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On the Motion of Gravity–Capillary Waves with Odd Viscosity

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Abstract

We develop three asymptotic models of surface waves in a non-Newtonian fluid with odd viscosity. This viscosity is also known as Hall viscosity and appears in a number of applications such as quantum Hall fluids or chiral active fluids. Besides the odd viscosity effects, these models capture both gravity and capillary forces up to quadratic interactions and take the form of nonlinear and nonlocal wave equations. Two of these models describe bidirectional waves, while the third PDE studies the case of unidirectional propagation. We also prove the well-posedness of these asymptotic models in spaces of analytic functions and in Sobolev spaces. Finally, we present a number of numerical simulations for the unidirectional model.

Keywords Waves · Odd viscosity · Hall viscosity · Moving interfaces · Free-boundary problems

Mathematics Subject Classification 35L75 · 35Q35 · 35Q31 · 35S10 · 35R35 · 76D03

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1 Introduction

The equations describing the motion of an incompressible fluid take the following form:

$$\rho\left(\frac{\partial}{\partial t}u+(u\cdot\nabla)u\right)=\nabla\cdot\mathscr{T}\quad \text{in } \Omega(t)\times[0,T],$$
$$\nabla\cdot u=0\quad \text{in } \Omega(t)\times[0,T],$$
$$\frac{\partial}{\partial t}\rho+\nabla\cdot(u\rho)=0\quad \text{in } \Omega(t)\times[0,T],$$

where u, ρ and $\mathscr{T}$ denote the velocity, density and stress tensor of the fluid, respectively. This stress tensor takes different forms depending on the physical properties of the fluid. On the one hand, we have the case of inviscid fluids where the previous equations reduce to the well-known Euler system with a stress tensor given by

$$\mathscr{T}_j^i=-p\delta_j^i.$$

On the other hand, we have the case of viscous Newtonian fluids where the tensor takes the classical form

$$\mathscr{T}_j^i=-p\delta_j^i+\nu_e\left(\frac{\partial}{\partial x_j}u^i+\frac{\partial}{\partial x_i}u^j\right).$$

Due to the symmetries of the tensor, this viscosity is called *even* or *shear* viscosity.

Fluids in which both time reversal and parity are broken can display a dissipationless viscosity that is odd under each of these symmetries. This viscosity is called *odd* or Hall viscosity. In this case, the stress tensor takes the following form (cf. Khain et al. 2020):

$$\mathscr{T}_j^i=-p\delta_j^i+\nu_o\left(\frac{\partial}{\partial x_i}(u^j)^\perp+\left(\frac{\partial}{\partial x_i}\right)^\perp u^j\right),$$

where for a vector $a=(a_1,a_2)$, we used the notation

$$a^\perp=(a_2,-a_1).$$

Situations where such a viscosity arise in a natural way are, for instance, the motion of quantum Hall fluids at low temperature (cf. Avron et al. 1995), the motion of vortices (cf. Wiegmann and Abanov 2014) or chiral active fluids (cf. Banerjee et al. 2017) among other applications (see Souslov et al. 2019 or Abanov et al. 2020 and the references therein).

Since the seminar works of Avron et al. (1995) (see also Avron 1998; Banerjee et al. 2017; Ganeshan and Abanov 2017; Ganeshan and Monteiro 2021; Lapa and Hughes 2014; Souslov et al. 2019), the effect of the odd viscosity has been an active research area. Fluids that experience odd or Hall viscosity have a rather counterintuitive behavior. For instance, while a rotating disk immersed in a fluid with even viscosity experiments a friction force that opposes the rotation, the same disk rotating in a fluid with odd viscosity feels a pressure in the radial direction (see Avron 1998, Section 5.3 and Lapa and Hughes 2014 for more details).

While in three dimensions, terms in the viscosity tensor with odd symmetry were known in the context of anisotropic fluids, Avron noticed that in two dimensions, odd viscosity and isotropy can hold at the same time (cf. Avron 1998). This motivated the study of two-dimensional incompressible flows with odd viscosity effects. This situation can be described by the following system (cf. Avron 1998; Ganeshan and Abanov 2017):

$$\rho\left(\frac{\partial}{\partial t}u + (u\cdot\nabla)u\right) = -\nabla p + \nu_o \Delta u^{\perp} \quad \text{in } \Omega(t)\times[0,T], \tag{1a}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega(t)\times[0,T], \tag{1b}$$

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(u\rho) = 0 \quad \text{in } \Omega(t)\times[0,T], \tag{1c}$$

where u, ρ and p denote the velocity, density and pressure of the fluid and ν_o is a positive constant reflecting the influence of odd viscosity.

2 The free boundary problem

2.1 Derivation

We observe that using the divergence-free condition, system (1) can be rewritten as:

$$\rho\left(\frac{\partial}{\partial t}u + (u\cdot\nabla)u\right) = -\nabla(p - \nu_o(\nabla\cdot u^{\perp})) \quad \text{in } \Omega(t)\times[0,T], \tag{2a}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega(t)\times[0,T], \tag{2b}$$

$$\frac{\partial}{\partial t}\rho + u\cdot\nabla\rho = 0 \quad \text{in } \Omega(t)\times[0,T]. \tag{2c}$$

This system needs to be supplemented with appropriate initial and boundary conditions.

We stress that the viscosity tensor takes the form of a gradient and, as a consequence, it can be absorbed into the pressure in the bulk of the fluid once we define the modified pressure

$$\tilde{p} = p - \nu_o(\nabla\cdot u^{\perp}).$$

However, the odd viscosity affects the boundary conditions (cf. Ganeshan and Abanov 2017). This is of particular importance when considering free boundary problems for flows under the effect of odd viscosity. Such free boundary flows have been reported in experiments by Soni, Bililign, Magkiriadou, Sacanna, Bartolo, Shelley & Irvine (see Soni et al. 2018, 2019). The purpose of this paper is to study the dynamics of a incompressible, irrotational and homogeneous fluid bounded above by a free interface under the effect of gravity, surface tension and, more importantly, odd viscosity.

We consider the (to be determined) moving domain

$$\Omega(t) = \left\{(x_1, x_2) \in \mathbb{R}^2, x_1 \in \mathbb{R}, x_2 < \eta(x_1, t)\right\},$$

for certain function η. The free interface is then

$$\Gamma(t) = \left\{(x_1, x_2) \in \mathbb{R}^2, x_1 \in \mathbb{R}, x_2 = \eta(x_1, t)\right\}.$$

This function η is advected by the fluid and as a consequence it satisfies a transport equation.

Since we consider surface tension effects, the stress tensor satisfies the following boundary condition:

$$\mathscr{T}^i_j n_j = \gamma \mathcal{K} n_i,$$

where $\vec{n}$ is the unit upward pointing normal to the surface wave, γ is the surface tension strength and $\mathcal{K}$ denotes the curvature of the free boundary

$$\mathcal{K} = \frac{\eta_{x_1 x_1}}{\left(1 + \eta_{x_1}^2\right)^{3/2}} \quad \text{on } \Gamma(t).$$

Since the fluid is homogeneous, we have that the density ρ remains constant. It is well-known that the occurrence of viscosity creates a boundary layer near the surface wave. However, the thickness of this boundary layer is small when the physical parameters are in a certain regime (see Dias et al. 2008; Lamb 1932 for instance). In particular, the vorticity is confined to this narrow boundary layer near the surface (see Abanov et al. 2018). Then, following the ideas in Abanov et al. (2018), Dias et al. (2008) and the references therein, we consider potential flow with a modified boundary conditions for the redefined pressure $\tilde{p}$. Let us emphasize that we are dealing with an approximate model of surface odd waves where the boundary layer is replaced by a modified Bernoulli's law instead of the original free boundary problem for the surface waves with odd viscosity.

As a consequence, we assume that

$$u = \nabla\theta,$$

where θ is the scalar velocity potential. This velocity potential, by virtue of the incompressibility condition, is harmonic. Then, the effects of boundary layer are captured by the odd viscosity modified pressure term (see Abanov et al. 2018; Ganeshan and

Monteiro 2021 for the reasoning behind this fact)

$$\tilde{p} = -\gamma\mathcal{K} - \frac{2\nu_o}{(1+\eta_{x_1}^2)^{1/2}}\,(u\cdot n)_{x_1}\,.$$

This new boundary condition for the modified pressure encodes the contribution of the narrow boundary layer and gives an accurate description of the physical problem.

Thus, to describe surface waves under the effect of gravity, capillary forces and odd viscosity in an accurate way, it is enough to consider the following free boundary problem (Abanov et al. 2018; Abanov and Monteiro 2019; Ganeshan and Monteiro 2021):

$$\Delta\theta = 0 \quad \text{in } \Omega(t)\times[0,T], \tag{3a}$$

$$\rho\left(\theta_t + \frac{\theta_{x_1}^2+\theta_{x_2}^2}{2} + G\eta\right) - \gamma\mathcal{K} = \frac{2\nu_o}{(1+\eta_{x_1}^2)^{1/2}}\left(\frac{\eta_t}{(1+\eta_{x_1}^2)^{1/2}}\right)_{x_1} \quad \text{on } \Gamma(t)\times[0,T], \tag{3b}$$

$$\eta_t = -\eta_{x_1}\theta_{x_1} + \theta_{x_2} \quad \text{on } \Gamma(t)\times[0,T], \tag{3c}$$

where θ is the scalar potential (units of $length^2/time$), η denotes the surface wave (units of $length$) and G (units of $length/time^2$) is the gravity acceleration. The constant ν_o reflects the odd viscosity contribution and has units of $(length^2/time)$.

As is customary (see Zakharov 1968), we use the trace of the velocity potential

$$\xi(t,x_1,x_2) = \theta(t,x_1,\eta(t,x_1)).$$

Thus, (3) can be written as

$$\Delta\theta = 0 \quad \text{in } \Omega(t)\times[0,T], \tag{4a}$$

$$\theta = \xi \quad \text{on } \Gamma(t)\times[0,T], \tag{4b}$$

$$\begin{aligned}\xi_t + \frac{\theta_{x_1}^2+\theta_{x_2}^2}{2} + G\eta &= \frac{\gamma}{\rho}\frac{\eta_{x_1x_1}}{\left(1+\eta_{x_1}^2\right)^{3/2}} + \theta_{x_2}\left(-\eta_{x_1}\theta_{x_1}+\theta_{x_2}\right)\\ &\quad + \frac{2\nu_o}{\rho}\frac{1}{(1+\eta_{x_1}^2)^{1/2}}\left(\frac{\eta_t}{(1+\eta_{x_1}^2)^{1/2}}\right)_{x_1} \quad \text{on } \Gamma(t)\times[0,T],\end{aligned} \tag{4c}$$

$$\eta_t = -\eta_{x_1}\theta_{x_1} + \theta_{x_2} \quad \text{on } \Gamma(t)\times[0,T]. \tag{4d}$$

The system (4) is supplemented with an initial condition for η and ξ:

$$\eta(x,0) = \eta_0(x), \tag{5}$$

$$\xi(x,0) = \xi_0(x). \tag{6}$$

2.2 Prior Works

System (4) is the odd viscosity analogue to the water waves system with (even) viscosity in the work of Dias, Dyachenko & Zakharov (see Dias et al. 2008). A careful inspection shows that the system with odd viscosity is of purely dispersive nature, while the system in Dias et al. (2008) is of cross-diffusion type. This free boundary problem with even viscosity has received a large amount of attention in recent years both from the applied mathematics viewpoint, which is interested in new and better mathematical models that capture the main dynamics in suitable regimes, and from the pure mathematics community that studies the dynamics of the underlying differential equations. We refer the reader to Dutykh (2009), Dutykh and Dias (2007b, a), Kakleas and Nicholls (2010), Ngom and Nicholls (2018), Ambrose et al. (2012), Granero-Belinchón and Scrobogna (2019a), Granero-Belinchón and Scrobogna (2020b, c).

Even if the odd viscosity effects are most visible at the free surface of the fluids (see Ganeshan and Abanov 2017; Abanov et al. 2018, 2020) and the phenomenon has been described experimentally (cf. Soni et al. 2018, 2019), the number of results studying surface waves with odd viscosity remains, to the best of our knowledge, small.

Let us briefly summarize the available literature on surface waves with odd viscosity. Very recently, Abanov et al. (2018) considered the free surface dynamics of a two-dimensional incompressible fluid with odd viscosity. Besides studying the dispersion relation of such waves derived a number of weakly nonlinear models. First, after neglecting gravity, surface tension and terms of cubic order, they obtained the following Craig-Sulem-type model (see equations (49) and (50) in Abanov et al. 2018)

$$
\begin{aligned}
u_t - [\mathcal{H}(u\mathcal{H}u)]_{x_1} + 2\nu_o \Lambda u_{x_1} &= -2\nu_o \left[[\![\mathcal{H}, h]\!] \mathcal{H}u \right]_{x_1 x_1 x_1}, \\
h_t + \mathcal{H}u &= -\left[[\![\mathcal{H}, h]\!] \mathcal{H}u \right]_{x_1},
\end{aligned}
$$

where $[\![\cdot, \cdot]\!]$ denotes the commutator and $\mathcal{H}$ and Λ stand for the Hilbert transform and the fractional Laplacian, respectively (see below for a proper definition). As the authors point out in their work, this system is Hamiltonian. In addition, the authors also considered the small surface angle approximation to conclude the following model (see equations (51) and (52) in Abanov et al. 2018)

$$
h_{tt} = -\left[(\mathcal{H}h_t) h_t \right]_{x_1} - 2\nu_o \mathcal{H}h_{t x_1 x_1}.
$$

This latter equation has the same nonlinearity as the $h-$model in Granero-Belinchón and Shkoller (2017) while keeping a linear operator akin to the classical Benjamin–Ono equation. Indeed, the previous model can be equivalently written as

$$
v_t = -\left[(\mathcal{H}v) v \right]_{x_1} - 2\nu_o \mathcal{H}v_{x_1 x_1}.
$$

In this new variable, we recognize a well-known nonlinearity already heavily studied in the literature (see Castro and Córdoba 2008; Bae and Granero-Belinchón 2015; Li and Rodrigo 2011 and the references therein).

In addition, Abanov et al. (2018), starting from the previous small angle approximation, derived and studied what they named the *chiral* Burgers equation

$$u_t + 2uu_{x_1} = 2i\nu_o u_{x_1x_1}.$$

We would like to emphasize that all these models are obtained with *heuristic* arguments instead of a more rigorous asymptotic approximation.

Later on, Abanov and Monteiro (2019) presented a variational principle which accounts for odd viscosity effects in free boundary incompressible flows.

Moreover, Ganeshan and Monteiro (2021) studied the case of waves with odd viscosity in a shallow fluid and derived the celebrated KdV equation as a model in the long wavelength weakly nonlinear regime.

2.3 Contributions and Main Results

The purpose of this paper is twofold. On the one hand, we obtain three new models for capillary–gravity surface waves with odd viscosity. These new models are obtained through a multiscale expansion in the steepness of the wave and extend the previous results in Aurther et al. (2019), Granero-Belinchón and Scrobogna (2019a), Granero-Belinchón and Scrobogna (2019b). Furthermore, our models consider both gravity and surface tension forces and, as a consequence, generalize those in Abanov et al. (2018). In particular, we obtain the model

$$\begin{aligned} f_{tt} &= -\Lambda f - \beta\Lambda^3 f + \alpha_o\Lambda f_{tx_1} + \varepsilon\left[-\mathcal{H}\left((\mathcal{H}f_t)^2\right) + \left([\![\mathcal{H}, f]\!]\Lambda f\right)\right]_{x_1} \\ &\quad + \varepsilon\left[-\alpha_o\left([\![\mathcal{H}, f]\!]\Lambda f_{tx_1}\right) + \beta[\![\mathcal{H}, f]\!]\Lambda^3 f\right]_{x_1} \quad \text{on } \Gamma\times[0,T]. \end{aligned} \tag{7}$$

Noticing that the terms that are $\mathcal{O}(\varepsilon\alpha_o)$ and $\mathcal{O}(\varepsilon\beta)$ are much smaller that the rest, we can also consider the following PDE:

$$f_{tt} = -\Lambda f - \beta\Lambda^3 f + \alpha_o\Lambda f_{tx_1} + \varepsilon\left[-\mathcal{H}\left((\mathcal{H}f_t)^2\right) + \left([\![\mathcal{H}, f]\!]\Lambda f\right)\right]_{x_1} \text{ on } \Gamma\times[0,T]. \tag{8}$$

Similarly, if we restrict ourselves to the study of unidirectional surface waves, we can derive the following dispersive equation:

$$\begin{aligned} 2f_t + \alpha_o\Lambda f_t &= \frac{1}{\varepsilon}\left\{f_{x_1} + \mathcal{H}f + (\alpha_o-\beta)\mathcal{H}f_{x_1x_1}\right\} \\ &\quad + \mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f + (\alpha_o-\beta)[\![\mathcal{H}, f]\!]\Lambda^3 f, \quad \text{on } \Gamma\times[0,T]. \end{aligned} \tag{9}$$

Conservation of mass is a limitation of some models of viscous fluids (Eeltink et al. 2020). In that regard, let us remark that the models here derived conserve the total mass of water for periodic domains and for waves that decay fast enough at infinity.

On the other hand, we prove a number of mathematical results establishing the well-posedness of our new models in appropriate functional spaces (see Granero-Belinchón and Scrobogna 2020a, c, 2021 for some related results).

Roughly speaking, we prove the following theorems (see below for the precise statements):

(1) Equation (7) is locally well-posed for analytic initial data. The proof is based on a Cauchy–Kowalevski-type argument that relies in finding appropriate bounds for a cascade of linear equations as in Aurther et al. (2019).
(2) Equation (8) is locally well-posed in $H^{4.5}(\mathbb{R}) \times H^3(\mathbb{R})$ when the Bond number $\beta > 0$. The required energy estimates exploit the commutator structure of the nonlinearity in a very precise way.
(3) Equation (9) is locally well-posed in $H^3(\mathbb{R})$ when the odd Reynolds number α_o is strictly positive regardless of the value of the Bond number β. Furthermore, when $0 < \alpha_o = \beta$, the problem admits a distributional solution in $H^{1.5}(\mathbb{R})$.

The plan of the paper is as follows. First, in Sect. 3, we write the dimensionless problem of gravity–capillary waves with odd viscosity in the arbitrary Lagrangian–Eulerian formulation. Next, in Sect. 4, we derive and study the case of a bidirectional gravity–capillary wave with odd viscosity. In particular, we obtain two new nonlinear and nonlocal wave equations (see Equations (7) and (8)). These PDEs describe the main dynamics in the weakly nonlinear regime and consider the case where the steepness parameter ε, the odd Reynolds number α_o and Bond number β are small. Moreover, we establish the local strong well-posedness of (7) and (8) in appropriate functional spaces. Then, in Sect. 5, we study the case of unidirectional waves and obtain the new nonlocal and nonlinear dispersive equation (9) in the case of right-moving waves. Furthermore, we establish the local strong well-posedness of (9) in Sobolev spaces provided that the odd Reynolds number is positive. Finally, we also prove a local in time existence of distributional solution with limited regularity for (9). In Sect. 6, we conclude with a brief discussion presenting the main novelties of our work.

2.4 Notation

Given a matrix A, we write A^i_j for the component of A, located on row i and column j. We will use the Einstein summation convention for expressions with indexes.

We write

$$f_{x_j} = \frac{\partial f}{\partial x_j}, \quad f_t = \frac{\partial f}{\partial t}$$

for the space derivative in the $j-$th direction and for a time derivative, respectively.

Let $f(x_1)$ denote a L^2 function on $\mathbb{R}$. We define the Hilbert transform $\mathcal{H}$ and the Dirichlet-to-Neumann operator Λ and its powers, respectively, using Fourier series

$$\widehat{\mathcal{H}f}(k) = -i\,\text{sgn}(k)\hat{f}(k)\,, \quad \widehat{\Lambda f}(k) = |k|\hat{f}(k)\,, \tag{10}$$

where

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_1)\, e^{-ikx_1} rmdx_1.$$

Finally, given an operator $\mathcal{T}$, we define the commutator as

$$[\![\mathcal{T}, f]\!] g = \mathcal{T}(fg) - f\mathcal{T}(g).$$

3 A Model of Gravity–Capillary Waves with Odd Viscosity

As mentioned above, we use the trace of the velocity potential

$$\xi(t, x_1, x_2) = \theta(t, x_1, \eta(t, x_1)).$$

Thus, (3) can be written in dimensionless form (see Granero-Belinchón and Shkoller 2017 for more details) as

$$\Delta\theta = 0 \quad \text{in } \Omega(t) \times [0, T], \tag{11a}$$

$$\theta = \xi \qquad \text{on } \Gamma(t) \times [0, T], \tag{11b}$$

$$\xi_t + \varepsilon \frac{\theta_{x_1}^2 + \theta_{x_2}^2}{2} + \eta = \frac{\beta \eta_{x_1 x_1}}{\left(1 + \left(\varepsilon \eta_{x_1}\right)^2\right)^{3/2}} + \varepsilon \theta_{x_2} \left(-\varepsilon \eta_{x_1} \theta_{x_1} + \theta_{x_2}\right) \tag{11c}$$

$$+ \alpha_o \frac{1}{(1 + \varepsilon^2 \eta_{x_1}^2)^{1/2}} \left(\frac{\eta_t}{(1 + \varepsilon^2 \eta_{x_1}^2)^{1/2}} \right)_{x_1} \qquad \text{on } \Gamma(t) \times [0, T], \tag{11d}$$

$$\eta_t = -\varepsilon \eta_{x_1} \theta_{x_1} + \theta_{x_2} \quad \text{on } \Gamma(t) \times [0, T], \tag{11e}$$

where ε is known as the *steepness parameter* and measures the ratio between the amplitude and the wavelength of the wave while α_o is a dimensionless parameter akin to the Reynolds number that represents the ratio between gravity and odd viscosity forces. Due to this similarity, we call it *odd Reynolds number*. Similarly, β is the Bond number comparing the gravity and capillary forces.

Now, we want to express system (11) on the reference domain Ω and reference interface Γ

$$\Omega = \mathbb{R} \times (-\infty, 0), \quad \Gamma = \mathbb{R} \times \{0\}. \tag{12}$$

In order to do that we define the following family of diffeomorphisms

$$\begin{aligned} \psi : \quad & [0, T] \times \Omega \to \Omega(t), \\ & (x_1, x_2, t) \mapsto \psi(x_1, x_2, t) = (x_1, x_2 + \varepsilon\eta(x_1, t)). \end{aligned}$$

We compute

$$\nabla\psi = \begin{pmatrix} 1 & 0 \\ \varepsilon\eta_{x_1}(x_1,t) & 1 \end{pmatrix}, \quad A = (\nabla\psi)^{-1} = \begin{pmatrix} 1 & 0 \\ -\varepsilon\eta_{x_1}(x_1,t) & 1 \end{pmatrix}. \tag{13}$$

We define the ALE variables

$$\Theta = \theta \circ \psi.$$

We can equivalently write (11) in the (fixed) reference domain as follows:

$$A_j^\ell \left(A_j^k \Theta_{x_k}\right)_{x_\ell} = 0 \quad \text{in } \Omega \times [0,T], \tag{14a}$$

$$\Theta = \xi \quad \text{on } \Gamma \times [0,T], \tag{14b}$$

$$\xi_t + \frac{\varepsilon}{2} A_j^k \Theta_{x_k} A_j^\ell \Theta_{x_\ell} + \eta = \frac{\beta\eta_{x_1x_1}}{\left(1 + (\varepsilon\eta_{x_1})^2\right)^{3/2}} + \varepsilon A_2^k \Theta_{x_k} A_j^\ell \Theta_{x_\ell} A_j^2 \tag{14c}$$

$$+ \alpha_o \frac{1}{(1+\varepsilon^2\eta_{x_1}^2)^{1/2}} \left(\frac{\eta_t}{(1+\varepsilon^2\eta_{x_1}^2)^{1/2}}\right)_{x_1} \quad \text{on } \Gamma \times [0,T], \tag{14d}$$

$$\eta_t = A_j^k \Theta_{x_k} A_j^2 \quad \text{on } \Gamma \times [0,T]. \tag{14e}$$

Using the explicit value of A_j^k, we can regroup terms and find that

$$\Delta\Theta = \varepsilon\left(\eta_{x_1x_1}\Theta_{x_2} + 2\eta_{x_1}\Theta_{x_1x_2}\right) - \varepsilon^2(\eta_{x_1})^2\Theta_{x_2x_2}, \quad \text{in } \Omega \times [0,T], \tag{15a}$$

$$\Theta = \xi \quad \text{on } \Gamma \times [0,T], \tag{15b}$$

$$\begin{aligned} \xi_t &= -\frac{\varepsilon}{2}\left[(\Theta_{x_1})^2 + (\varepsilon\eta_{x_1}\Theta_{x_2})^2 + (\Theta_{x_2})^2 - 2\varepsilon\eta_{x_1}\Theta_{x_2}\Theta_{x_1}\right] \\ &\quad - \eta + \varepsilon\Theta_{x_2}\left(-\varepsilon\eta_{x_1}\Theta_{x_1} + \varepsilon^2(\eta_{x_1})^2\Theta_{x_2} + \Theta_{x_2}\right) \\ &\quad + \frac{\beta\eta_{x_1x_1}}{\left(1 + (\varepsilon\eta_{x_1})^2\right)^{3/2}} \\ &\quad + \alpha_o \frac{1}{(1+\varepsilon^2\eta_{x_1}^2)^{1/2}} \left(\frac{\eta_t}{(1+\varepsilon^2\eta_{x_1}^2)^{1/2}}\right)_{x_1} \quad \text{on } \Gamma \times [0,T], \end{aligned} \tag{15c}$$

$$\eta_t = -\varepsilon\eta_{x_1}\Theta_{x_1} + \varepsilon^2(\eta_{x_1})^2\Theta_{x_2} + \Theta_{x_2} \quad \text{on } \Gamma \times [0,T]. \tag{15d}$$

4 The Bidirectional Asymptotic Model for Waves with Odd Viscosity

4.1 Derivation

We are interested in a model approximating the dynamics up to $\mathcal{O}(\varepsilon^2)$. As a consequence a number of terms can be neglected with this order of approximation, and we find

$$\Delta\Theta = \varepsilon\left(\eta_{x_1x_1}\,\Theta_{x_2} + 2\eta_{x_1}\Theta_{x_1x_2}\right), \quad \text{in } \Omega\times[0,T]\,, \tag{16a}$$
$$\Theta = \xi \quad \text{on } \Gamma\times[0,T], \tag{16b}$$
$$\xi_t = -\frac{\varepsilon}{2}\left[(\Theta_{x_1})^2 + (\Theta_{x_2})^2\right] - \eta + \varepsilon\Theta_{x_2}^2 + \beta\eta_{x_1x_1} + \alpha_o\eta_{tx_1} \quad \text{on } \Gamma\times[0,T], \tag{16c}$$
$$\eta_t = -\varepsilon\eta_{x_1}\Theta_{x_1} + \Theta_{x_2} \quad \text{on } \Gamma\times[0,T]. \tag{16d}$$

In order to obtain the asymptotic model for the interface under the effect of odd viscosity, we will assume the following form for the unknowns:

$$\Theta\left(x_1,x_2,t\right) = \sum_n \varepsilon^n\Theta^{(n)}\left(x_1,x_2,t\right),$$
$$\xi\left(x_1,t\right) = \sum_n \varepsilon^n\xi^{(n)}\left(x_1,t\right),$$
$$\eta\left(x_1,t\right) = \sum_n \varepsilon^n\eta^{(n)}\left(x_1,t\right). \tag{17}$$

For the case $n=0$, we have that

$$\Delta\Theta^{(0)} = 0, \quad \text{in } \Omega\times[0,T]\,, \tag{18a}$$
$$\Theta^{(0)} = \xi^{(0)} \quad \text{on } \Gamma\times[0,T], \tag{18b}$$
$$\xi_t^{(0)} = -\eta^{(0)} + \beta\eta_{x_1x_1}^{(0)} + \alpha_o\eta_{tx_1}^{(0)} \quad \text{on } \Gamma\times[0,T], \tag{18c}$$
$$\eta_t^{(0)} = \Theta_{x_2}^{(0)} \quad \text{on } \Gamma\times[0,T]. \tag{18d}$$

The solution of the associated elliptic problem for the first term of the velocity potential is given by

$$\widehat{\Theta^{(0)}}\left(k,x_2,t\right) = \xi^{(0)}(k,t)e^{|k|x_2} \qquad \text{in } \Omega\times[0,T],$$

so

$$\Theta_{x_2}^{(0)} = \Lambda\xi^{(0)} \qquad \text{on } \Gamma.$$

Then, we find that the linear problem for the first term of the series for the interface is

$$\eta_{tt}^{(0)} = -\Lambda\eta^{(0)} - \beta\Lambda^3\eta^{(0)} + \alpha_o\Lambda\eta_{tx_1}^{(0)} \quad \text{on } \Gamma\times[0,T].$$

The second term in the expansion is

$$\Delta\Theta^{(1)} = \eta^{(0)}_{x_1x_1}\,\Theta^{(0)}_{x_2} + 2\eta^{(0)}_{x_1}\Theta^{(0)}_{x_1x_2}, \quad \text{in } \Omega\times[0,T], \tag{19a}$$

$$\Theta^{(1)} = \xi^{(1)} \quad \text{on } \Gamma\times[0,T], \tag{19b}$$

$$\xi^{(1)}_t = -\frac{1}{2}\left[(\Theta^{(0)}_{x_1})^2 + (\Theta^{(0)}_{x_2})^2\right] - \eta^{(1)} + (\Theta^{(0)}_{x_2})^2 + \beta\eta^{(1)}_{x_1x_1} + \alpha_o\eta^{(1)}_{tx_1} \quad \text{on } \Gamma\times[0,T], \tag{19c}$$

$$\eta^{(1)}_t = -\eta^{(0)}_{x_1}\Theta^{(0)}_{x_1} + \Theta^{(1)}_{x_2} \quad \text{on } \Gamma\times[0,T]. \tag{19d}$$

We recall now the following lemma:

Lemma 4.1 (Granero-Belinchón and Scrobogna 2019a) *Let us consider the Poisson equation*

$$\begin{cases} \Delta u\,(x_1,x_2) & = b\,(x_1,x_2)\,,\ (x_1,x_2)\in\mathbb{R}\times(-\infty,0)\,, \\ u\,(x_1,0) & = g\,(x_1)\,, \qquad x_1\in\mathbb{R}, \\ \lim_{x_2\to-\infty}\partial_2 u\,(x_1,x_2) & = 0, \qquad x_1\in\mathbb{R}, \end{cases} \tag{20}$$

where we assume that the forcing b and the boundary data g are smooth and decay fast enough at infinity. Then,

$$u_{x_2}\,(x_1,0) = \int_{-\infty}^{0} e^{y_2\Lambda} b\,(x_1,y_2)\,\mathrm{d}y_2 + \Lambda g(x_1). \tag{21}$$

Using this lemma, we find that

$$\Theta^{(1)}_{x_2}\Big|_{x_2=0} = \Lambda\xi^{(1)} - \left[\!\left[\Lambda,\eta^{(0)}\right]\!\right]\Lambda\xi^{(0)} \tag{22}$$

and we can write system (19) as

$$\xi^{(1)}_t = -\frac{1}{2}\left[(\xi^{(0)}_{x_1})^2 + (\Lambda\xi^{(0)})^2\right] - \eta^{(1)} + (\Lambda\xi^{(0)})^2 + \beta\eta^{(1)}_{x_1x_1} + \alpha_o\eta^{(1)}_{tx_1} \quad \text{on } \Gamma\times[0,T], \tag{23a}$$

$$\eta^{(1)}_t = -\eta^{(0)}_{x_1}\xi^{(0)}_{x_1} + \Lambda\xi^{(1)} - \left[\!\left[\Lambda,\eta^{(0)}\right]\!\right]\Lambda\xi^{(0)} \quad \text{on } \Gamma\times[0,T]. \tag{23b}$$

Recalling that

$$\eta^{(0)}_t = \Lambda\xi^{(0)},$$

and Tricomi's identity

$$(\mathcal{H}f)^2 - f^2 = 2\mathcal{H}\,(f\mathcal{H}f)\,, \tag{24}$$

we can compute

$$\xi^{(1)}_t = \beta\eta^{(1)}_{x_1x_1} - \mathcal{H}\left(\eta^{(0)}_t\mathcal{H}\eta^{(0)}_t\right) - \eta^{(1)} + \alpha_o\eta^{(1)}_{tx_1} \quad \text{on } \Gamma\times[0,T],$$

$$\eta_t^{(1)} = \eta_{x_1}^{(0)} \mathcal{H} \eta_t^{(0)} + \Lambda \xi^{(1)} - \left[\!\left[\Lambda, \eta^{(0)} \right]\!\right] \eta_t^{(0)} \quad \text{on } \Gamma \times [0, T].$$

Taking a time derivative and substituting the value of $\Lambda \xi_t^{(1)}$, we compute that

$$\begin{aligned}\eta_{tt}^{(1)} = &-\Lambda \eta^{(1)} - \beta \Lambda^3 \eta^{(1)} + \alpha_o \Lambda \eta_{tx_1}^{(1)} - \Lambda \left((\mathcal{H} \eta_t^{(1)})^2 \right) + \left(\left[\!\left[\mathcal{H}, \eta^{(0)} \right]\!\right] \Lambda \eta^{(0)} \right)_{x_1} \\ &+ \beta \left(\left[\!\left[\mathcal{H}, \eta^{(0)} \right]\!\right] \Lambda^3 \eta^{(0)} \right)_{x_1} - \alpha_o \left(\left[\!\left[\mathcal{H}, \eta^{(0)} \right]\!\right] \Lambda \eta_{tx_1}^{(0)} \right)_{x_1} \quad \text{on } \Gamma \times [0, T].\end{aligned}$$

If we now define

$$f(x_1, t) = \eta^{(0)}(x_1, t) + \varepsilon \eta^{(1)}(x_1, t),$$

after neglect terms of order $\mathcal{O}(\varepsilon^2)$, we conclude the following bidirectional model of gravity–capillary waves with odd viscosity

$$\begin{aligned}f_{tt} = &-\Lambda f - \beta \Lambda^3 f + \alpha_o \Lambda f_{tx_1} + \varepsilon \left[-\mathcal{H} \left((\mathcal{H} f_t)^2 \right) + \left([\![\mathcal{H}, f]\!] \Lambda f \right) \right]_{x_1} \\ &+ \varepsilon \left[-\alpha_o \left([\![\mathcal{H}, f]\!] \Lambda f_{tx_1} \right) + \beta [\![\mathcal{H}, f]\!] \Lambda^3 f \right]_{x_1} \quad \text{on } \Gamma \times [0, T]. \qquad (25)\end{aligned}$$

With an appropriate choice of the parameters, (25) recovers the quadratic $h-$model in Aurther et al. (2019), Matsuno (1992), Matsuno (1993b), Matsuno (1993a), Akers and Milewski (2010), Akers and Nicholls (2010).

Furthermore, we observe that some of the terms are $\mathcal{O}(\varepsilon\alpha_o)$ and $\mathcal{O}(\varepsilon\beta)$, *i.e.,* they are much smaller than the rest of the nonlinear contributions. Then, one can expect that they can be neglected to find

$$f_{tt} = -\Lambda f - \beta \Lambda^3 f + \alpha_o \Lambda f_{tx_1} + \varepsilon \left[-\mathcal{H} \left((\mathcal{H} f_t)^2 \right) + \left([\![\mathcal{H}, f]\!] \Lambda f \right) \right]_{x_1} \text{ on } \Gamma \times [0, T]. \qquad (26)$$

A similar equation was obtained in Granero-Belinchón and Scrobogna (2019a), Granero-Belinchón and Scrobogna (2020c, 2021) for the case of damped waves under the effect of even viscosity.

4.2 Well-Posedness for Analytic Initial Data

We recall the definition of the Wiener spaces in the real line (see Gancedo et al. 2020 for more properties)

$$\mathbb{A}_\tau(\mathbb{R}) = \left\{ h \in L^1(\mathbb{R}) \text{ s.t. } \|h\|_{\mathbb{A}_\tau} = \int_{\mathbb{R}} e^{\tau|n|} |\hat{h}(n)| \mathrm{d}n < \infty \right\}.$$

This section is devoted to the proof of the following result:

Theorem 1 *Let $\beta, \alpha_o \geq 0$ be fixed constants. Assume that the initial data for equation (25) satisfies*

$$(f_0, f_1) \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$$

and

$$f_0(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-D}^{D} \hat{f}_0(k) e^{ikx_1} \mathrm{d}k,$$
$$f_1(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-D}^{D} \hat{f}_1(k) e^{ikx_1} \mathrm{d}k,$$

for some $1 < D < +\infty$. Then, there exists $0 < T^*$ and a unique solution to equation (25) such that

$$(f, f_t) \in L^\infty(0, T^*; \mathbb{A}_1(\mathbb{R})) \times L^\infty(0, T^*; \mathbb{A}_1(\mathbb{R})) \cap C(0, T^*; \mathbb{A}_{0.5}(\mathbb{R})) \times C(0, T^*; \mathbb{A}_{0.5}(\mathbb{R})).$$

Proof We look for a solution of the form

$$f(x, t) = \sum_{\ell=0}^{\infty} \lambda^{\ell+1} f^{(\ell)}(x, t) \tag{27}$$

for some λ to be fixed later. By substituting this expression into (25) and matching terms, we get that $f^{(\ell)}$ satisfies the equation

$$\begin{aligned} f_{tt}^{(\ell)} &= -\Lambda f^{(\ell)} - \beta \Lambda^3 f^{(\ell)} + \alpha_o \Lambda f_{tx_1}^{(\ell)} \\ &\quad + \sum_{j=0}^{\ell-1} \Big[-\mathcal{H}\left(\mathcal{H} f_t^{(j)} \mathcal{H} f_t^{(\ell-1-j)}\right) + \mathcal{H}\left(f^{(j)} \Lambda f^{(\ell-1-j)}\right) - f^{(j)} \mathcal{H} \Lambda f^{(\ell-1-j)} \Big]_{x_1} \\ &\quad - \alpha_o \sum_{j=0}^{\ell-1} \Big[\left(\mathcal{H}\left(f^{(j)} \Lambda f_{tx_1}^{(\ell-1-j)}\right) - f^{(j)} \mathcal{H} \Lambda f_{tx_1}^{(\ell-1-j)} \right) \Big]_{x_1} \\ &\quad + \beta \sum_{j=0}^{\ell-1} \Big[\mathcal{H}\left(f^{(j)} \Lambda^3 f^{(\ell-1-j)}\right) - f^{(j)} \mathcal{H} \Lambda^3 f^{(\ell-1-j)} \Big]_{x_1} \end{aligned} \tag{28}$$

with initial conditions

$$f^{(l)}(x_1, 0) = \begin{cases} 0 & \text{if } \ell \neq 0, \\ \frac{f_0}{\lambda} & \text{if } \ell = 0. \end{cases} \quad \text{and} \quad f_t^{(l)}(x_1, 0) = \begin{cases} 0 & \text{if } \ell \neq 0, \\ \frac{f_1}{\lambda} & \text{if } \ell = 0. \end{cases}$$

Using the Fourier series expansion, from (28) we get that each $\widehat{f}^{(\ell)}(x_1, t)$ satisfies the differential equation

$$\widehat{f_{tt}}^{(\ell)}(k, t) = -|k| \widehat{f}^{(\ell)}(k, t) - \beta |k|^3 \widehat{f}^{(\ell)}(k, t) + i\, \alpha_o k |k| \widehat{f_t}^{(\ell)}(k, t) + F(k, t)$$

where

$$\begin{aligned} F(k,t) &= |k| \sum_{j=0}^{\ell-1} \int_{-\infty}^{\infty} \operatorname{sgn}(m) \widehat{f_t}^{(j)}(m,t) \operatorname{sgn}(k-m) \widehat{f_t}^{(\ell-1-j)}(k-m,t) \mathrm{d}m \\ &+ \sum_{j=0}^{\ell-1} \int_{-\infty}^{\infty} \widehat{f}^{(j)}(m,t) \widehat{f}^{(\ell-1-j)}(k-m,t) \Big[|k||k-m| - k(k-m) \Big] \mathrm{d}m \\ &+ i\alpha_o \sum_{j=0}^{\ell-1} \int_{-\infty}^{\infty} \widehat{f}^{(j)}(m,t) \widehat{f_t}^{(\ell-1-j)}(k-m,t)(k-m) \Big[|k||k-m| - k(k-m) \Big] \mathrm{d}m \\ &+ \beta \sum_{j=0}^{\ell-1} \int_{-\infty}^{\infty} \widehat{f}^{(j)}(m,t) \widehat{f}^{(\ell-1-j)}(k-m,t)(k-m)^2 \Big[|k||k-m| - k(k-m) \Big] \mathrm{d}m. \end{aligned} \tag{29}$$

Solving (28) for $\ell = 0$, we get

$$\widehat{f}^{(0)}(k,t) = \frac{1}{i(r^+ - r^-)} \left\{ [\hat{f}_1 - ir^- \hat{f}_0] e^{ir^+ t} - [\hat{f}_1 - ir^+ \hat{f}_0] e^{ir^- t} \right\}$$

where

$$r^\pm = r^\pm(k) = \frac{\alpha_o k|k| \pm \sqrt{\alpha_o^2 k^4 + 4(|k| + \beta |k|^3)}}{2}.$$

Similarly, for $\ell > 0$, we obtain

$$\widehat{f}^{(\ell)}(k,t) = \frac{1}{i(r^+ - r^-)} \int_0^t F(k,s) \left\{ e^{ir^+(t-s)} - e^{ir^-(t-s)} \right\} \mathrm{d}s$$

with $F(k,t)$ given in (29). Thus,

$$\widehat{f_t}^{(\ell)}(k,t) = \frac{1}{(r^+ - r^-)} \int_0^t F(k,s) \left\{ r^+ e^{ir^+(t-s)} - r^- e^{ir^-(t-s)} \right\} \mathrm{d}s.$$

Let us exploit the commutator structure of the nonlinearity (cf. Granero-Belinchón and Scrobogna 2020a). In particular, we note that

$$[|k||k-m| - k(k-m)] = |k||k-m|[1 - \operatorname{sgn}(k)\operatorname{sgn}(k-m)] \le 2|k||k-m|$$

for $k < m$ and

$$[|k||k-m| - k(k-m)] = 0$$

otherwise. Hence,

$$\frac{1}{r^+ - r^-}[|k||k-m| - k(k-m)] \le 2\sqrt{|m|}|k-m| \le 2(1+|m|)|k-m|. \tag{30}$$

Let us fix $1 < D < R \in \mathbb{Z}_+$ such that

$$\frac{D(R+1)}{1+R^2} \leq 1.$$

Since the series (27) are respectively bounded by

$$\sup_{0\leq t\leq T} \sum_{l=0}^{\infty} \lambda^{\ell+1} \| f^{(\ell)}(t)\|_{\mathbb{A}_1} \quad \text{and} \quad \sup_{0\leq t\leq T} \sum_{l=0}^{\infty} \lambda^{\ell+1} \| f_t^{(\ell)}(t)\|_{\mathbb{A}_1} \tag{31}$$

by proving the boundedness of (31), we get the absolute convergence of (27) and, hence, the existence of solutions. We start by considering the truncated series

$$S_R^1 := \sum_{\ell=0}^{R} \lambda^{\ell+1} f^{(\ell)}(x_1, t) \quad \text{and} \quad S_R^2 := \sum_{\ell=0}^{R} \lambda^{\ell+1} f_t^{(\ell)}(x_1, t). \tag{32}$$

Thus, because of (29) and (30), we obtain

$$\begin{aligned}
\| f^{(\ell)}(t)\|_{\mathbb{A}_{R+1-l}} \leq & \int_{-\infty}^{\infty} e^{(R+1-\ell)|k|} \int_0^t \sum_{j=0}^{\ell-1} \Big[\int_{-\infty}^{\infty} \sqrt{|k|} |\hat{f}_t^{(j)}(m,s)||\hat{f}_t^{(\ell-1-j)}(k-m,s)| \mathrm{d}m \\
& + 2\int_{-\infty}^{\infty} |k-m|(1+|m|)|\hat{f}^{(j)}(m,s)||\hat{f}^{(\ell-1-j)}(k-m,s)|\mathrm{d}m \\
& + 2\alpha_o \int_{-\infty}^{\infty} |k-m|^2(1+|m|)|\hat{f}^{(j)}(m,s)||\hat{f}_t^{(\ell-1-j)}(k-m,s)|\mathrm{d}m \\
& + 2\beta \int_{-\infty}^{\infty} |k-m|^3(1+|m|)|\hat{f}^{(j)}(m,s)||\hat{f}^{(\ell-1-j)}(k-m,s)|\mathrm{d}m \Big] \mathrm{d}s\mathrm{d}k \\
\leq & C_1(\alpha_o, \beta) \int_0^t \sum_{j=0}^{\ell-1} \Big[\| f_t^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \| f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \\
& + \| f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \| f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \\
& + \| f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \| f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \\
& + \| f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \| f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \Big] \mathrm{d}s
\end{aligned}$$

where we have used Fubini's theorem together with $\hat{f}^{(l)}(0,t) = 0$ and the inequalities

$$\begin{aligned}
1 + |k| &\leq e^{|k|} \ \forall k \in \mathbb{R} \\
|k|^n &\leq n! e^{|k|} \ \forall k \in \mathbb{R} \\
|k| &\leq c e^{\frac{|k|}{c}} \leq c e^{\frac{|k-m|+|m|}{c}} \ \forall c \in \mathbb{Z}_+.
\end{aligned}$$

Moreover, using the commutator structure again, we obtain that

$$\left|\frac{r^{\pm}}{r^{+}-r^{-}}\right|[|k||k-m|-k(k-m)] \leq 2(|m|+\alpha_o|m|^2+\alpha_o|m|^3)|k-m|$$

for $k < m$ and, hence, we also find that

$$\begin{aligned}\|f_t^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}} \leq C_2(\alpha_o,\beta)\int_0^t \sum_{j=0}^{\ell-1}\Big[&\|f_t^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\\ &+\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\\ &+\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\\ &+\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\Big]\,\mathrm{d}s.\end{aligned}$$

On the other hand, since $R+2-\ell \leq R+2-\ell+j = R+1-(\ell-1-j)$, we have

$$\|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}} \leq \|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+1-(\ell-1-j)}}.$$

Furthermore, since $R+2-\ell = R+1-(\ell-1) \leq R+1-j$ for $j \leq \ell-1$, we also have

$$\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-l}} \leq \|f^{(j)}(s)\|_{\mathbb{A}_{R+1-j}}.$$

As a consequence, we find

$$\begin{aligned}&\|f^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}}+\|f_t^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}}\\ &\leq C(\alpha_o,\beta)\int_0^t\sum_{j=0}^{\ell-1}\Big[\|f_t^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\\ &\quad+\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\\ &\quad+\|f^{(j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+2-\ell}}\Big]\,\mathrm{d}s\\ &\leq C(\alpha_o,\beta)\int_0^t\sum_{j=0}^{\ell-1}\Big[\|f_t^{(j)}(s)\|_{\mathbb{A}_{R+1-j}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+1-(\ell-1-j)}}\\ &\quad+\|f^{(j)}(s)\|_{\mathbb{A}_{R+1-j}}\|f_t^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+1-(\ell-1-j)}}\\ &\quad+\|f^{(j)}(s)\|_{\mathbb{A}_{R+1-j}}\|f^{(\ell-1-j)}(s)\|_{\mathbb{A}_{R+1-(\ell-1-j)}}\Big]\,\mathrm{d}s\end{aligned}$$

with $C(\alpha_o,\beta) = 2\max\{C_1(\alpha_o,\beta), C_2(\alpha_o,\beta)\}$. Let us define

$$\mathcal{A}_\ell(t) = C(\alpha_o,\beta)e^{-\frac{\ell+1}{1+\ell R^2}D(R+1)}\left[\|f^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}}+\|f_t^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}}\right].$$

First we observe that, since $R > 1$, for $0 \leq j \leq \ell - 1$,

$$\frac{\ell+1}{1+\ell R^2} + 2 \geq \frac{(\ell-1-j+1)}{1+(\ell-1-j)R^2} + \frac{(j+1)}{1+jR^2}, \tag{33}$$

so that

$$e^{-\frac{l+1}{1+lR^2}D(R+1)} \leq e^2 e^{-\frac{l-1-j+1}{1+(l-1-j)R^2}D(R+1)} e^{-\frac{j+1}{1+jR^2}D(R+1)}.$$

Hence, the former recursion for $\|f^{(l)}(t)\|_{\mathbb{A}_{R+1-l}} + \|f_t^{(l)}(t)\|_{\mathbb{A}_{R+1-l}}$ can be equivalently written as

$$\mathcal{A}_\ell(t) \leq e^2 \int_0^t \sum_{j=0}^{\ell-1} \mathcal{A}_j(s)\mathcal{A}_{\ell-1-j}(s)\mathrm{d}s.$$

We define now

$$\mathcal{B}_\ell(t) = e^2 \mathcal{A}_\ell(t),$$

and find that $\mathcal{B}_\ell$ satisfies

$$\mathcal{B}_\ell(t) \leq \int_0^t \sum_{j=0}^{\ell-1} \mathcal{B}_j(s)\mathcal{B}_{\ell-1-j}(s)\mathrm{d}s.$$

First, let us observe that

$$\begin{aligned}
\mathcal{B}_0(t) &= e^2 C(\alpha_o, \beta) e^{-D(R+1)} \left[\|f^{(0)}(t)\|_{\mathbb{A}_{R+1}} + \|f_t^{(0)}(t)\|_{\mathbb{A}_{R+1}} \right] \\
&\leq \frac{e^2}{\sqrt{2\pi}\lambda} C(\alpha_o, \beta) e^{-D(R+1)} \int_{-D}^{D} e^{(R+1)|k|} \left(|\hat{f}_0| + \left| \frac{\hat{f}_1}{r^+ - r^-} \right| \right) \mathrm{d}k \\
&\leq \frac{e^2}{\sqrt{2\pi}\lambda} C(\alpha_o, \beta) \int_{-D}^{D} \left(|\hat{f}_0| + \left| \frac{\hat{f}_1}{\sqrt{|k|}} \right| \right) \mathrm{d}k \\
&\leq \frac{C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1})}{\lambda}.
\end{aligned}$$

We fix

$$\lambda = C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1})$$

then, we prove by induction that

$$\mathcal{B}_\ell(t) \leq \mathcal{C}_\ell t^\ell \tag{34}$$

with $\mathcal{C}_l$ being the Catalan numbers,

$$\mathcal{C}_l = \sum_{j=0}^{l-1} \mathcal{C}_j \mathcal{C}_{l-1-j},$$

which behave as

$$\mathcal{C}_l \sim O(l^{-\frac{3}{2}}4^l) \text{ for } l >> 1. \tag{35}$$

Since we have already seen that (34) holds for $\ell = 0$, we continue with the induction step. For $\ell \geq 1$, we have

$$\begin{aligned}
\mathcal{B}_\ell &\leq \int_0^t \sum_{j=0}^{\ell-1} \mathcal{B}_j(s)\mathcal{B}_{\ell-1-j}(s)\mathrm{d}s \\
&\leq \int_0^t \sum_{j=0}^{\ell-1} \mathcal{C}_j s^j \mathcal{C}_{\ell-1-j} s^{l-1-j}\mathrm{d}s \\
&= \mathcal{C}_\ell \int_0^t s^{\ell-1}\mathrm{d}s \\
&= \mathcal{C}_\ell \frac{t^\ell}{\ell}.
\end{aligned}$$

Therefore, because of (35), we find

$$\begin{aligned}
\|f^{(\ell)}(t)\|_{\mathbb{A}_1} + \|f_t^{(\ell)}(t)\|_{\mathbb{A}_1} &\leq \|f^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}} + \|f_t^{(\ell)}(t)\|_{\mathbb{A}_{R+1-\ell}} \\
&\leq [C(\alpha_o,\beta)]^{-1} e^{-2} e^{\frac{\ell+1}{1+\ell R^2}D(R+1)} 4^\ell t^\ell
\end{aligned}$$

In a similar way,

$$\|f^{(0)}(t)\|_{\mathbb{A}_1} + \|f_t^{(0)}(t)\|_{\mathbb{A}_1} \leq C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1})$$

Then, the truncated series (32) satisfies the estimates

$$\begin{aligned}
\|S_R^1\|_{\mathbb{A}_1} &\leq C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1}) + 2\frac{\lambda}{e^2 C(\alpha_o,\beta)} \sum_{\ell=1}^R e^{\frac{D(R+1)}{1+\ell R^2}} \left(4e^{\frac{D(R+1)}{1+\ell R^2}} \lambda t\right)^\ell \\
&\leq C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1}) + 2\frac{\lambda}{C(\alpha_o,\beta)} \sum_{\ell=1}^R (4e\lambda t)^\ell,
\end{aligned}$$

as well as

$$\|S_R^2\|_{\mathbb{A}_1} \leq C(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1}) + 2\frac{\lambda}{C(\alpha_o,\beta)} \sum_{\ell=1}^R (4e\lambda t)^\ell,$$

Thus, we conclude that, if

$$t \leq T^* < \frac{1}{4eC(\|f_0\|_{\mathbb{A}_0}, \|f_1\|_{\mathbb{A}_0}, \|f_1\|_{L^1})},$$

we can take the limit as $R \to +\infty$ in (32) and we obtain the existence of

$$f(x_1,t) = S^1_\infty \quad \text{and} \quad f_t(x_1,t) = S^2_\infty.$$

We also observe that the above estimates ensure $f, f_t \in L^\infty\left(0, T^*; \mathbb{A}_1\right)$. In addition, since both f and f_t are analytic functions in space, using the Cauchy product of power series, we also find that $f, f_t \in C\left(0, T^*; \mathbb{A}_{0.5}\right)$. At this level of regularity, we can perform standard energy estimates and find the estimate

$$\|f_t - g_t\|_{L^2} + \|\Lambda^{1.5}(f-g)\|_{L^2} \leq (\|f_1 - g_1\|_{L^2} + \|\Lambda^{1.5}(f_0 - g_0)\|_{L^2})C(t,{}^f_0, f_1, g_0, g_1)$$

where f and g are two solutions of the equation (25) with different initial data compactly supported on Fourier space. The uniqueness in this space of analytic functions follows from a standard contradiction argument. □

4.3 Well-Posedness for Sobolev Initial Data

We recall the definition of the standard L^2-based Sobolev spaces

$$H^s(\mathbb{R}) = \left\{h \in L^2(\mathbb{R}) \text{ s.t. } \|h\|^2_{H^s} = \int_{\mathbb{R}} (1+|n|^{2s})|\hat{h}(n)|^2 \mathrm{d}n < \infty\right\}.$$

In this section, we prove the well-posedness of equation (26) with periodic boundary conditions. In order to do that, we will make extensive use of the following commutator estimate (Dawson et al. 2008):

$$\left\|\partial_x^\ell [\![\mathcal{H}, u]\!] \partial_x^m v\right\|_{L^p} \leq C \left\|\partial_x^{\ell+m} u\right\|_{L^\infty} \|v\|_{L^p}, \quad p \in (1,\infty), \quad \ell, m \in \mathbb{N}, \tag{36}$$

and the fractional Leibniz rule (see Grafakos and Seungly 2014; Kato and Ponce 1988; Kenig et al. 1993):

$$\|\Lambda^s(uv)\|_{L^p} \leq C\left(\|\Lambda^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|\Lambda^s v\|_{L^{p_3}}\|u\|_{L^{p_4}}\right),$$

which holds whenever

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \qquad \text{where } 1/2 < p < \infty, 1 < p_i \leq \infty,$$

and $s > \max\{0, 1/p - 1\}$.

Our result reads as follows:

Theorem 2 *Let $\beta > 0$ be a constant and $(f_0, f_1) \in H^{4.5}(\mathbb{R}) \times H^3(\mathbb{R})$ be the initial data for equation (26). Then, there exists $0 < T^*$ and a unique solution*

$$(f, f_t) \in L^\infty(0, T^*, H^{4.5}(\mathbb{R})) \times L^\infty(0, T^*, H^3(\mathbb{R})).$$

Proof Without loss of generality, we fix $\alpha_o = \varepsilon = 1$. The proof follows from appropriate energy estimates after a standard approximation using mollifiers (see Granero-Belinchón and Shkoller 2017). As a consequence, we will focus in obtaining the *a priori* estimates. We define the energy

$$\mathcal{E}(t) = \beta\|f(t)\|_{H^{4.5}} + \|f(t)\|_{H^{3.5}} + \|f_t(t)\|_{H^3}.$$

In order to estimate the low-order terms, we test the equation against f_t. Integrating by parts and using that

$$(\mathcal{H}f_t)_{x_1} = \Lambda f_t,$$

we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f_t\|_{L^2}^2 + \|f\|_{H^{0.5}}^2 + \beta\|f\|_{H^{1.5}}^2\right) = \int_{\mathbb{R}} \left([\![\mathcal{H}, f]\!]\Lambda f\right)_{x_1} f_t \mathrm{d}x_1 \le \mathcal{E}(t)^3.$$

Using the fundamental theorem of calculus, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^2}^2 = 2\int_{\mathbb{R}} f f_t \mathrm{d}x_1 \le 2\|f\|_{L^2}\|f_t\|_{L^2} \le C\mathcal{E}(t)^2$$

To bound the high-order terms, we test the equation against $\Lambda^6 f_t$. Then, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f_t\|_{H^3}^2 + \|f\|_{H^{3.5}}^2 + \beta\|f\|_{H^{4.5}}^2\right) = I_1 + I_2,$$

where

$$I_1 = -\int_{\mathbb{R}} \mathcal{H}\left((\mathcal{H}f_t)^2\right)_{x_1} \Lambda^6 f_t \mathrm{d}x_1,$$
$$I_2 = \int_{\mathbb{R}} \left([\![\mathcal{H}, f]\!]\Lambda f\right)_{x_1} \Lambda^6 f_t \mathrm{d}x_1.$$

We integrate by parts and find that

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}} (\mathcal{H}f_t)^2 \Lambda^7 f_t \mathrm{d}x_1 \\ &= \int_{\mathbb{R}} (\mathcal{H}f_t)^2 \partial_{x_1}^6 \Lambda f_t \mathrm{d}x_1 \\ &= -\int_{\mathbb{R}} \partial_{x_1}^3 (\mathcal{H}f_t)^2 \partial_{x_1}^3 \Lambda f_t \mathrm{d}x_1 \\ &= -2\int_{\mathbb{R}} (\mathcal{H}f_t \Lambda f_{x_1x_1t} + 3\Lambda f_t \Lambda f_{tx_1}) \partial_{x_1}^3 \Lambda f_t \mathrm{d}x_1 \\ &= J_1^1 + J_1^2, \end{aligned}$$

with

$$
\begin{aligned}
J_1^1 &= -2\int_{\mathbb{R}} \mathcal{H} f_t \Lambda f_{x_1x_1t} \partial_{x_1} \Lambda f_{x_1x_1t} \mathrm{d}x_1, \\
J_1^2 &= -6\int_{\mathbb{R}} \Lambda f_t \Lambda f_{tx_1} \partial_{x_1} \Lambda f_{x_1x_1t} \mathrm{d}x_1.
\end{aligned}
$$

Integrating by parts and using Hölder's inequality, we find that

$$
\begin{aligned}
J_1^1 &= \int_{\mathbb{R}} \Lambda f_t (\Lambda f_{x_1x_1t})^2 \mathrm{d}x_1 \leq C\mathcal{E}(t)^3, \\
J_1^2 &= 6\int_{\mathbb{R}} \partial_{x_1}(\Lambda f_t \Lambda f_{tx_1}) \Lambda f_{x_1x_1t} \mathrm{d}x_1 \\
&= 6\int_{\mathbb{R}} (\Lambda f_{x_1t} \Lambda f_{tx_1} + \Lambda f_t \Lambda f_{tx_1x_1}) \Lambda f_{x_1x_1t} \mathrm{d}x_1, \\
&\leq C\mathcal{E}(t)^3.
\end{aligned}
$$

We have to handle the second nonlinear contribution. We compute that

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}} \left(\Lambda(f\Lambda f) + (ff_{x_1})_{x_1}\right) \partial_{x_1}^6 f_t \mathrm{d}x_1 \\
&= -\int_{\mathbb{R}} \partial_{x_1}^3 \left(\Lambda(f\Lambda f) + (ff_{x_1})_{x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1 \\
&= J_2^1 + J_2^2 + J_2^3 + J_2^4,
\end{aligned}
$$

with

$$
\begin{aligned}
J_2^1 &= -\int_{\mathbb{R}} \left(\Lambda(f\Lambda f_{x_1x_1x_1}) + (ff_{x_1x_1x_1x_1})_{x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1, \\
J_2^2 &= -\int_{\mathbb{R}} \left(\Lambda(f_{x_1}\Lambda f_{x_1x_1}) + (f_{x_1}f_{x_1x_1x_1})_{x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1 \\
J_2^3 &= -\int_{\mathbb{R}} \left(\Lambda(f_{x_1x_1}\Lambda f_{x_1}) + (f_{x_1x_1}f_{x_1x_1})_{x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1 \\
J_2^4 &= -\int_{\mathbb{R}} \left(\Lambda(f_{x_1x_1x_1}\Lambda f) + (f_{x_1x_1x_1}f_{x_1})_{x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1.
\end{aligned}
$$

Using Hölder's inequality and the fractional Leibniz rule, we find that

$$
J_2^2 + J_2^3 + J_2^4 \leq C\mathcal{E}(t)^3.
$$

We observe that we can find a commutator structure in J_2^1. Indeed, we have that

$$
J_2^1 = -\int_{\mathbb{R}} \left([\![\Lambda, f]\!]\Lambda f_{x_1x_1x_1} + f_{x_1}f_{x_1x_1x_1x_1}\right) f_{tx_1x_1x_1} \mathrm{d}x_1
$$

$$\leq \|f_t\|_{H^3}\|[\![\Lambda, f]\!]\Lambda f_{x_1x_1x_1}\|_{L^2} + \mathcal{E}(t)^3.$$

This commutator in Fourier variables takes the following form:

$$\widehat{[\![\Lambda, f]\!]g} = \int_{\mathbb{R}} (|n| - |n-m|)\hat{f}(m)\hat{g}(n-m)\mathrm{d}m.$$

In particular, using Young's inequality for convolution, Sobolev inequality and Plancherel theorem, we conclude that

$$\|\widehat{[\![\Lambda, f]\!]g}\|_{L^2} \leq \||\cdot|\hat{f}\|_{L^1}\|g\|_{L^2} \leq C\|f\|_{H^2}\|g\|_{L^2}.$$

Inserting this commutator estimate in J_2^1, we conclude that

$$J_2^1 \leq C\|f_t\|_{H^3}\|f\|_{H^2}\|f\|_{H^4} + \mathcal{E}(t)^3 \leq C\mathcal{E}(t)^3.$$

As a consequence, we find the following differential inequality for the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq C\mathcal{E}(t)^2 + \mathcal{E}(t),$$

and we can ensure a uniform time of existence T^* such that

$$\mathcal{E}(t) \leq 2\mathcal{E}(0).$$

Once this uniform time of existence has been obtained, the rest of the proof is standard so we only give a sketch of the argument. First, we define approximate problems using mollifiers. These mollifiers are such that the previous energy estimates also holds for the regularized PDE. Then, we repeat the previous computations and find the uniform time of existence T^* for the sequence of regularized problems. Finally, we can pass to the limit. The uniqueness follows from a contradiction argument and we omit it. □

5 The Unidirectional Asymptotic Model for Waves with Odd Viscosity

5.1 Derivation

Let us consider the following 'far-field' variables,

$$\chi = x - t, \quad \tau = \varepsilon t.$$

An application of the chain rule leads to

$$\begin{aligned}\frac{\partial^2}{\partial t^2} f(\chi(x,t), \tau(t)) &= -f_{\chi\chi}\frac{\partial \chi}{\partial t} - f_{\chi\tau}\frac{\partial \tau}{\partial t} + \varepsilon f_{\tau\chi}\frac{\partial \chi}{\partial t} + \varepsilon f_{\tau\tau}\frac{\partial \tau}{\partial t} \\ &= f_{\chi\chi} - \varepsilon f_{\chi\tau} - \varepsilon f_{\tau\chi} + \varepsilon^2 f_{\tau\tau}.\end{aligned}$$

So that, neglecting terms of order $\mathcal{O}(\varepsilon^2)$, equation (25) reads

$$\left(f_\chi - 2\varepsilon f_\tau\right)_\chi = \Big(-\mathcal{H}f - \varepsilon\mathcal{H}\left[(\Lambda f)^2\right] + \alpha_o\mathcal{H}\Lambda^2 f + \alpha_o\varepsilon\Lambda f_\tau + \varepsilon[\![\mathcal{H}, f]\!]\Lambda f - \varepsilon\alpha_o[\![\mathcal{H}, f]\!]\Lambda^3 f - \beta\mathcal{H}\Lambda^2 f + \varepsilon\beta[\![\mathcal{H}, f]\!]\Lambda^3 f\Big)_\chi \quad \text{on } \Gamma\times[0,T].$$

Integrating on χ, reordering terms and abusing notation by using x_1 and t as variables again, we are led to the equation

$$2f_t + \alpha_o\Lambda f_t = \frac{1}{\varepsilon}\left\{f_{x_1} + \mathcal{H}f + (\alpha_o - \beta)\mathcal{H}f_{x_1x_1}\right\} + \mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f + (\alpha_o-\beta)[\![\mathcal{H}, f]\!]\Lambda^3 f, \quad \text{on } \Gamma\times[0,T]. \tag{37}$$

This equation reminds the classical Benjamin–Ono equation (cf. Brooke Benjamin 1967; Ono 1975) and Burgers–Hilbert equation (cf. Biello and Hunter 2010) (see also Riaño 2021). It is also similar to the equation derived in Durán (2020). This similarity is not only due to the linear operators. Indeed, in the new variable

$$u = \Lambda f,$$

we find that (37) contains the classical Burgers term:

$$u_t = \frac{1}{\varepsilon}\frac{1}{2+\alpha_o\Lambda}\left\{u_{x_1} + \mathcal{H}u + (\alpha_o-\beta)\mathcal{H}u_{x_1x_1}\right\} + \frac{1}{2+\alpha_o\Lambda}\left\{-\partial_{x_1}(u)^2 - \Lambda[\![\mathcal{H}, f]\!]u + (\alpha_o-\beta)\Lambda[\![\mathcal{H}, f]\!]\Lambda^2 u\right\}, \quad \text{on } \Gamma\times[0,T]. \tag{38}$$

5.2 Well-Posedness for Sobolev Initial Data

In this section, we study the well-posedness of equation (37).

Theorem 3 *Let $\alpha_o > 0$ and $\beta \geq 0$ be two constants and $f_0 \in H^3(\mathbb{R})$ be the initial data for equation (37). Then, there exists $0 < T^*$ and a unique solution*

$$f \in L^\infty(0, T^*, H^3(\mathbb{R})).$$

Proof As before, the proof follows from appropriate energy estimates and a suitable sequence of approximate problems. Thus, we start with the energy estimates. We test the equation against $\Lambda^6 f$ and we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{H^3}^2 = \int_{\mathbb{R}} \frac{1}{2+\alpha_o\Lambda}\left\{\mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f + (\alpha_o-\beta)[\![\mathcal{H}, f]\!]\Lambda^3 f\right\}\Lambda^6 f\,\mathrm{d}x_1.$$

Integrating by parts we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{H^3}^2 &= -\int_{\mathbb{R}} \frac{1}{2+\alpha_o\Lambda}\left\{\mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f + (\alpha_o-\beta)[\![\mathcal{H}, f]\!]\Lambda^3 f\right\}\partial_{x_1}^2\Lambda^4 f \mathrm{d}x_1 \\ &= -\int_{\mathbb{R}} \partial_{x_1}^2\left\{\mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f + (\alpha_o-\beta)[\![\mathcal{H}, f]\!]\Lambda^3 f\right\}\frac{\Lambda^4}{2+\alpha_o\Lambda} f \mathrm{d}x_1.\end{aligned}$$

Using the bound

$$\left\|\frac{\Lambda^4}{2+\alpha_o\Lambda} f\right\|_{L^2} \leq \|f\|_{H^3},$$

together with the commutator estimate for the Hilbert transform (36), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{H^3}^2 \leq C\|f\|_{H^3}^3.$$

This differential inequality leads to a uniform time of existence. Equipped with the uniform time of existence, the existence of solution can be obtained using a sequence of regularized problems (see Granero-Belinchón and Shkoller 2017). The uniqueness is a consequence of a contradiction argument and the regularity of the solution. □

5.3 Distributional Solution with Limited Regularity

Let us consider the case $\alpha_o > 0$ and $\beta = \alpha_o$.

Before stating our result, we define our concept of distributional solution: we say that f is a distributional solution of (9) if and only if

$$\begin{aligned} &-\int_{\mathbb{R}}(2+\alpha_o\Lambda)\varphi(x_1,0) f_0(x_1)\mathrm{d}x_1\mathrm{d}s - \int_0^T\int_{\mathbb{R}}(2+\alpha_o\Lambda)\varphi_t(x_1,s) f(x_1,s)\mathrm{d}x_1\mathrm{d}s \\ &\quad= -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}} f(x_1,s)\varphi_{x_1}(x_1,s) + \mathcal{H}\varphi(x_1,s) f(x_1,s)\mathrm{d}x_1\mathrm{d}s \\ &\qquad+ \int_0^T\int_{\mathbb{R}}\left\{\mathcal{H}(\Lambda f)^2 - [\![\mathcal{H}, f]\!]\Lambda f\right\}\varphi(x_1,s)\mathrm{d}x_1\mathrm{d}s,\end{aligned}$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R})$.

Theorem 4 *Let $\alpha_o > 0$ and $\beta = \alpha_o$ be two constants and $f_0 \in H^{1.5}(\mathbb{R})$ be the initial data for equation (37). Then, there exists $0 < T^*$ and at least one distributional solution*

$$f \in L^\infty(0, T^*, H^{1.5}(\mathbb{R})).$$

Proof We consider the regularized problem

$$2f_t^{(n)}+\alpha_o\Lambda f_t^{(n)}-\frac{1}{n}f_{x_1x_1}^{(n)}=\frac{1}{\varepsilon}\left\{f_{x_1}^{(n)}+\mathcal{H}f^{(n)}\right\}\\+\mathcal{H}\left(\Lambda f^{(n)}\right)^2-\left[\!\left[\mathcal{H},f^{(n)}\right]\!\right]\Lambda f^{(n)},\quad\text{on }\Gamma\times[0,T],\tag{39}$$

with the mollified initial data

$$f^{(n)}(x_1,0)=\rho_n*f_0(x_1),$$

where ρ_n is a standard Friedrich mollifier.

We test the equation against $f^{(n)}$ and use Hölder and Sobolev inequalities to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f^{(n)}\|_{L^2}^2+\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{0.5}f^{(n)}\|_{L^2}^2+\frac{1}{n}\|f_{x_1}^{(n)}\|_{L^2}^2\le C\|f_{x_1}^{(n)}\|_{L^2}^3.$$

Now, we test the equation against $\Lambda^2f^{(n)}$. We find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda f^{(n)}\|_{L^2}^2+\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1.5}f^{(n)}\|_{L^2}^2+\frac{1}{n}\|f_{x_1x_1}^{(n)}\|_{L^2}^2=\int_{\mathbb{R}}\mathcal{H}\left(\Lambda f^{(n)}\right)^2\Lambda^2f^{(n)}\mathrm{d}x_1\\-\int_{\mathbb{R}}\left[\!\left[\mathcal{H},f^{(n)}\right]\!\right]\Lambda f^{(n)}\Lambda^2f^{(n)}\mathrm{d}x_1.$$

The first nonlinear contribution vanishes. Indeed,

$$\int_{\mathbb{R}}\mathcal{H}\left(\Lambda f^{(n)}\right)^2\Lambda^2f^{(n)}\mathrm{d}x_1=\int_{\mathbb{R}}\left(\Lambda f^{(n)}\right)^2\Lambda f_{x_1}^{(n)}\mathrm{d}x_1\\=0.$$

The second nonlinear contribution can be handled as follows:

$$\begin{aligned}-\int_{\mathbb{R}}\left[\!\left[\mathcal{H},f^{(n)}\right]\!\right]\Lambda f^{(n)}\Lambda^2f^{(n)}\mathrm{d}x_1&=-\int_{\mathbb{R}}(\mathcal{H}(f^{(n)}\Lambda f^{(n)})+f^{(n)}f_{x_1}^{(n)}))\Lambda^2f^{(n)}\mathrm{d}x_1\\&=\int_{\mathbb{R}}(\mathcal{H}(f^{(n)}\Lambda f^{(n)})+f^{(n)}f_{x_1}^{(n)}))f_{x_1x_1}^{(n)}\mathrm{d}x_1\\&\le C\|f_{x_1}^{(n)}\|_{L^3}^3\\&\le C\|f_{x_1}^{(n)}\|_{H^{1/6}}^3\\&\le C\|f_{x_1}^{(n)}\|_{L^2}^2\|f_{x_1}^{(n)}\|_{H^{0.5}}.\end{aligned}$$

As a consequence, if we define

$$\mathcal{E}(t)=\|f^{(n)}\|_{L^2}^2+\|\Lambda^{0.5}f^{(n)}\|_{L^2}^2+\|\Lambda f^{(n)}\|_{L^2}^2+\|\Lambda^{1.5}f^{(n)}\|_{L^2}^2,$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq C\mathcal{E}(t)^2.$$

We thus conclude the uniform-in-n time of existence T^* such that

$$f^{(n)} \in L^\infty(0, T^*, H^{1.5}(\mathbb{R})),$$

with a bound that is independent of n. This implies that

$$f^{(n)} \stackrel{*}{\rightharpoonup} f \in L^\infty(0, T^*, H^{1.5}(\mathbb{R})).$$

Furthermore, using the regularity of f together with

$$\|(\Lambda f)^2\|_{L^2}^2 = \|\Lambda f\|_{L^4}^4 \leq C\|\Lambda f\|_{H^{0.25}}^4 \leq C\|\Lambda f\|_{L^2}^2\|\Lambda f\|_{H^{0.5}}^2,$$

and we can compute

$$f_t^{(n)} \in L^\infty(0, T^*, L^2(\mathbb{R})),$$

with a bound that is independent of n.

In particular,

$$f^{(n)} \in L^\infty(0, T^*, H^{1.5}([-1, 1])),$$
$$f_t^{(n)} \in L^\infty(0, T^*, L^2([-1, 1])).$$

Then, a standard application of the Aubin–Lions theorem ensures that we can obtain a subsequence such that

$$f^{(n_1(j))} \to f_{(1)} \in L^2(0, T^*, H^1([-1, 1])).$$

Similarly, the elements in this sequence satisfy

$$f^{(n)} \in L^\infty(0, T^*, H^{1.5}([-2, 2])),$$
$$f_t^{(n)} \in L^\infty(0, T^*, L^2([-2, 2])),$$

so, we can extract another subsequence such that

$$f^{(n_2(j))} \to f_{(2)} \in L^2(0, T^*, H^1([-2, 2])).$$

Due to the uniqueness of the limit, we have that

$$f_{(1)} = f_{(2)}$$

at least in the common interval $[-1, 1]$. Then, for each m, we can repeat this procedure and find different subsequences $f^{(n_m(j))}$ of the original sequence $f^{(n)}$. Now, we use Cantor's diagonal argument. Then, we define the sequence

$$f^{(\ell)} = f^{(n_\ell(\ell))}$$

These $f^{(\ell)}$ are a subsequence of the original sequence $f^{(n)}$ and then

$$f^{(\ell)} \in L^\infty(0, T^*, H^{1.5}(\mathbb{R})),$$
$$f_t^{(\ell)} \in L^\infty(0, T^*, L^2(\mathbb{R})).$$

Now, we observe that for each interval $[-k, k]$, we have that

$$f^{(\ell)} \to f \in L^2(0, T^*, H^1([-k, k])).$$

Indeed, it is enough to note that the elements $f^{(\ell)}$ for $\ell \geq k+1$ are elements of a subsequence that converges in $[-k, k]$ and that the resulting limit must be unique. In addition, if we fix an arbitrary compact set $\mathcal{U} \subset \mathbb{R}$, we have that

$$f^{(\ell)} \to f \in L^2(0, T^*, H^1(\mathcal{U})).$$

Now, if we fix a test function φ, we have that the distributional form of the approximate problems reads

$$\begin{aligned}
&-\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi(x_1,0)\rho_\ell * f_0(x_1)\mathrm{d}x_1\mathrm{d}s - \int_0^T\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi_t(x_1,s) f^{(\ell)}(x_1,s)\mathrm{d}x_1\mathrm{d}s \\
&= -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}} f^{(\ell)}(x_1,s)\varphi_{x_1}(x_1,s) + \mathcal{H}\varphi(x_1,s) f^{(\ell)}(x_1,s)\mathrm{d}x_1\mathrm{d}s \\
&\quad + \int_0^T\int_{\mathbb{R}} \left\{\mathcal{H}\left(\Lambda f^{(\ell)}\right)^2 - \left[\!\left[\mathcal{H}, f^{(\ell)}\right]\!\right]\Lambda f^{(\ell)}\right\}\varphi(x_1,s)\mathrm{d}x_1\mathrm{d}s + \int_0^T\int_{\mathbb{R}} \frac{f^{(\ell)}}{n(\ell)}\varphi_{x_1x_1}\mathrm{d}x_1\mathrm{d}s.
\end{aligned}$$

Due to weak-$*$ convergence, it is easy to see that the linear terms converge

$$\begin{aligned}
&-\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi(x_1,0)\rho_\ell * f_0(x_1)\mathrm{d}x_1\mathrm{d}s - \int_0^T\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi_t(x_1,s) f^{(\ell)}(x_1,s)\mathrm{d}x_1\mathrm{d}s \\
&\quad \to -\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi(x_1,0) f_0(x_1)\mathrm{d}x_1\mathrm{d}s - \int_0^T\int_{\mathbb{R}} (2+\alpha_o\Lambda)\varphi_t(x_1,s) f(x_1,s)\mathrm{d}x_1\mathrm{d}s, \\
&-\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}} f^{(\ell)}(x_1,s)\varphi_{x_1}(x_1,s) + \mathcal{H}\varphi(x_1,s) f^{(\ell)}(x_1,s)\mathrm{d}x_1\mathrm{d}s \\
&\quad \to -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}} f(x_1,s)\varphi_{x_1}(x_1,s) + \mathcal{H}\varphi(x_1,s) f(x_1,s)\mathrm{d}x_1\mathrm{d}s, \\
&\int_0^T\int_{\mathbb{R}} \frac{f^{(\ell)}}{n(\ell)}\varphi_{x_1x_1}\mathrm{d}x_1\mathrm{d}s \to 0.
\end{aligned}$$

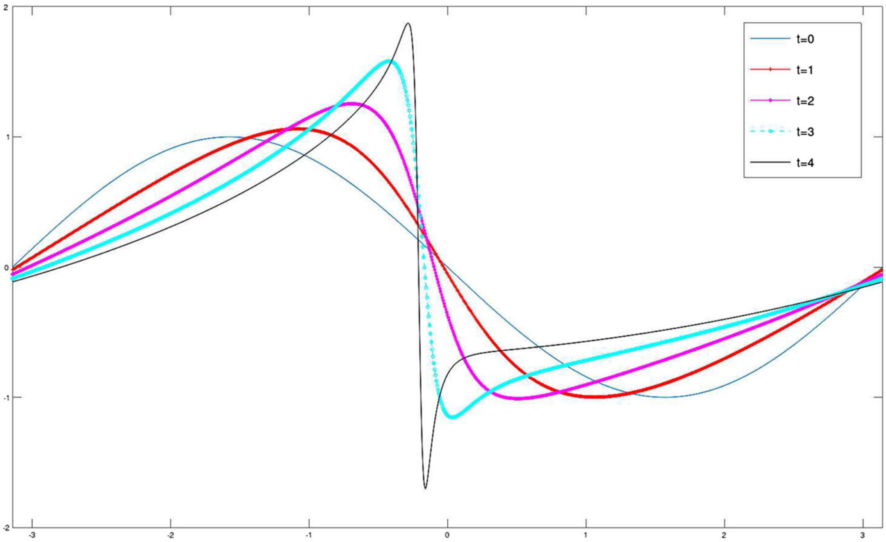

Fig. 1 Evolution in the case $\epsilon = \alpha = \beta = 1$ and $N = 2^{10}$

The first nonlinear contribution can be handled as follows:

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}} \mathcal{H}\left[\left(\Lambda f^{(\ell)} + \Lambda f\right)\left(\Lambda f^{(\ell)} - \Lambda f\right)\right] \varphi \mathrm{d}x_1 \mathrm{d}s \\ &= \int_0^T \int_{-M}^{M} \left(\Lambda f^{(\ell)} + \Lambda f\right)\left(\Lambda f^{(\ell)} - \Lambda f\right) \mathcal{H}\varphi \mathrm{d}x_1 \mathrm{d}s \\ &\le C_\varphi \|f^{(\ell)}\|_{H^1(\mathbb{R})} \|f^{(\ell)} - f\|_{H^1([-M,M])} \to 0. \end{aligned}$$

The commutator term is of lower order and can be handled similarly. Then, passing to the limit, we conclude that the limit function f satisfies the distributional form (39). □

Remark 5.1 We would like to emphasize that it is possible to find uniform-in-time energy estimates for the H^1 norm (instead of the $H^{1.5}$ norm). However, the notion of solution seems unclear at that level of regularity.

5.4 Numerical Study

In this section, we report a preliminary numerical study of equation (38) with periodic boundary conditions in $[-\pi, \pi]$.

These numerical results have been obtained using a spectral method to simulate both the differential and the singular integral operators. In particular, in order to simulate (38), we use the Fourier collocation method. This method considers a discretization of the spatial domain with N uniformly distributed points. Then, we use the fast Fourier transform and inverse fast Fourier transform (IFFT) routines already implemented in

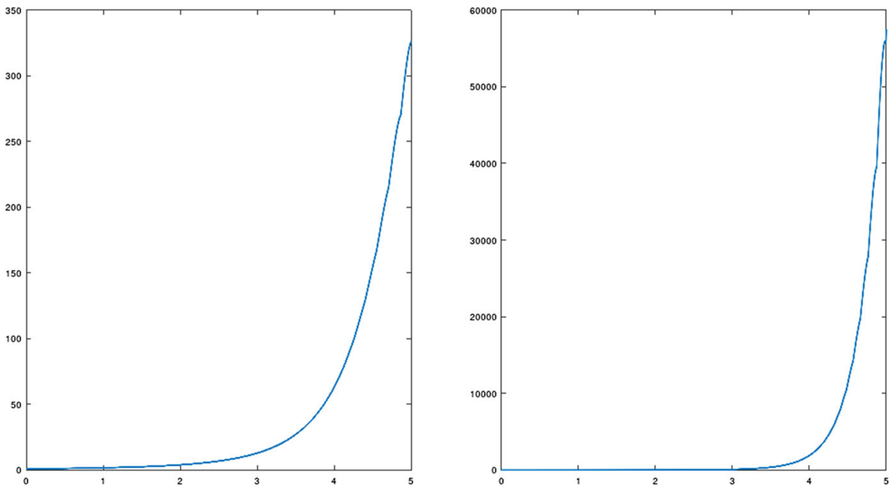

Fig. 2 $\|u_{x_1}(t)\|_{L^\infty}$ (left) and $\|u_{x_1x_1}(t)\|_{L^\infty}$ (right) in the case $\epsilon = \alpha = \beta = 1$ and $N = 2^{11}$

Octave to jump between the physical and the frequency spaces. In this way, we can take advantage of the fact that in Fourier variables, the differential operators and the Hilbert transform are defined by multipliers. With this method, the problem reduces to a system of ODEs in Fourier space. To advance in time, we used the standard adaptative Runge–Kutta scheme implemented in the Octave function *ode45*.

Case 1: We consider the initial data for (38) given by

$$u(x_1, 0) = -\sin(x_1).$$

The physical parameters are $\epsilon = \alpha = \beta = 1$. Here, we see that the solution is getting steeper and steeper (see Fig. 1). However, when we compute $\|u_{x_1}(t)\|_{L^\infty}$ and $\|u_{x_1x_1}(t)\|_{L^\infty}$, we cannot conclude the existence of a finite time singularity (see Fig. 2). In particular, both the first and the second derivative grow, however, they seem to remain $\mathcal{O}(10^2)$ and $\mathcal{O}(10^4)$, respectively.

Case 2: We consider the initial data for (38) given by

$$u(x_1, 0) = -10\sin(10x_1).$$

The physical parameters are $\epsilon = 0.1$, $\alpha = \beta = 1$. Here, we see that the solution oscillates. In particular, the solution is getting steeper but then depletes (see Fig. 3).

6 Discussion

In this paper, we have obtained new asymptotic models for both bidirectional and unidirectional gravity–capillary odd waves. Besides the derivation, we have also stud-

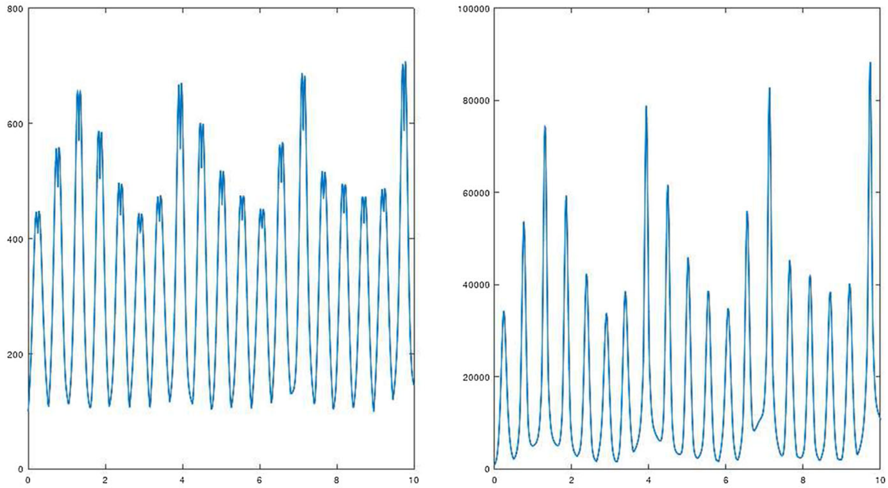

Fig. 3 $\|u_{x_1}(t)\|_{L^\infty}$ (left) and $\|u_{x_1x_1}(t)\|_{L^\infty}$ (right) in the case $\epsilon = 0.1$, $\alpha = \beta = 1$ and $N = 2^{11}$

ied some of their mathematical properties rigorously. In particular, we have proved a number of local in time well-posedness results in appropriate spaces. Furthermore, we have also studied the unidirectional model numerically trying to find a numerical scenario of finite time singularities. At this point, this scenario remains undetermined and the question of finite time singularities or the global existence of smooth solutions remains as open problems.

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