operator defined as

$$\mathcal{D}(\tilde{h}_{\nu,k}) = C_c^\infty\big((0,+\infty);\mathbb{C}^2\big) \quad \tilde{h}_{\nu,k}u = \begin{pmatrix} 0 & -\partial_r - \frac{\tilde{\lambda}_k}{r} \\ \partial_r - \frac{\tilde{\lambda}_k}{r} & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Let us first assume that $\lambda_k^2 - \nu^2 \ne 0$. In this case, the matrices M_1 and M_2 are invertible; thus, $u \in \mathcal{D}(\overline{h_{\nu,k}})$ if and only if $M_2 u \in \mathcal{D}(\overline{\tilde{h}_{\nu,k}})$. We get the desired result applying Lemma A.1 in Ref. 22. Although this result is stated when $\tilde{\lambda}_k$ is real, the same approach can be used for purely imaginary constants.

Let us now assume $\lambda_k^2 - \nu^2 = 0$. In this case, one can easily see that

$$M_2^* \cdot M_1 = 0_{2\times 2} \quad \text{and} \quad M_1 + M_2 = 2\lambda_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, for any $u \in \mathcal{D}(h_{\nu,k})$, thanks to (2.5) and by the one-dimensional Hardy-inequality, we have that

$$\|h_{\nu,k}u\|_{L^2}^2 = \frac{1}{4\lambda_k^2}\|M_1 h_{\nu,k}\|_{L^2}^2 + \frac{1}{4\lambda_k^2}\|M_2 h_{\nu,k}\|_{L^2}^2 \ge \frac{1}{4\lambda_k^2}\|\tilde{h}_{\nu,k}M_2 u\|_{L^2}^2 = \frac{1}{4\lambda_k^2}\|\partial_r(M_2 u)\|_{L^2}^2 \ge C\left\|\frac{M_2}{r}u\right\|_{L^2}^2.$$

Thanks to this and by the definition of $h_{\nu,k}$, we have that

$$\|\partial_r u\|_{L^2} = \left\|\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot h_{\nu,k} + \frac{M_2}{r} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)u\right\|_{L^2} \le C\|h_{\nu,k}u\|_{L^2}.$$

This proves that

$$\mathcal{D}(\overline{h_{\nu,k}}) \subset H_0^1((0,+\infty);\mathbb{C}^2),$$

and it concludes the proof. □

Proof of Proposition II.3. Let us denote by $h_{\nu,k}^*$ the adjoint operator of $h_{\nu,k}$. Then, by definition, we have that

$$\mathcal{D}(h_{\nu,k}^*) := \{u \in L^2((0,+\infty);\mathbb{C}^2) : d_{\nu,k}u \in L^2((0,+\infty);\mathbb{C}^2)\},$$
$$h_{\nu,k}^* u := d_{\nu,k}u,$$

where $d_{\nu,k}u$ has to be read in the distributional sense.

From the analysis of Ref. 27, it turns out that $\delta := \lambda_k^2 - \nu^2$ is the parameter ruling essential self-adjointness of $h_{\nu,k}$. Indeed, from Theorem 1.1 in Ref. 27, we know that $h_{\nu,k}$ is essentially self-adjoint if $\delta \geq \frac{1}{4}$. This consideration together with the explicit characterization of the closure of Proposition II.4 yields precisely *(i)*.

Let us now assume $0 < \delta < \frac{1}{4}$. By Ref. 27, Theorem 1.2, *(i)* the operator $h_{\nu,k}$ is not essentially self-adjoint and it admits a one-parameter family of self-adjoint extensions $\left\{t_{\nu,k}^{(\alpha)}\right\}_{\alpha\in[0,\pi)}$. Moreover, $u \in \mathcal{D}(t_{\nu,k}^{(\alpha)})$ if and only if $u \in \mathcal{D}(h_{\nu,k}^*)$ and there exists $(A^+, A^-) \in \mathbb{C}^2$ such that

$$A^+ \sin(\alpha) + A^- \cos(\alpha) = 0, \tag{2.6}$$

$$u(r) = P_{\nu,k}\begin{pmatrix} A^+ r^{\sqrt{\delta}} \\ A^- r^{-\sqrt{\delta}} \end{pmatrix} + o(r^{1/2}) \qquad \text{for } r \to 0. \tag{2.7}$$

Let us decompose

$$u(r) = P_{\nu,k}\begin{pmatrix} A^+ r^{\sqrt{\delta}} \\ A^- r^{-\sqrt{\delta}} \end{pmatrix}\chi(r) + \left[u(r) - P_{\nu,k}\begin{pmatrix} A^+ r^{\sqrt{\delta}} \\ A^- r^{-\sqrt{\delta}} \end{pmatrix}\chi(r)\right] =: v(r) + w(r).$$

Thanks to (2.6), we have that $v \in \operatorname{span}\left(u_{\nu,k}^{(\alpha)}\right)$.

Let us focus on w. By definition, $w \in \mathcal{D}(h_{\nu,k}^*)$. Moreover, by (2.7), $w(r) = o(r^{1/2})$ as $r \to 0$. Thanks to this and applying Eq. (3.3) in Ref. 27, we have that

$$\int_0^\infty \frac{|w(r)|^2}{r^2}\, dr < +\infty.$$

For this reason,

$$\begin{pmatrix} 0 & -\partial_r \\ \partial_r & 0 \end{pmatrix} w = h_{\nu,k}^* w - \begin{pmatrix} \nu & -\lambda_k \\ -\lambda_k & \nu \end{pmatrix} \frac{w}{r} \in L^2((0,+\infty);\mathbb{C}^2),$$

which proves that $w \in H_0^1((0,+\infty);\mathbb{C}^2)$. This implies that

$$\mathcal{D}(t_{\nu,k}^{(\alpha)}) \subset \operatorname{span}\left\{u_{\nu,k}^{(\alpha)}\right\} + H_0^1((0,+\infty);\mathbb{C}^2).$$

The other inclusion is obvious, and this concludes the proof of *(ii)*.

Last two points are proved analogously, but they rely on Theorems 1.2 *(ii)* and 1.3 in Ref. 27, respectively. □

We are now ready to prove Theorems I.7 and I.9.

Proof of Theorem I.7. Looking for self-adjoint extensions of $H_{\min}$ is equivalent to looking for self-adjoint extensions of the massless problem (as their difference is a bounded operator). The latter can be conveniently expressed in the unitary equivalent form of a direct sum

(Proposition II.1), which can be exploited for the computation of deficiency indices (see Ref. 49, Sec. 1.6). From Proposition II.3, we know that the deficiency indices of $h_{\nu,k}$ are as follows:

$$\dim\ker(h^*_{\nu,k} \pm i) = \begin{cases} 0 & \textit{if} \quad \lambda_k^2 - \nu^2 \geq \dfrac{1}{4}, \\ 1 & \textit{if} \quad \lambda_k^2 - \nu^2 < \dfrac{1}{4}. \end{cases}$$

It follows immediately that if $\lambda_0^2 - \nu^2 = \frac{\pi^2}{4\omega^2} - \nu^2 \geq \frac{1}{4}$, then $\dim\ker(h^*_{\nu,k} \pm i) = 0$ for all $k \in \mathbb{N}$, and thus, by the basic criterion of essential self-adjointness (Ref. 45, Corollary to Theorem VIII.3), $H_{\min}$ is essentially self-adjoint.

Using the characterization of Proposition II.4, one completes the proof of Theorem I.7. For any $k \in \mathbb{N}$ such that $\lambda_k^2 - \nu^2 < \frac{1}{4}$, the operator $h_{\nu,k}$ is not essentially self-adjoint. Since each non-essentially self-adjoint radial operator contributes to increasing the deficiency indices of 1, the deficiency index of the total operator will be equal to the number of non-self-adjoint radial operators. For fixed ν and ω, the condition on k for $h_{\nu,k}$ being non-essentially self-adjoint is

$$k < \frac{\omega}{\pi}\sqrt{\nu^2 + \frac{1}{4}} - \frac{1}{2}.$$

Calling d the maximum of such k, deficiency indices of $H_{\min}$ are $d+1$, and then, from von Neumann's theorem of self-adjoint extensions (Ref. 45, Theorem X.2), we get the thesis. □

The Proof of Theorem I.9 follows from Propositions II.1 and II.3 and the analysis of analogous properties on the reduced operators $h_{\nu,k}$. We study them in the following proposition.

Proposition II.5. Let $h_{\nu,k}$ be defined as in (2.3), and assume that $\lambda_k^2 - \nu^2 < 1/4$. Let $\{t^{(\alpha)}_{\nu,k}\}_{\alpha\in[0,\pi)}$ be the family of all the self-adjoint extensions of $h_{\nu,k}$ defined, respectively, as in Proposition II.3 (ii)–(iv). Then, the following holds:

(i) *If $0 < \lambda_k^2 - \nu^2$, $h_{\nu,k}$ admits a unique self-adjoint extension $t^{(D)}_{\nu,k}$ that verifies the property*

$$\mathcal{D}(t^{(D)}_{\nu,k}) \subset \mathcal{D}(r^{-a}\chi_{\{r\leq 1\}}) \qquad \textit{for all } 0 \leq a < \frac{1}{2} + \sqrt{\lambda_k^2 - \nu^2}. \tag{2.8}$$

(ii) *If $0 = \lambda_k^2 - \nu^2$, $h_{\nu,k}$ admits a unique self-adjoint extension $t^{(D)}_{\nu,k}$ that verifies the property*

$$\mathcal{D}(t^{(D)}_{\nu,k}) \subset \mathcal{D}((r^a \log^2 r)^{-1}\chi_{\{r\leq 1\}}) \qquad \textit{for all } 0 \leq a \leq \frac{1}{2}. \tag{2.9}$$

(iii) *If $\lambda_k^2 - \nu^2 < 0$, all the self-adjoint extensions $t^{(\alpha)}_{\nu,k}$ of $h_{\nu,k}$ verify*

$$\mathcal{D}(t^{(\alpha)}_{\nu,k}) \subset \mathcal{D}((r^a \log^2 r)^{-1}\chi_{\{r\leq 1\}}) \qquad \textit{for all } 0 \leq a \leq \frac{1}{2}. \tag{2.10}$$

Proof. Thanks to Proposition II.3, with explicit computation, we have that condition (2.8) is verified for $t^{(\alpha)}_k$ if and only if $\alpha = 0$. Analogously condition (2.9) is verified if and only if $\alpha = -(\operatorname{sign}\nu)\pi/4$. Finally, (2.10) is verified for any $\alpha \in [0,\pi)$. □

Proof of Theorem I.9. Let us first assume that $\frac{\pi^2-\omega^2}{4\omega^2} < \nu^2 < \frac{\pi^2}{4\omega^2}$. This implies that d defined in (1.6) is equal to 0. Thus, by Theorem I.7, the operator $H_{\min}$ has infinitely many self-adjoint extensions, and there exists a one-to-one correspondence between the self-ajdoint extensions of $H_{\min}$ and the space $\mathcal{U}(1) \sim [0,\pi)$. Since $0 \leq \lambda_0 - \nu^2 < 1/4$ and $\lambda_k^2 - \nu^2 \geq 1/4$ for all $k \in \mathbb{N}\backslash\{0\}$, thanks to Proposition II.3, we have that for any $\alpha \in [0,\pi)$,

$$T^{(\alpha)} \cong t^{(\alpha)}_0 \oplus \bigoplus_{k\in\mathbb{N}\backslash\{0\}} \overline{h_{\nu,k}}.$$

From Proposition II.5 *(i)*, we have that the self-adjoint realization defined through the unitary map in (A3) as

$$T^{(D)} :\cong t^{(D)}_0 \oplus \bigoplus_{k\in\mathbb{N}\backslash\{0\}} \overline{h_{\nu,k}}$$

is the unique one whose domain in included in $\mathcal{D}(|x|^{-a}, B_1)$. Last two points are proved analogously, but they rely on Proposition II.5 *(ii)* and *(iii)*, respectively. □

III. SPECTRAL PROPERTIES, HARDY–DIRAC, AND PROOF OF THEOREM I.11

Proof of Theorem I.3. Let $\psi \in C_c^\infty\big(\overline{S_\omega}\backslash\{0\};\mathbb{C}^2\big) \subset \mathcal{D}(H_\omega)$. Thanks to Proposition A.3 for $\nu = 0$, there exist $u_k^+, u_k^- \in C_c^\infty\big((0,+\infty);\mathbb{C}^2\big)$, $k \in \mathbb{N}$, such that (A4) holds true. Thanks to (A6) and (A7), we explicitly compute

$$\begin{aligned}\int_{S_\omega} |\boldsymbol{\sigma}\cdot\nabla\psi|^2\,dx &= \sum_{k\in\mathbb{N}}\bigg[\int_0^\infty \Big|\Big(\partial_r + \frac{\lambda_k}{r}\Big)u_k^-(r)\Big|^2\,dr + \int_0^\infty \Big|\Big(\partial_r - \frac{\lambda_k}{r}\Big)u_k^+(r)\Big|^2\,dr\bigg]\\ &= \sum_{k\in\mathbb{N}}\bigg[\int_0^\infty \Big|\partial_r\Big(r^{\lambda_k}u_k^-(r)\Big)\Big|^2 r^{-2\lambda_k}\,dr + \int_0^\infty \Big|\partial_r\Big(r^{-\lambda_k}u_k^+(r)\Big)\Big|^2 r^{2\lambda_k}\,dr\bigg].\end{aligned} \tag{3.1}$$

Thanks to Proposition 2.4 (i) and (ii) in Ref. 27 and using that $u_k^\pm(0) = u_k^\pm(\infty) = 0$, we have that for any $k\in\mathbb{N}$,

$$\begin{aligned}\int_0^\infty \Big|\partial_r\Big(r^{-\lambda_k}u_k^+(r)\Big)\Big|^2 r^{2\lambda_k}\,dr &\ge \Big(\lambda_k - \frac{1}{2}\Big)^2\int_0^\infty \frac{|u_k^+(r)|^2}{r^2}\,dr,\\ \int_0^\infty \Big|\partial_r\Big(r^{\lambda_k}u_k^-(r)\Big)\Big|^2 r^{-2\lambda_k}\,dr &\ge \Big(\lambda_k + \frac{1}{2}\Big)^2\int_0^\infty \frac{|u_k^-(r)|^2}{r^2}\,dr.\end{aligned} \tag{3.2}$$

Since $\min_{k\in\mathbb{N}}\big(\lambda_k \pm \frac{1}{2}\big)^2 = \frac{(\pi-\omega)^2}{4\omega^2}$, combining (3.1) with (3.2), we have that

$$\begin{aligned}\int_{S_\omega} |\boldsymbol{\sigma}\cdot\nabla\psi|^2\,dx &\ge \frac{(\pi-\omega)^2}{4\omega^2}\sum_{k\in\mathbb{N}}\bigg[\int_0^\infty \frac{|u_k^+(r)|^2}{r^2}\,dr + \int_0^\infty \frac{|u_k^-(r)|^2}{r^2}\,dr\bigg]\\ &= \frac{(\pi-\omega)^2}{4\omega^2}\int_{S_\omega}\frac{|\psi|^2}{|x|^2}\,dx.\end{aligned}$$

Finally, thanks to the fact that ψ is supported far away from the origin and that the curvature of a straight line is null, applying Ref. 8, Eq. (2.5), we have that

$$\int_{S_\omega} |\boldsymbol{\sigma}\cdot\nabla\psi|^2\,dx = \int_{S_\omega} |\nabla\psi|^2\,dx.$$

This implies that the H_ω-norm is equivalent to the H^1-norm on S_ω, and so, by a density argument, we conclude the proof. □

From now on, we denote by $H_\omega^{(D)}$ the self-adjoint operator H_ω defined in (1.3) when $0 < \omega \le \pi$ and its distinguished self-adjoint extension (the unique one whose domain is included in the Sobolev space $H^{1/2}$) when $\pi < \omega \le 2\pi$. We recall a known result, namely, Ref. 7, Proposition 1.12, that states that

$$\sigma_{\mathrm{ess}}(H_\omega^{(D)}) = (-\infty,-m]\cup[m,+\infty). \tag{3.3}$$

We are ready now to prove Theorem I.11.

Proof of Theorem I.11. We can always assume that T verifies $\mathcal{D}(T)\subset H^s(S_\omega;\mathbb{C}^2)$ for $s<1/2$; see Remark I.10. Indeed, if $\tilde{T}$ is an extension that does not verify this property, then $\tilde{T}$ is a finite rank perturbation of T in the sense of resolvent differences, and so they have the same essential spectrum. We use the strategy of Ref. 2, Sec. 4.3.4, exploiting the Weyl theorem (Ref. 50, Theorem XIII.14): we prove that for any $z\in\mathbb{C}\backslash\mathbb{R}$, the operator $(T-z)^{-1} - (H_\omega^{(D)}-z)^{-1}$ is compact. Consequently, from (3.3), we have that

$$\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(H_\omega^{(D)}) = (-\infty,-m]\cup[m,+\infty).$$

For this purpose, let χ be defined as in (2.4), and for any $n\in\mathbb{N}$, set $\chi_n(x) := \chi(|x|/n)$ and $\zeta_n(x) := 1-\chi_n(x)$. Then,

$$\begin{aligned}(T-z)^{-1} - (H_\omega^{(D)}-z)^{-1} &= (T-z)^{-1}\chi_n - (H_\omega^{(D)}-z)^{-1}\chi_n\\ &\quad + (T-z)^{-1}\zeta_n - \zeta_n(H_\omega^{(D)}-z)^{-1}\\ &\quad + \zeta_n(H_\omega^{(D)}-z)^{-1} - (H_\omega^{(D)}-z)^{-1}\zeta_n\\ &=: S_n^{(1)} + S_n^{(2)} + S_n^{(3)}.\end{aligned}$$

We claim that $S_n^{(1)} = \left(\chi_n(T-\bar{z})^{-1} - \chi_n(H_\omega^{(D)}-\bar{z})^{-1}\right)^*$ is compact. Indeed, both operators $\chi_n(T-\bar{z})^{-1}$ and $\chi_n(H_\omega^{(D)}-\bar{z})^{-1}$ are compact since they are both bounded from $L^2(S_\omega;\mathbb{C}^2)$ to $H^s(S_\omega\cap B_{2n};\mathbb{C}^2)$ (being B_{2n} the ball of radius $2n$) and $H^s(S_\omega\cap B_{2n};\mathbb{C}^2)$ is compactly embedded in $L^2(S_\omega;\mathbb{C}^2)$.

Let us analyze $S_n^{(2)}$. We observe that $V_n := \frac{\nu}{|x|}\zeta_n \in L^\infty$, and so

$$\begin{aligned}S_n^{(2)} &= -(T-z)^{-1}(\zeta_n(H_\omega - z) - (T-z)\zeta_n)(H_\omega - z)^{-1}\\ &= -(T-z)^{-1}(V_n)(H_\omega - z)^{-1}\\ &\quad -(T-z)^{-1}(-i\boldsymbol{\sigma}\cdot\nabla\zeta_n)(H_\omega^{(D)}-z)^{-1}.\end{aligned}$$

Using the fact that $\|V_n\|_\infty \le \frac{C}{n}$ and $\|-i\boldsymbol{\sigma}\cdot\nabla\zeta_n\|_\infty \le \frac{C}{n}$, we can conclude that $S_n^{(2)} \to 0$ in the operator norm for $n\to+\infty$.

Let us finally estimate $S_n^{(3)}$. Reasoning as above, we have

$$S_n^{(3)} = -(H_\omega^{(D)}-z)^{-1}(-i\boldsymbol{\sigma}\cdot\nabla\zeta_n)(H_\omega^{(D)}-z)^{-1} \to 0 \quad \text{in the operator norm for } n\to+\infty.$$

This concludes the first part of the proof.

Having proved that $\sigma_{\mathrm{ess}}(T) = (-\infty,-m]\cup[m,+\infty)$, one has immediately that $\sigma_{\mathrm{d}}(T)\subset(-m,m)$. Recalling that $\sigma_{\mathrm{d}}(T)$, the discrete spectrum of T, is the set of isolated eigenvalues with finite multiplicity, we note the following. Pick up $\lambda\in(-m,m)$; then, there are two possibilities: either $\lambda\in\sigma_{\mathrm{d}}(T)$ or λ is in the resolvent set $\rho(T)$, and so there exists a neighborhood of λ that is contained in $\rho(T)$. If the latter is the case and the operator $H_{\min}$ is not essentially self-adjoint, since the dimension of deficiency subspaces is constant in the resolvent set, $\dim\ker(H_{\min}^* - \lambda)\ge 1$. Using the Kreĭn–Višik–Birman–Grubb extension scheme (see Ref. 49, Theorem 2.13), one can consider the self-adjoint extension $H_{\mathbb{O}} - \lambda$ of $H_{\min}-\lambda$ corresponding to the Birman parameter $\mathbb{O}$ that has domain $\mathcal{D}(H_{\mathbb{O}}) = \mathcal{D}(H_{\min})\dot{+}\ker((H_{\min})^*-\lambda)$. Shifting the operator by λ does not change its domain, and thus, $\mathcal{D}(H_{\mathbb{O}})$ is a domain of a self-adjoint extension of $H_{\min}$ with eigenvalue λ. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: PROPERTIES OF THE ANGULAR OPERATOR

In this appendix, we decompose the Hilbert space $L^2(S_\omega;\mathbb{C}^2)$ into the direct sum of *partial wave subspaces*, namely, invariant subspaces for the action of the Dirac operator with a potential having spherical symmetry. The topic is very well known, so we give here a light presentation: for complete details, the reader can see, e.g., Refs. 7–9 and 22 or Ref. 2, Sec. 4.6 for the analogous three-dimensional reduction.

We use the standard notation for polar coordinates: for $x=(x_1,x_2)\in\mathbb{R}^2\setminus\{0\}$,

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \end{aligned} \qquad \text{being} \qquad \begin{aligned} r &:= \sqrt{x_1^2+x_2^2}\in(0,+\infty),\\ \theta &:= \operatorname{sign}(x_2)\arccos(x_1/r)\in[0,2\pi).\end{aligned}$$

For all $\psi\in L^2(S_\omega;\mathbb{C})$, let $\varphi=\varphi(r,\theta):(0,+\infty)\times(0,\omega)\to\mathbb{C}$ be defined as follows:

$$\varphi(r,\theta) := \sqrt{(x_1(r,\theta))^2+(x_2(r,\theta))^2}\,\psi(x_1(r,\theta),x_2(r,\theta)) \qquad \textit{for all } r\in(0,+\infty),\theta\in(0,\omega).$$

The map $\psi \mapsto \varphi$ is a unitary map $L^2(S_\omega;\mathbb{C}) \to L^2((0,+\infty);\mathbb{C}) \otimes L^2((0,\omega);\mathbb{C})$ since

$$\int_{S_\omega} |\psi(x)|^2 \, \mathrm{d}x = \int_0^\omega \int_0^{+\infty} |\varphi(r,\theta)|^2 \, \mathrm{d}r \, \mathrm{d}\theta.$$

Repeating this reasoning for every component of the wave-function, we obtain the following decomposition:

$$L^2(S_\omega;\mathbb{C}^2) \cong L^2((0,+\infty),\mathrm{d}r) \otimes L^2((0,\omega);\mathbb{C}^2), \tag{A1}$$

where "$\cong$" means unitarily equivalent.

It is useful to express the Dirac operators in polar coordinates: setting

$$\boldsymbol{e_r} := (\cos\,\theta, \sin\,\theta) = \frac{x}{r}, \quad \boldsymbol{e_\theta} := (-\sin\,\theta, \cos\,\theta) = \frac{\partial \boldsymbol{e_r}}{\partial\theta},$$

we abbreviate

$$\partial_r = \boldsymbol{e_r} \cdot \nabla \quad \text{and} \quad \partial_\theta = \boldsymbol{e_\theta} \cdot \nabla.$$

By means of elementary computations, it is easy to see that

$$\boldsymbol{\sigma} \cdot \boldsymbol{e_r} = \begin{pmatrix} 0 & e^{-\mathrm{i}\theta} \\ e^{\mathrm{i}\theta} & 0 \end{pmatrix}$$

and that the identity $\boldsymbol{\sigma} \cdot \boldsymbol{e_\theta} = i\boldsymbol{\sigma} \cdot \boldsymbol{e_r}\sigma_3$ holds. We obtain

$$-\mathrm{i}\boldsymbol{\sigma} \cdot \nabla = -\mathrm{i}\boldsymbol{\sigma} \cdot \left(\boldsymbol{e_r}\partial_r + \frac{1}{r}\boldsymbol{e_\theta}\partial_\theta \right) = -\mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{e_r}\left(\partial_r + \frac{1}{2r} - \frac{K_\omega}{r} \right), \tag{A2}$$

where K_ω is the *spin–orbit operator, which is given by*

$$K_\omega := \frac{1}{2}\mathbb{1} - i\sigma_3\partial_\theta.$$

In order to decompose appropriately $L^2((0,\omega);\mathbb{C}^2)$, we recall Lemma 2.4 in Ref. 8 about the properties of K_ω.

Proposition A.1 (properties of the spin–orbit operator). Let $\omega \in (0,2\pi]$, S_ω be as in (1.2), and $\{\lambda_k\}_{k\in\mathbb{N}}$ be as in (2.1). Set

$$f_k^+(\theta) := \frac{1}{\sqrt{2\omega}}\begin{pmatrix} e^{\mathrm{i}(\lambda_k - \frac{1}{2})\theta} \\ e^{-\mathrm{i}(\lambda_k - \frac{1}{2})\theta} \end{pmatrix}, \quad f_k^-(\theta) := \frac{-\mathrm{i}}{\sqrt{2\omega}}\begin{pmatrix} e^{-\mathrm{i}(\lambda_k + \frac{1}{2})\theta} \\ e^{+\mathrm{i}(\lambda_k + \frac{1}{2})\theta} \end{pmatrix} \quad \textit{for } \theta \in (0,\omega). \tag{A3}$$

The spin–orbit operator with infinite mass boundary conditions

$$\begin{aligned} K_\omega &:= \frac{1}{2}\mathbb{1} - \mathrm{i}\sigma_3\partial_\theta, \\ \mathcal{D}(K_\omega) &:= \left\{ \phi = (\phi_1,\phi_2) \in H^1((0,\omega),\mathbb{C}^2) \, : \, \phi_2(\omega) = -e^{\mathrm{i}\omega}\phi_1(\omega), \, \phi_1(0) = \phi_2(0) \right\} \end{aligned}$$

has the following properties:

(i) *K_ω is self-adjoint and has a compact resolvent.*
(ii) *$\{f_k^+, f_k^-\}_{k\in\mathbb{N}}$ is an orthonormal basis of eigenfunctions of $L^2((0,\omega);\mathbb{C}^2)$ with eigenvalues $\{\lambda_k, -\lambda_k\}_{k\in\mathbb{N}}$.*
(iii) *$-\mathrm{i}(\boldsymbol{\sigma} \cdot \boldsymbol{e_r})f_k^\pm = \pm f_k^\mp$ for all $k \in \mathbb{N}$.*

In the following proposition, we finally decompose the space $L^2(S_\omega;\mathbb{C}^2)$ into partial wave subspaces.

Proposition A.2 (decomposition in partial wave subspaces). Let $\omega \in (0,2\pi]$ and S_ω be defined as in (1.2), and for all $k\in\mathbb{N}$, let λ_k as in (2.1) and $f_k^\pm$ as in (A3). Then,

$$L^2(S_\omega;\mathbb{C}^2) \cong \bigoplus_{k\in\mathbb{N}}\big[L^2((0,+\infty),\mathrm{d}r)\otimes \operatorname{span}\{f_k^+,f_k^-\}\big],$$

i.e., for any $\psi\in L^2(S_\omega;\mathbb{C}^2)$, there exists $\{(u_k^+,u_k^-)\}_{k\in\mathbb{N}}\in L^2((0,\infty),\mathrm{d}r)\oplus L^2((0,\infty),\mathrm{d}r)$ such that

$$\begin{aligned}\psi(r,\theta) &= \frac{1}{\sqrt{r}}\sum_{k\in\mathbb{N}}\big[u_k^+(r)f_k^+(\theta)+u_k^-(r)f_k^-(\theta)\big] \quad \text{for a.a. } r\in(0,+\infty),\theta\in(0,\omega),\\ \|\psi\|^2_{L^2(S_\omega;\mathbb{C}^2)} &= \sum_{k\in\mathbb{N}}\big[\|u_k^+\|^2_{L^2((0,+\infty),\mathrm{d}r)}+\|u_k^-\|^2_{L^2((0,+\infty),\mathrm{d}r)}\big].\end{aligned} \tag{A4}$$

Proof. The proof is immediate from (A1) and (ii) of Proposition A.1. □

Thanks to Proposition A.2, it is possible to decompose the Dirac operator $H_{\min}$ defined in (1.5) as the direct sum of the one-dimensional Dirac operators on the half-line with Coulomb potentials $h_{\nu,k}$ defined in (2.3). The following proposition implies Proposition II.1.

Proposition A.3. Let $\omega\in(0,2\pi]$, S_ω be defined as in (1.2), and $H_{\min}$ be defined as in (1.5), and for all $k\in\mathbb{N}$, let $h_{\nu,k}$ be defined as in (2.3). Then, if $m=0$,

$$H_{\min} \cong \bigoplus_{k\in\mathbb{N}} h_{\nu,k},$$

where "≅" means that the operators are unitarily equivalent.

In detail, for $\psi\in L^2(S_\omega;\mathbb{C}^2)$, there exists $\{(u_k^+,u_k^-)\}_{k\in\mathbb{N}}\subset L^2((0,\infty),\mathrm{d}r)\oplus L^2((0,\infty),\mathrm{d}r)$ such that (A4) holds true and

$$\psi\in\operatorname{dom} H_{\min} \iff (u_k^+,u_k^-)\in\operatorname{dom} h_{\nu,k} = C_c^\infty\big((0,+\infty);\mathbb{C}^2\big) \quad \text{for all } k\in\mathbb{N}. \tag{A5}$$

Moreover, for a.a. $r\in(0,+\infty),\theta\in(0,\omega)$,

$$\left(D_0+\frac{\nu}{|x|}\mathbb{1}_2\right)\psi(r,\theta) = \frac{1}{\sqrt{r}}\sum_{k\in\mathbb{N}}\big[\widetilde{u_k^+}(r)f_k^+(\theta)+\widetilde{u_k^-}(r)f_k^-(\theta)\big], \tag{A6}$$

with

$$\begin{pmatrix}\widetilde{u_k^+}\\ \widetilde{u_k^-}\end{pmatrix} = d_{\nu,k}\begin{pmatrix}u_k^+\\ u_k^-\end{pmatrix}. \tag{A7}$$

Proof. The equivalence in (A5) is immediate since $f_k^\pm\in C_c^\infty([0,\omega];\mathbb{C}^2)$. Using (A2) and the fact that

$$\left(\partial_r+\frac{1}{2r}\right)\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r}}\partial_r,$$

we compute

$$\begin{aligned}\left(D_0+\frac{\nu}{|x|}\mathbb{1}_2\right)\psi(r,\theta) &= -\mathrm{i}(\boldsymbol{\sigma}\cdot\boldsymbol{e}_r)\left(\partial_r+\frac{1}{2r}-\frac{K_\omega}{r}\right)\frac{1}{\sqrt{r}}\sum_{k\in\mathbb{N}}\big[u_k^+(r)f_k^+(\theta)+u_k^-(r)f_k^-(\theta)\big]\\ &= \frac{-\mathrm{i}(\boldsymbol{\sigma}\cdot\boldsymbol{e}_r)}{\sqrt{r}}\sum_{k\in\mathbb{N}}\left[\partial_r u_k^+(r)f_k^+(\theta)+\partial_r u_k^-(r)f_k^-(\theta)-\frac{\lambda_k}{r}u_k^+(r)f_k^+(\theta)+\frac{\lambda_k}{r}u_k^-(r)f_k^-(\theta)\right]\\ &= \frac{1}{\sqrt{r}}\sum_{k\in\mathbb{N}}\left[\partial_r u_k^+(r)f_k^-(\theta)-\partial_r u_k^-(r)f_k^+(\theta)-\frac{\lambda_k}{r}u_k^+(r)f_k^-(\theta)-\frac{\lambda_k}{r}u_k^-(r)f_k^+(\theta)\right].\end{aligned}$$

From this, (A6) and (A7) follow. □

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