Stability and phase-noise analysis

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***Abstract* — This contribution emphasizes the relevance of Prof. Vittorio Rizzoli's works on stability and phase-noise analysis and describes how they have impacted more recent investigations. Regarding the stability analysis, in 1985 he developed a frequency-domain formulation that provided unvaluable insight into the way how the perturbed system should be described and analyzed. This formulation enabled the application, for the first time, of the Nyquist criterion to circuits simulated with harmonic balance (HB). In 1994, he derived a HB formulation for phase-noise analysis, which considered both the frequency modulation, associated with the timing noise, and the frequency conversion effects; it provided a complete prediction of the noisy oscillator spectrum at small and large offset frequencies from the carrier. Departing from these relevant contributions, more recent advances in the two topics will be described.**

***Keywords* —stability analysis, phase-noise analysis.**

I. Introduction

The instability of nonlinear circuits is a common problem that often invalidates the prototypes and increases the production costs. If the simulated solution is unstable, the measured solution will be qualitatively different and will usually contain self-generated oscillations. Harmonic balance (HB) is insensitive to the stability properties and, by default, will converge to unstable solutions, less involved from a computation point of view. Thus, checking the stability of the HB solution is essential to validate its physical existence, which requires the introduction into the system of a small perturbation. In 1985 [1], Prof. Rizzoli and his co-author developed a frequency-domain formulation that provided unvaluable insight into the way how the perturbed system should be described and analyzed. It enabled the application, for the first time, of the Nyquist criterion [1]-[2] to circuits simulated with harmonic balance (HB). Prof. Rizzoli also addressed the qualitative stability changes in the physical solutions under the continuous variation of a circuit parameter [3]-[7], such as a bias voltage, the input power, or the input frequency, and related these qualitative changes to bifurcation phenomena [8]. A local bifurcation is obtained when a real pole or a pair of complex-conjugate poles of the perturbed solution cross the imaginary axis under the continuous variation of the parameter. He derived mathematical conditions for the fundamental types of local bifurcation based on the evaluation of the HB characteristic determinant at the (real) frequencies of the crossing poles. All these major contributions were published in 1988 in his famous paper "State of the Art and Present Trends in Nonlinear Microwave CAD Techniques" [9], which also included aspects such as optimization, frequency conversion and noise.

In 1994, Prof. Rizzoli and his co-workers presented a complete methodology for the noise analysis [10] of forced and autonomous circuits, including the fundamental aspect of phase noise. This is an undesired characteristic of microwave sources that degrades the timing accuracy and gives rise to a broadening of the carrier spectrum, susceptible to cause interferences and demodulation errors. The work [10] derived a HB formulation for phase-noise analysis, which considered both the frequency modulation, associated with the timing noise, and the frequency conversion; this enabled a complete prediction of the oscillator spectrum at small and large offset frequencies from the carrier.

II. Stability Analysis

The stability analysis predicts the solution response to the small perturbations that are always present in real life. This analysis is involved, and for years it has been addressed in non-rigorous manners. In small-signal conditions, one common mistake is the application of the Rollet stability criteria to two-port networks that do not satisfy the Rollet's proviso: when unloaded, the network must not have any poles in the right-hand side (RHS) of the complex plane. Another source of error is the application of the Nyquist criterion [11] to functions that can exhibit both RHS zeroes and poles.

Prof. Rizzoli's work [1] presented a rigorous perturbation analysis of nonlinear circuits described with harmonic balance [9], [12]-[13]. In [9] a piecewise formulation is used, in terms of the voltages $\bar{V}$ and currents $\bar{I}$ of the nonlinear subnetwork. Considering the fundamental frequency ω_o, at each harmonic frequency $k\omega_o$, we have [9]:

$$\bar{H} = A(k\omega_o)\bar{V}_k + B(k\omega_o)\bar{I}_k + \bar{D}_k(k\omega_o) = 0 \quad (1)$$

where A and B are the frequency-dependent matrixes that describe the passive-linear circuitry and $\bar{D}$ are the driving functions. For the stability analysis of any solution of (1), a small amplitude perturbation of complex frequency was introduced in [1], expressed in terms of a complex exponential $\exp\{(\sigma+\omega)t\}$. Two fundamental aspects were considered: (i) due to the small amplitude of the perturbation it is possible to linearize the nonlinear devices about the steady-state solution (in the case of periodic regimes, this linearization should be carried out with the conversion-matrix approach), (ii) there are no sources at the perturbation frequencies. Thus, the following homogeneous system was derived [9]:

$$\left[diag\left\{Y(\omega - j\sigma + k\omega_o)\right\} + Y_c\right]\Delta\bar{V} = 0 \quad (2)$$

where Y is the passive-linear admittance that relates the voltage and currents and Y_c is the admittance conversion matrix that relates the current and voltage increments as: $\Delta\bar{I} = Y_c\Delta\bar{V}$. An analogous formulation in terms of impedance is also possible. Because the system (2) is homogeneous, its determinant is zero:

$$\det\left[diag\left\{Y(\omega - j\sigma + k\omega_o)\right\} + Y_c\right] = \det\left[\omega - j\sigma\right] = 0 \quad (3)$$

and corresponds to the system characteristic determinant. The roots of this determinant provide the generalized eigenvalues or poles associated to the steady-state solution [1]. As written, equation (3) (involving all the circuit reactive elements) should be solved in terms of a complex frequency, which is only feasible for very simple circuits. Instead, Prof. Rizzoli proposed the application of the Nyquist criterion [1]-[2], [9] to the determinant in (3). This criterion provides the difference between the number of zeroes and poles of a complex function (the determinant in this case) located in the RHS, that is, $N = Z - P$. We are interested in Z, which agrees the number of generalized eigenvalues located in the RHS. The crucial advantage of the determinant (3) is that one can be sure that $P = 0$. This is because neither the conversion matrix Z_c nor the passive network impedance can exhibit any RHS poles. Thus, one will obtain Z from the number of clockwise encirclements about the origin of the function:

$$\det\left[diag\left\{Y(\omega + k\omega_o)\right\} + Y_c\right] \quad (4)$$

The solution may be stable for a given set of circuit-element values or parameters. However, when varying continuously one of these parameters, ρ, one may obtain a qualitative change of the stability properties, this corresponding to a bifurcation [8]. In [1], [3] Prof. Rizzoli presented a formal derivation of the mathematical conditions fulfilled by the characteristic determinant at the main types of local bifurcations from DC and periodic regime. In a local bifurcation either a real pole or a pair of complex conjugate poles crosses the imaginary axis, which implies the fulfilment, at the bifurcation parameter value ρ_o, of the following condition:

$$\det\left[\omega, \rho_o\right] = 0 \quad (5)$$

From a periodic regime at ω_o, there are three types of local bifurcations [1]-[3], [5]-[7], [14]: D-type, for $\omega = 0$, flip-type for $\omega = \omega_o/2$ and Hopf-type for $\omega = \alpha\omega_o$, where α is a non-rational number. In the first two cases the determinant is real, and the only unknown will be ρ_o. In the third case the determinant is complex, and the two unknowns will be ω and ρ_o. Though calculated in a different manner [15], Fig. 1 presents an example of bifurcation loci of a test-bench amplifier at 1.5 GHz, traced in the plane defined by the gate-bias voltage and input power. The amplifier is unstable in the shadowed regions because of two different phenomena: Hopf bifurcations, leading to an undesired autonomous quasi-periodic regime, and flip bifurcations, leading to a frequency-division by two.

The HB analysis of oscillatory solutions is more involved than that of forced ones; this is because there are no input sources at the oscillation frequency. Thus, the HB system admits a solution with zero value at the oscillation frequency and its multiples, to which HB will converge by default (e.g., the DC solution of a free-running oscillator). Moreover, the oscillation frequency depends on the circuit elements. In [16]-[17], a mixed HB formulation was presented, which included the oscillation frequency as an unknown. This was done by replacing the imaginary part of a state variable with the oscillation frequency. In fact, this imaginary part can be arbitrarily set to zero due to the oscillation invariance versus phase shifts (since there are no independent sources at the oscillation frequency). Because of the coexistence of solutions, convergence to the oscillatory one will require a proper initialization procedure [16].

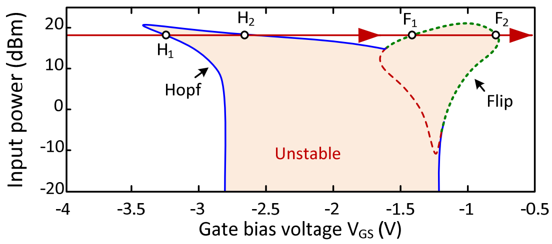

Fig. 1. Bifurcation loci of a test-bench power amplifier.

The work [18] proposed a formulation able to deal with free-running and injection locked oscillators in a systematic manner. Because the HB problem is the lack of sources at the oscillation frequency, a probe is introduced in the HB formulation. This is an artificial generator at the oscillation frequency that must fulfill a non-perturbation condition of the steady state [18]. This condition is added to the HB system and solved jointly with this system through a Newton-Raphson algorithm. Note that the probe introduces two extra unknowns: its amplitude and frequency (in a free-running solution) or its amplitude and phase (in an injection-locked solution). The work [18] continued the investigation of bifurcation phenomena in injection-locked oscillators [4] and frequency dividers; it provided an efficient method to trace the bifurcation loci since the probes helped remove some singularities of the circuit. In [19], the probe non-perturbation equation was solved in a two-tier fashion, that is, applying a Newton-Raphson algorithm to the probe equation (outer tier) in terms of its own variables (amplitude and frequency/phase), and considering the HB system as an inner tier. The bifurcation conditions were formulated in a different way, making use of the properties fulfilled by the probe at each kind of bifurcation [19]. This two-tier analysis paved the way to introduce probes (in the form auxiliary generators [20]) in commercial simulators, where the non-perturbation condition is achieved with the software optimization tools. Even these days, the analysis of oscillatory solutions suffers from limitations; for instance, in the case of a system with several subcircuits in an oscillatory state [20], or in the presence of coexistent oscillation modes.

In commercial software, there is also a lack of stability analysis tools, and numerous efforts have been devoted to the derivation of stability-analysis methods compatible with commercial HB. The work [21] proposed the use of a normalized determinant that satisfies $P = 0$ and is obtained by calculating a sequence of open-loop transfer functions from the device intrinsic terminals. In [22], this procedure was extended to large-signal regimes. A fully different approach is the one proposed in [23], which relies on the fact that all the closed-loop transfer functions that can be defined in a linear system share the same denominator and, thus, will exhibit the same poles, which should agree with the roots of the characteristic determinant in (3). The poles are calculated [23] by obtaining a closed-loop transfer function in commercial software through the introduction, for instance, of a small-signal current source, and applying pole-zero identification to this function. This way the possible coexistence of RHS zeroes and poles is no longer a problem since both the poles and zeroes are calculated and, thus, distinguished in the identification procedure. This method is very reliable and has become widely used. Nevertheless, in the case of complex topologies there can be problems of observability. This is because, unlike the poles, the zeroes depend on the selected transfer function and may lead to a cancellation or quasi-cancellation of RHS poles. Recently, a method for the stability analysis of circuits with complex topologies [24] has been proposed, which combines the calculation of the characteristic determinant (as in [1]), with pole-zero identification [23]. The characteristic determinant with $P = 0$ is obtained by splitting the topology into smaller blocks that are open-circuit (short-circuit) stable, and obtaining the total impedance (admittance) matrix at the ports defined in the partition. Pole-zero identification [23] is used to verify the stability of the smaller-size blocks. Fig. 2 presents the analysis of a system of three power amplifiers (considered blocks) at 800 MHz under output-coupling effects. Because $P = 0$, one can safely apply the Nyquist criterion [1] to the determinant of the impedance matrix that accounts for the whole system.

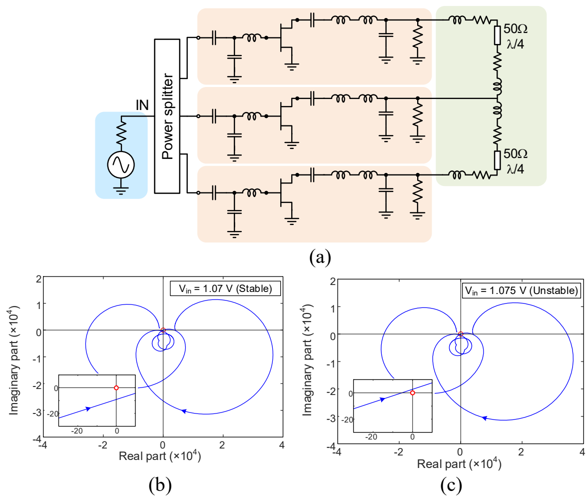

Fig. 2. Test-bench of three power amplifiers under output coupling effects. The Nyquist criterion is applied to the characteristic determinant for two input-amplitude values. (a) Stable. (b) Unstable.

III. PHASE-NOISE ANALYSIS

As shown in [10], [25], a noisy oscillator can be described in the frequency domain by introducing noise sources $\bar{J}_k(\omega)$ about the components $k\omega_o$ in the perturbed HB system:

$$\frac{\partial \bar{H}_k(k\omega_o+\omega)}{\partial \bar{X}}\Delta\bar{X} = \left[A(k\omega_o+\omega)\frac{\partial \bar{V}_k(\bar{X})}{\partial \bar{X}} + B(k\omega_o+\omega)\frac{\partial \bar{I}_k(\bar{X})}{\partial \bar{X}}\right]\Delta\bar{X} = \bar{J}_{B,k}(\omega) \quad (6)$$

where $\bar{X}$ is the vector of state variables. System (6) accounts for the frequency conversion occurring in the nonlinear devices [10]. However, and due to the invariance of the oscillatory solution with respect to phase shifts, the HB Jacobian matrix is singular at the free-running oscillation ($\omega = 0$). As a result, system (6) may become ill-conditioned for low ω. This ill-conditioning is less severe for a higher accuracy in the calculation of the steady-state oscillation [26]. Prof. Rizzoli combined (6) with a different type of formulation: the carrier-modulation approach [10], in which the imaginary part of one of the state variables is replaced with ω_o, which should become modulated by the noise perturbations.:

$$\left[\frac{\partial \bar{H}_k(k\omega_o)}{\partial \bar{X}'}, \frac{\partial \bar{H}_k(k\omega_o)}{\partial \omega_o}\right]\begin{bmatrix}\Delta\bar{X}'\\ \Delta\omega\end{bmatrix} = \bar{J}_{H,k}(\omega) \quad (7)$$

where the imaginary part of one of the state variables is suppressed in $\bar{X}'$. In (7), the noise sources are modulated sinusoids located at the carrier harmonics, with random pseudo-sinusoidal phase and amplitude modulation laws of frequency ω [10]. For many years, the combination of the two approaches circumvented the ill-conditioning problems of (6). However, as shown in [27], the carrier modulation approach has an extraordinary conceptual value, as it can be used to connect time- and frequency-domain methods for phase-noise analysis.

The work [28] presented a time-domain analysis of the noisy oscillator based on a decomposition of its perturbed solution as $\bar{x}_p[t+\theta(t)] = \bar{x}[t+\theta(t)] + \Delta\bar{x}(t+\theta(t))$. This distinguishes the perturbations in the direction of the oscillation limit cycle, or timing noise $\theta(t)$, and the orbital-deviation perturbations $\Delta\bar{x}(t)$ [29]. Due to lack of a phase reference, $\theta(t)$ can grow unboundedly, unlike $\Delta\bar{x}(t)$, which will necessary remain small due to the restoring mechanism of the stable periodic oscillation. The works [28]-[29] derive a single scalar nonlinear differential equation in $\theta(t)$, fully decoupled from $\Delta\bar{x}(t)$, to which a detailed stochastic characterization is applied. Then, the oscillator power spectral density (PSD) due to the timing noise is obtained from $\bar{x}[t+\theta(t)]$, which remains bounded despite the unbounded growth of $\theta(t)$. The work [27] presented a translation of the procedures in [28]-[29] to the frequency-domain. In fact, the carrier modulation and the timing noise are related as $\Delta\omega(t) = \omega_o\dot{\theta}(t)$ and at a sufficient frequency distance from the carrier, the phase noise spectrum obtained in [28]-[29] analytically agrees with the one provided by the carrier modulation approach [10]. As a result, it is possible to use this approach to determine the variance $\sigma_\theta^2(t)$ of the timing noise. This depends on the noise sources, white and colored, and can be expressed as $\sigma_\theta^2(t) = c_w t + \sum_{i=1}^{M}\sigma_{\gamma,i}^2(t)$ [29], where c_w is due to the combined effect of all the white noise sources and $\sigma_{\gamma,i}$ is associated to each of the M colored noise sources. As shown in [30], both c_w and the constant coefficients in the expressions of $\sigma_{\gamma,i}$ can be extracted from the application of the carrier modulation approach under different noise excitations. For instance, the coefficient c_w can be extracted by considering only the set of white noise sources. When applying the carrier-modulation approach, the phase-noise noise spectrum will have the form: $S_f = c_w f_o^2 / f^2$. Then, the PSD of $\overline{x}[t+\theta(t)]$, with a Lorentzian shape about each $k\omega_o$ [28]-[29], will be:

$$S(f) = \sum_{k=-NH}^{NH} X_k X_k^* \frac{f_o^2 k^2 c_w}{\pi^2 f_o^4 k^4 c_w^2 + (f + kf_o)^2} \qquad (8)$$

Fig. 3 shows an example of the procedure, applied to an oscillator at 5 GHz in the presence of white-noise sources only.

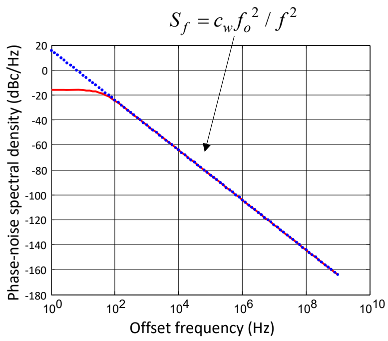

Fig. 3. Calculation of the PSD of $\overline{x}[t+\theta(t)]$.

IV. Conclusion

The works by Prof. Rizzoli have provided the microwave community with rigorous and practical methods for nonlinear circuit analysis. This document focuses on some relevant aspects of his work on stability and phase noise, in which he made outstanding conceptual contributions that paved the way for other more recent research.

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References

[1] V. Rizzoli, A. Lipparini, “General stability analysis of periodic steady-state regimes in nonlinear microwave circuits,” *IEEE Trans. Microw. Theory Tech*., vol. 33, no. 1, pp. 30 – 37, Jan. 1985.

[2] V. Rizzoli, A. Neri and D. Masotti, "Local stability analysis of microwave oscillators based on Nyquist's theorem," *IEEE Microwave and Guided Wave Letters*, vol. 7, no. 10, pp. 341-343, Oct. 1997.

[3] V. Rizzoli and A. Neri, "Global Stability Analysis of Microwave Circuits by a Frequency-Domain Approach," 1987 *IEEE MTT-S International Microwave Symposium Digest*, 1987, pp. 689-692.

[4] V. Rizzoli, A. Neri and D. Masotti, "The application of harmonic-balance methodology to the analysis of injection locking," 1992 *IEEE MTT-S Microwave Symposium Digest*, 1992, pp. 1591-1594, vol. 3.

[5] V. Rizzoli, A. Neri and A. Costanzo, "Microwave Oscillator Design by State-Of-The-Art Nonlinear CAD Techniques," 1988 18th *European Microwave Conference*, 1988, pp. 231-236.

[6] V. Rizzoli, A. Neri and G. Righi, "Analysis of Spurious Tones in Microwave Oscillators via the Hopf Bifurcation Concept," 1994 24th *European Microwave Conference*, 1994, pp. 836-841.

[7] V. Rizzoli, A. Neri and A. Costanzo, "Microwave Oscillator Design by State-Of-The-Art Nonlinear CAD Techniques," 1988 *18th European Microwave Conference*, 1988, pp. 231-23

[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, 1990.

[9] V. Rizzoli and A. Neri, "State of the art and present trends in nonlinear microwave CAD techniques," *IEEE Trans. Microwave Theory Tech*., vol. 36, no. 2, pp. 343–356, Feb., 1988.

[10] V. Rizzoli, F. Mastri, and D. Masotti, “General noise analysis of nonlinear microwave circuits by the piecewise harmonic balance technique,” *IEEE Trans. Microw. Theory Tech*., vol. 42, no. 5, pp. 807–819, May 1994.

[11] K. Ogata, *Modern Control Engineering*. Englewood Cliffs, NJ,: Prentice-Hall, 1980.

[12] C. Camacho-Peñalosa “NumerlcaI steady state analysis of non-linear microwave circuits with periodic exitation” *IEEE Trans. Microw. Theory Tech*. vol. 31, 1983. pp.724-730.

[13] M. Gayral, E. Ngoya, R. Quere, J. Rousset and J. Obregon, "The Spectral Balance: A General Method for Analysis of Nonlinear Microwave Circuits Driven by Non-Harmonically Related Generators," 1987 *IEEE MTT-S International Microwave Symposium Digest*, pp. 119-121.

[14] D. Masotti, F. Mastri, V. Rizzoli and A. Costanzo, "Power-handling capabilities optimization of MEMS-reconfigurable antennas," 2011 *Workshop on Integrated Nonlinear Microwave and Millimetre-Wave Circuits*, 2011, pp. 1-4

[15] A. Suárez, "Check the Stability: Stability Analysis Methods for Microwave Circuits," *IEEE Microwave Magazine*, vol. 16, no. 5, pp. 69-90, June 201.

[16] V. Rizzoli and A. Neri, "A Fast Newton Algorithm for the Analysis and Design of Microwave Oscillators and VCOs," 1989 19th *European Microwave Conference*, 1989, pp. 386-391.

[17] V. Rizzoli and A. Neri, "Harmonic-balance analysis of multitone autonomous nonlinear microwave circuits," 1991 *IEEE MTT-S International Microwave Symposium Digest*, 1991, pp. 107-110 vol. 1.

[18] R. Quéré, E. Ngoya, M. Camiade, A. Suárez, M. Hessane and J. Obregon, "Large signal design of broadband monolithic microwave frequency dividers and phase-locked oscillators," *IEEE Trans. Microw. Theory Tech*., vol. 41, no. 11, pp. 1928–1938, Nov., 1993.

[19] A. Suárez, J. Morales, R. Quéré, "Synchronization analysis of autonomous microwave circuits using new global-stability analysis tools," *IEEE Trans. Microw. Theory Tech*., vol. 46, no. 5, pp. 494–504, May 1998.

[20] F. Ramirez, M. Ponton, S. Sancho and A. Suárez, "Stability Analysis of Oscillation Modes in Quadruple-Push and Rucker's Oscillators," *IEEE Trans. Microw. Theory Tech*., vol. 56, no. 11, pp. 2648-2661, Nov. 2008.

[21] W. Struble, A. Platzker, “A rigorous yet simple method for determining stability of linear N-port networks [and MMIC application],” *15th GaAs IC Symposium*, pp. 251 – 254, 1993.

[22] S. Mons, J.-C. Nallatamby, R. Queré, P. Savary, and J. Obregon, "A unified approach for the linear and nonlinear stability analysis of microwave circuits using commercially available tools," *IEEE Trans. Microw. Theory Tech*., vol. 47, no. 12, pp. 2403–2409, Dec. 1999.

[23] N. Ayllon, J. M. Collantes, A. Anakabe, I. Lizarraga, S. Soubercaze-Pun, S. Forestier, “Systematic approach to the stabilization of multitransistor circuits”, *IEEE Trans. Microw. Theory Tech*., vol. 59, no. 8, pp. 2073–2082, Aug. 2011.

[24] A. Suárez and F. Ramírez, "Two-Level Stability Analysis of Complex Circuits," *IEEE Trans. Microw. Theory Tech*., vol. 69, no. 1, pp. 132-146, Jan. 2021.

[25] J. M. Paillot, J. C. Nallatamby, M. Hessane, R. Quere, M. Prigent and J. Rousset, "A general program for steady state, stability, and FM noise analysis of microwave oscillators," 1990 *IEEE MTT-S International Microwave Symposium Digest*, pp. 1287-1290 vol.3.

[26] P. Bolcato, J. C. Nallatamby, R. Larcheveque, M. Prigent and J. Obregon, "A unified approach of PM noise calculation in large RF multitone autonomous circuits," 2000 *IEEE IMS*, 2000, pp. 417-420 vol.1

[27] A. Suárez, S. Sancho, S. Ver Hoeye and J. Portilla, "Analytical comparison between time- and frequency-domain techniques for phase-noise analysis," *IEEE Trans. Microw. Theory Tech*., vol. 50, no. 10, pp. 2353-2361, Oct. 2002.

[28] F. X. Kaertner, “Analysis of white and f-α noise in oscillators,” *Int.J. Circuit Theory Appl.*, vol. 18, pp. 485–519, 1990.

[29] A. Demir, A. Mehrotra, and J. Roychowdhury, “Phase noise in oscillators: A unifying theory and numerical methods for characterization,” *IEEE Trans. Circuits Syst. I*, vol. 47, no. 5, pp.655–674, May 2000.

[30] S. Sancho, A. Suárez, J. Dominguez, and F. Ramírez, "Analysis of Near-Carrier Phase-Noise Spectrum in Free-Running Oscillators in the Presence of White and Colored Noise Sources," *IEEE Trans. Microw. Theory Tech*., vol. 58, no. 3, pp. 587-601, March 2010.