Strategies Used by Students with Autism when Solving Multiplicative Problems: An Exploratory Study

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Strategies used by students with autism when solving multiplicative problems: an exploratory study

This work studies the strategies used by ten students diagnosed with autism when solving multiplication and division problems, since these operations are rarely studied in students with this condition. We conducted an exploratory study with 10 students diagnosed with autism in order to explore and describe the strategies used in solving equal group problems. We also describe in detail the case of a student whom we deem to be representative due to the reasoning he employed. The informal strategies that they used are described, as well as the difficulties observed in the various problems, depending on the operation required to solve them. The strategies used include direct modeling with counting and others that relied on incorrect additive relationships, with strategies based on multiplication and division operations being scarce. Difficulties were observed in several problems, with measurement division being particularly challenging for the study participants. The detailed description of the strategies used by the students revealed the meanings that they associate with the operations they are executing, and brought to light potential difficulties, which can help teachers plan their instruction. This research supplements other studies focusing on mathematical problem solving with autistic students.

Keywords: Autism; division; mathematical problem solving; multiplication; strategies

Introduction

Of all the areas of mathematics, problem solving stands out as a particularly important field through which students learn about operations (Verschaffel and De Corte, 1997). Some research works of a cognitive nature have examined the skills that are involved in the problem-solving process (e.g., Daroczy *et al.*, 2015). Some of these cognitive abilities are frequently impaired in autistic people (Happé *et al.*, 2006), which affects their understanding of the problems and their choice and execution of suitable strategies (Bae *et al.*, 2015). Consequently, understanding their learning processes in this area is essential in order to offer effective instruction that is tailored to their needs (Wei *et al.*, 2014), the goal being to achieve improved performance and, therefore, greater autonomy and quality of life in adulthood.

Since mathematical content is particularly difficult for autistic students (Bullen *et al.*, 2020), the amount of research involving this group has grown in recent years (Gevarter *et al.*, 2016). Most of it focuses on evaluating the effectiveness of instructional methodologies, thus there is little research that focuses on solving problems with a multiplicative structure (Polo-Blanco *et al.*, 2022), and particularly on studying the strategies used by students with autism when solving these problems (Polo-Blanco et al, 2019; 2021-b).

Taking this into account, in this paper we propose studying in detail the problem-solving strategies employed by students diagnosed with autism so as to understand their difficulties, as well as to provide information to help plan teaching proposals adapted to their needs. Specifically, we focus on multiplication and division problems, as these operations are rarely studied in autistic students.

Mathematical word problem solving. A focus on the multiplicative structure

The literature agrees in classifying problems with a multiplicative structure (that is, those that require a multiplication or division operation to be solved) into three large groups (Nesher, 1992): equal groups, multiplicative comparison and cartesian product. Some researchers have analysed the various levels of difficulty that multiplicative structure problems entail for typically developing (TD) children in early grades. Nesher (1992) concluded that the easiest problems to solve were equal group problems, followed by comparison problems. Cartesian product problems are considered the most difficult for primary school children (6-12 years old) (Mulligan and Mitchellmore, 1997; Nesher 1992; Ivars and Fernández, 2016).

This article focuses on equal group problems (e.g. *I have 7 marbles in each bag. If I have 4 bags, how many marbles do I have?*), since they are the simplest with a multiplicative structure (Nesher, 1992) and because they require multiplication and division arithmetic operations, which are rarely studied in autistic students (Polo-Blanco *et al.*, 2019). There are three types of equal group problems depending on the location of the unknown (Nesher, 1992): (1) multiplication: the unknown quantity is the product; (2) partitive division: the unknown quantity is the number of elements in each set; (3) measurement division: the unknown quantity is the number of sets.

Within the category of equal group problems, Hart (1981) concluded, in her study with students aged 11 to 16, that it was more difficult to identify a multiplication problem than a division problem. Bell *et al.* (1984), in their study with 12-13-year-old students, noted that measurement division problems were more difficult than partitive division problems. These findings match those of Ivars and Fernández (2016) starting from the 4th grade (9-10-year-old students). However, students between the ages of 6 and 9 were more successful solving multiplication problems than division problems and obtained a higher percentage in measurement division problems than in partitive division problems.

Problem-solving strategies

The development of mathematical competence begins at an early age as an informal cognitive activity, and gradually evolves as the individual's cognitive development advances to more complex levels. Accordingly, research on elementary arithmetic operations indicates that children build a wide range of strategies by themselves before receiving formal education, which serves as a basis for the subsequent development of formal mathematics (Ginsburg and Baroody, 2007). The way informal strategies emerge and develop allows us to determine how children organize and process information, as well as to understand how subsequent understanding of arithmetic operations develops (Ginsburg and Baroody, 2007).

Research on the strategies for solving mathematical problems, and specifically those with a multiplicative structure, has traditionally focused on TD students. For example, Mulligan and Mitchelmore (1997) carried out a two-year longitudinal study with a random sample of 70 students in 2nd and 3rd grades. These authors found that the repeated-addition model was the most frequent correctly used model in almost every occasion for all semantic structures except comparison. The results also showed that the vast majority of incorrect responses resulted from superficial strategies, adding the two given numbers, or from incorrect models of the problem situation. Moreover, large-number problems seemed to make demands on information retrieval or processing capacity that forced many students to revert to a more primitive and less demanding model (Mulligan and Mitchelmore, 1997).

In the work by Carpenter *et al.* (1993), the authors concluded that the less difficult problems in their sample of 70 kindergarten students involved multiplication, followed by measurement division and partitive division, with direct modeling being the most frequent strategy. The study by Ivars and Fernández (2016) aimed at characterizing the development of the strategies used by Spanish students between ages 6 and 12 to solve multiplicative structure problems. The results showed that students aged 6 to 8 mostly used modeling and counting strategies, although starting with the third grade, the most used strategy relied on algorithms. However, that use did not entail a decrease in incorrect strategies but was associated with the appearance of an incorrect strategy: the use of the inverse algorithm.

Mathematical problem solving in students with autism

Problem solving pose a challenge for many students, particularly for those with cognitive difficulties, since it requires not only mathematical skills but also reading comprehension and reasoning skills, and the ability to transform words and numbers into the appropriate operation (Daroczy *et al.*, 2015). In this sense, there are certain cognitive traits of the autistic population that directly interfere with problem solving, such as impaired executive functions or reading comprehension difficulties (Happé *et al.*, 2006).

Autism spectrum disorder (ASD) is a neurobiological developmental disorder which manifest itself during the first years of life and lasts throughout the entire life cycle (APA, 2013). The main symptoms are persistent shortcomings in communication and in social interactions and restrictive and repetitive patterns of behaviour, interests, or activities. Over the last few years, the number of people diagnosed with autism has increased. The manual *Diagnostic and Statistical Manual of Mental Disorders-5* (DSM-5) estimates that around 1% of the population has ASD, the diagnosis being more frequent in men (APA, 2013).

Because some of the specific characteristic of this disorder have a direct impact on learning, there has recently been an increase in looking into the academic performance of autistic students. For instance, Goñi-Cervera *et al.* (2020) explored the formal and informal mathematical knowledge of eight autistic students and showed that the participants presented a lower mathematical knowledge than the one corresponding to their chronological age, observing more deficiencies in formal than in informal mathematical knowledge. Wei *et al.* (2014), based on a longitudinal study on the mathematical and reading performance of autistic students between 6 and 9 years of age, noted that the participants, on average, performed worse in mathematical problem solving and calculation skills. Bae *et al.* (2015) confirmed this result, noting that the factors involved in this observation had to do with the difficulty in identifying the arithmetic operations needed to solve the problem. Thus, students with autism tended to use less advanced strategies for solving mathematical problems and yielded a greater number of incorrect solutions.

As for the research that has been carried out on problem solving and mathematical learning in autistic students, most of its focus is on evaluating the effectiveness of different teaching methods. For instance, Cox and Root (2020) have employed a modified schema-based approach (MSBI) with autistic students to improve verbal mathematical problem-solving skills. Gevarter *et al.* (2016) provide a review of mathematics interventions in autistic students. Other authors such as Bae *et al.* (2015) compare the factors involving problem solving in autistic and TD students. Most of the existing research on students with autism has focused mainly on the additive structure (Rockwell *et al.*, 2011; Polo-Blanco *et al.*, 2021-b), with research on the multiplicative structure being much less common (Polo-Blanco *et al.*, 2022).

Concerning problem-solving strategies by students with autism, the literature is very scarce (Bae *et al.*, 2015; Polo-Blanco *et al.*, 2019; 2021-a). In their work, Polo-Blanco *et al.* (2019) describe the strategies and representations of an autistic student when solving multiplicative structure problems. These authors conclude that before receiving formal instruction on division, the measurement division problems were easier to solve for the student than partitive division problems. Subsequently, in the case of partitive division, they observe a clear preference for the one-to-many correspondence strategy in the problems with support material, whereas the student mainly resorted to the sharing one-by-one strategy when he did not have the material.

Polo-Blanco *et al.* (2021-a) evaluate how the context (of special interest, familiar and non-familiar) of the multiplication and division arithmetic verbal problems influence the solution process in a case study with an 11-year-old autistic student with intellectual disability. The results showed that when the problem was contextualized in a special interest theme, the student seemed to be more engaged in its resolution, though this did not yield an effective improvement with respect to the familiar contexts.

In light of the above, more studies are needed that analyse problem-solving strategies in students with autism (Bae *et al.*, 2015), especially those which focus on the multiplicative structure. Given how important studying these strategies is to provide adequate interventions, the goal of this research is to analyse aforementioned strategies based on the operation in the problem, as well as the difficulties observed. Specifically, the following research questions are posed:

(1) What strategies do students with autism use when solving equal group problems?
(2) What type of strategies and success rates are obtained for the different meanings of the operation in the problems (multiplication, partitive division, measurement division)?

Methodology

This research is exploratory and descriptive. We conducted a case study with 10 students diagnosed with autism in order to explore and describe the strategies used in solving equal group problems. The case study with a reduced sample allowed us to thoroughly analyse the strategies and the problem-solving process in a way that would have been more complex with a larger sample. Thus, the case study was used as an exploratory methodology to collect detailed information on the participants' responses

(Yin, 2017). We conducted a predominantly qualitative mixed data analysis. A critical case study design was also used describing in detail the case of a student whom we deem to be representative in the sense that he used strategies that other participants also used. In addition, student S1 was chosen since, among all the participants, he was the one who gave us the most detailed answers, as he described the procedure that he followed to achieve the results in more detail than the other participants. The data were collected from the audio-visual recordings of the individual sessions with each of the participants, and by reviewing the booklets used by the students.

Participants

The subjects participating in this work were involved in a larger project of which the study is a part. The participants had been recruited through associations for people with disabilities and/or autism, school guidance teams and hospital outpatient clinics. For the present study, ten students between 8 and 13 years old were selected from ten different schools, of which eight of them were mainstream and two were special education centres. All participants met the following inclusion criteria: (1) to be diagnosed with autism according to DSM-5 (APA, 2013), (2) to exhibit a lag in mathematical competence of at least 6 years as per TEMA-3 (Ginsburg and Baroody, 2007), to ensure the pre-requisite knowledge of numerical addition and subtraction facts, (3) to correctly solve at least 80% of the additive informal verbal problem items in TEMA-3 and (4) to present difficulties when solving mathematical word problems involving multiplication and division according to the criteria of their teachers. All participants benefited from special education services at their school.

The Full Scale Intelligence Quotient (FSIQ) and the Verbal Comprehension Index (VCI) were measured with WISC-V. This last index reflects the ability to verbalize meaningful concepts, think about verbal information and express herself using words.

Table I shows the characteristics of the students taking part in this study.

Table I. Data on the study participants.

Student	Age	Equivalent Mathematical Age*	FSIQ**	VCI**
S1	8:4	7:1	82	76
S2	8:9	8:2	79	73
S3	9:3	6:4	88	111
S4	9:6	6:3	65	62
S5	10:9	7:5	80	93
S6	10:9	6:10	65	65
S7	10:10	7:9	75	45
S8	11:1	>9	110	130
S9	11:4	8:5	91	89
S10	13:4	7:3	54	55

Note: *: Test of Early Mathematics Ability TEMA-3 (Gingsburg and Baroody, 2007); **: Wechsler Intelligence Scale for Children (WISC-V); FSIQ: Full Scale Intelligence Quotient; VCI: Verbal Comprehension Index

Information gathering tool

Based on previous works (Mulligan and Mitchelmore, 1997), a questionnaire with six equal group problems was designed, separated into: (1) multiplication problems, (2) partitive division problems and (3) measurement division problems. Within each category, the problems were classified into those with small numbers P(S) and large numbers P(L).

The problems were solved by each participant in a classroom with no distractions. The interviewer first gave the student a stapled booklet with a single small-number problem on each page (see problem statements in Tables II, III and IV), written concisely and with ample blank space to represent and solve it. Then, the student was asked to read the problem aloud, and if the interviewer observed any difficulty, they read it again together. The student was asked to find the solution and was told that he or she could write, use manipulatives (interlocking blocks) or answer orally. Once the small-number problems were completed, they proceeded to solve the large-number ones. In keeping with similar studies (Mulligan and Mitchelmore, 1997) and so as not to tire the participants, the student had to solve the large-number problem only when the corresponding small-number problem was answered correctly. The entire process was videotaped and later transcribed for analysis.

Analysis categories

For this study, we adhered to the following system for classifying the strategies used to solve multiplicative structure problems (Ivars and Fernández, 2016; Mulligan and Mitchelmore, 1997): incorrect strategies (level 0), direct modeling with counting (level 1), counting (level 2) and operation strategies (level 3).

The incorrect strategies (level 0) considered were: inverse algorithm (divide instead of multiply or vice versa); inappropriate additive relationships (addition or subtraction instead of multiplication or division); given number (one of the numbers in the problem is given as the answer); random number (a random number, close to the answer or not, is given as a quick answer).

Direct modeling with counting strategies (level 1) are those in which concrete manipulatives or drawings are used to model the problem situation, and objects are counted with no obvious reference to the multiplicative structure. With regard to multiplication, a general distinction is made between: one-by-one representation (the student forms groups and places the elements one by one in each group before finally counting everything) and representation by multiples (the student places, for example, four blocks on one side and four blocks on the other side and counts the total: "Eight"). As for division, a distinction is made between sharing one-by-one (the manipulatives are distributed one by one in containers and at the end, the number of objects in a container is counted); sharing by multiples (the manipulatives are distributed by twos, by threes, etc.); grouping (the student makes groups the same size as the divisor and counts the number of groups); and trial and error (the student makes groups and adjusts them later as needed).

Counting strategies (level 2) are those in which the same actions are performed as in the previous level, but without the use of manipulatives. Some of the most frequent types of counting include: rhythmic counting backward or forward (the counting follows the structure of the problem while the number of groups is counted simultaneously); skip counting backward or forward (the student counts in multiples); repeated adding or subtracting (the student repeatedly adds or subtracts the same number).

Finally, operation strategies (level 3) are those that use multiplication or division as the operation. They can take the form of internalized known facts, such as the use of multiplication tables, for example.

Interobserver reliability

Interobserver reliability data were collected on all the students and all the problem types. A mathematics education teacher, who was blind to the study's hypotheses, recoded 38% of the students' solutions. Interobserver agreement was calculated for each student by dividing the number of agreements by the number of agreements plus disagreements and multiplying by 100. Interobserver reliability agreement for each student across strategy was 94% and across success was 100%.

Results

The results for the students' performance in the equal group problems are presented below.

Problem-solving strategies used by the participants

Multiplication Problems

Table II shows the results of the multiplication problems solved by each of the students, as well as the type of strategy used

Table II. Participants' solutions to problems P1 (S) and P1 (L).

P1: *There are n tables in the class, and there are m children sitting at each table. How many children are in the class in total?*						
	P1(S): *n*=2, *m*=4			P1(L): *n*=4, *m*=7		
Student	Answer	Strategy Level	Type	Answer	Strategy Level	Type
S1	6*	L_0	AR	–	–	–
S2	6*	L_0	AR	–	–	–
S3	8	L_3	M	28	L_2	RA
S4	'A sum'*	L_0	AR	–	–	–
S5	8	L_1	RM	22*	L_1	RM
S6	6*	L_0	AR	–	–	–
S7	6*	L_0	AR	–	–	–
S8	8	L_3	M	28	L_3	M
S9	8	L_2	RA	28	L_3	M
S10	8	L_1	RM	28	L_1	RM

Note: *: Incorrect response; L_0: Incorrect strategy; L_1: Direct modeling with counting; L_2: Counting; L_3: Operation; AR: Inappropriate additive relationships; M: Multiplication; RM: Representation by multiples; RA: Repeated adding or subtracting; – : not applied.

As Table II shows, five of the 10 participants correctly solved problem P1(S). All of the students who did not solve the problem incorrectly used an additive strategy, adding the given numbers instead of multiplying them. Of the five students who used this incorrect strategy, two of them (S1 and S2) relied on manipulatives to do the calculation. Student S7, after reading the problem, answered: “Six children”. He then drew an elliptical shape (see Figure 1, left) and drew six children around it as he counted them (“one, two…, six”). Student S6 also used the same incorrect additive strategy, which he reflected in writing in the form of a vertical algorithm (see Figure 1, right).

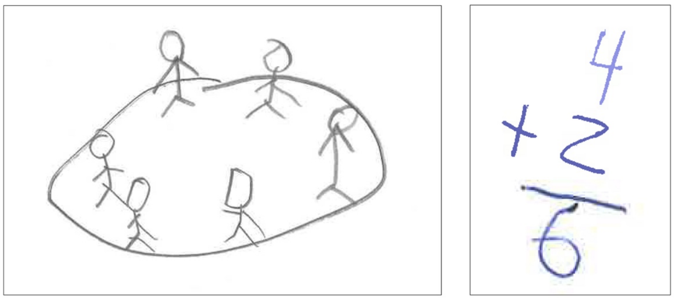

Figure 1. Incorrect strategy of additive relationships in the solution to P1 (S): with modeling (S5, left) and with addition algorithm (S6, right).

Two of the participants who successfully solved problem P1(S) resorted to a modeling strategy. S5 combined three blocks to represent a table and arranged four individual blocks around the three blocks to depict four chairs. He then placed red tokens on top of the individual blocks to represent four children around each table. He repeated the above process for the second table (see Figure 2, left), thus employing a modeling strategy using representation by multiples. Finally, he said: “Four children at each table. So, four children would make eight.” Student S10 relied on the same modeling strategy, using a detailed drawing of the situation described in the problem (see Figure 2, right).

Figure 2. Modeling strategies for solving problem P1(S): with manipulatives (S7, left) and detailed drawing (S10, right)

Student S9 used a counting strategy based on repeated addition to solve the problem, writing the addition “4+4=8” horizontally. Two students (S3 and S8) used an operations strategy based on multiplication facts. For example, S3 wrote down the operation “2x4=8” using the vertical algorithm, and verbally expressed: “Eight, twice four is eight.”

In the case of problem P1(L), of the five students who solved the problem, four found the right solution. The two students who had resorted to a modeling strategy in the previous problem repeated this strategy. Figure 3 shows how S5 modeled the situation using manipulatives (see Figure 3, left) and the detailed drawing of S10 (see Figure 3, right).

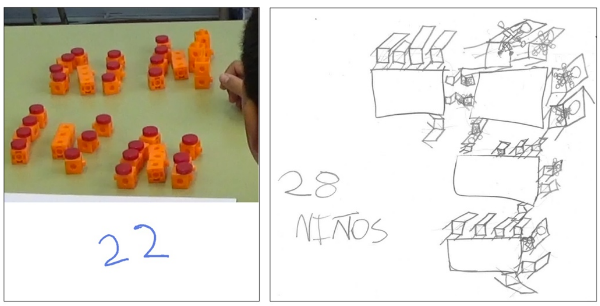

Figure 3. Modeling strategies for solving problem P1(L): with manipulatives (S7, left) and detailed drawing (S10, right).

Student S5 did a count after modeling the situation (see Figure 3, left) but made a counting mistake and answered “22.” In the case of S10, after finishing the drawing, he counted all the chairs until he obtained the correct result: “28.”

As for S3, he resorted to a counting strategy based on the repeated addition of the multiplier 7 (see Figure 4, left). In contrast, S9 went from using a counting strategy in P1(S) to one of multiplication facts in P1(L) (see Figure 4, right). Student S8 again resorted to multiplication facts.

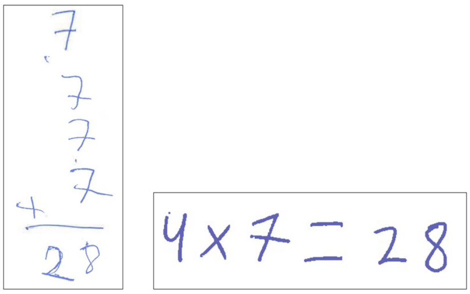

Figure 4. Counting strategy through repeated addition to solve problem P1(L) by S3 (left) and multiplication facts by S9 (right).

Partitive division

The results of the partitive division problems solved by each of the students are depicted in Table III, as well as the type of strategy used.

Table III. Participants’ solutions to problems P2(S) and P2 (L).

P2: *There are n children and m tables in the class. If the same number of children are sitting at each table, how many children are seated at each table?*

	P2(S): *n*=10, *m*=2			P2(L): *n*=28, *m*=4		
Student	Answer	Strategy Level	Type	Answer	Strategy Level	Type
S1	5	L_1	RM	‘4x28’*	L_0	IA
S2	5	L_1	RM	32*	L_0	AR
S3	5	L_3	M	4*	L_2	SC
S4	10*	L_0	GN	–	–	–

S5	21*	L_0	IA	–	–	–
S6	12*	L_0	AR	–	–	–
S7	12*	L_0	AR	–	–	–
S8	5	L_3	D	7	L_3	D
S9	5	L_3	D	7	L_3	D
S10	20*	L_0	IA	–	–	–

Note: *: Incorrect response; L_0: Incorrect strategy; L_1: Direct modeling with counting; L_3: Operation; RM: Representation by multiples; M: Multiplication as inverse of division; GN: Given number; AR: Inappropriate additive relationships; IA: Inverse Algorithm; D: Division; SC: Skip counting backward or forward; – : not applied.

As it can be observed in Table III, five of the 10 participants correctly solved the small-number problem P2(S). Students S7 and S6 used an incorrect strategy of additive relationships by adding the given numbers instead of multiplying them (see S6's solution in Figure 5, left). Two of the participants resorted to an incorrect inverse operation strategy by multiplying both numbers instead of dividing them. S5 modeled the situation with manipulatives, as in P1 (S and L), although he again made a counting mistake. S10 also used an inverse operation strategy, first by modeling with a drawing (see Figure 5, right), which he finally solved with a mental calculation (10x2=20).

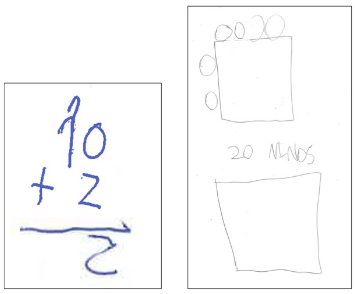

Figure 5. Incorrect strategies when solving problem P2(S): using additive relationships (S6, left) and inverse operation (S10, right).

Two (S1 and S2) of the five participants who successfully solved problem P2(S) resorted to a modeling strategy. The other three students who correctly solved the problem relied on a strategy of operation (see Figure 6).

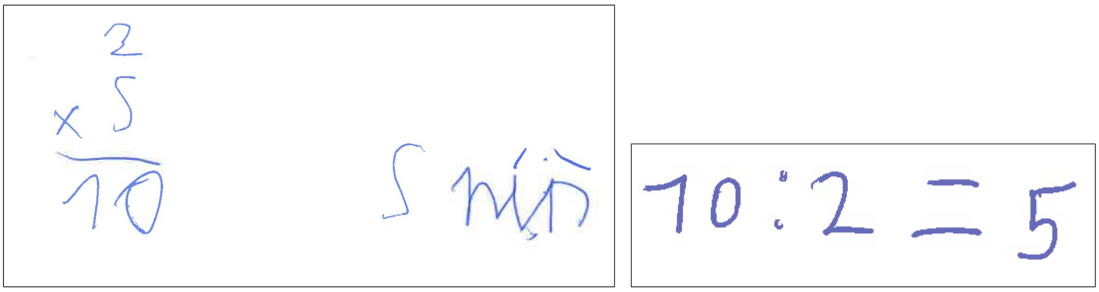

Figure 6. Correct solutions to problem P2(S) using a strategy of known multiplication facts (S3, left) and known division facts (S9, right).

In the case of the large-number problem P2(L), three of the five students who attempted it were unable to solve it correctly. For example, S3 used a counting strategy, first descending and then ascending, reasoning: “If there are 4 tables and 28 students, what I have to do is subtract by fours, which equals 0. Multiply it by… if we add four to… four times four, it equals 16? 16.” While he chose a suitable strategy (counting), he executed it incompletely by not subtracting the divisor from the dividend until reaching zero, yielding the answer “four.”

The only right answers to this problem were provided by S8 and S9, who again used a strategy of operation by known division facts, writing “28:4=7”.

Measurement division

The results of the measurement division problems are shown below.

Table IV. Participants’ solutions to problems P3(S) and P3(L).

	P3: *In class there are n toys to distribute equally among several children. If each child receives m toys, how many children are there in the class?*					
	P3(S): *n*=15, *m*=3			P3(L): *n*=24, *m=6*		
Student	Answer	Strategy Level	Type	Answer	Strategy Level	Type
S1	12*	L_0	AR	–	–	–

S2	18*	L_0	AR	–	–	–
S3	45*	L_0	IA	–	–	–
S4	‘one sum’*	L_0	AR	–	–	–
S5	18*	L_0	AR	–	–	–
S6	18*	L_0	AR	–	–	–
S7	18*	L_0	AR	–	–	–
S8	5	L_3	D	4	L_3	D
S9	5	L_3	D	144*	L_0	IA
S10	3*	L_0	GN	–	–	–

Note: *: Incorrect response; L_0: Incorrect strategy; L_3: Operation; AR: Inappropriate additive relationships; GN: Given number; IA: Inverse Algorithm; D: Division; – : not applied

Only two of the ten participants correctly solved small-number problem P3(S). Of the eight participants who did not correctly solve the problem, six used inappropriate additive relationships using the given numbers. For example, S6 automatically replied “18” and wrote the algorithm (see Figure 7, left). Student S3 employed another type of incorrect inverse algorithm strategy, multiplying rather than dividing the given numbers (Figure 7, center). Student S10 used a given number strategy, automatically repeating one of the numbers from the problem: “three children” (Figure 7, right).

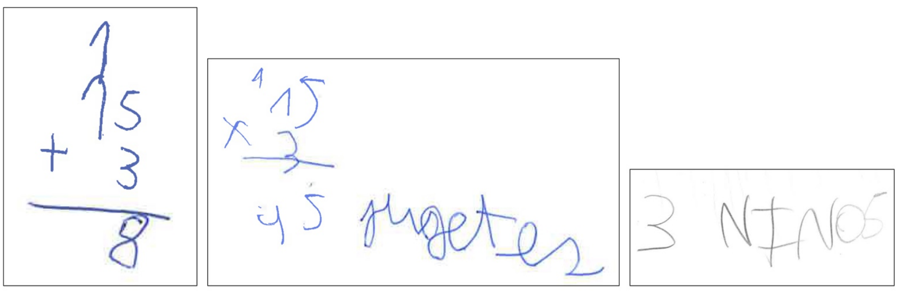

Figure 7: Incorrect strategies when solving problem P3(S): additive relationships (S6, left), inverse operation (S3, center), and given number (S10, right).

The only student who was able to answer the problem correctly used a strategy of operations (known division facts and a mental calculation), writing “15:3=5”.

In the case of large-number problem P3(L), only one of the two students (S8 and S9) who did the problem P3(S) solved it correctly. The student S9 was unsuccessful when solving the equivalent large-number problem as he employed an inverse algorithm strategy, multiplying the numbers in the problem instead of dividing them, writing “24x6=144”. Student S8 did successfully solve the problem by using the division algorithm, writing “24:6 = 4 children in the class.”

Student S1 Critical Case Study

The following subsection describes in detail the performance by the student S1 in terms of the strategies that he used and the difficulties he encountered. As Table II shows, student S1 did not correctly solve problem P1(S), which he attempted using an incorrect additive strategy. S1 combined a manipulative and symbolic representation (see Figure 8, left) while he argued:

> S1: [Placing four blocks on top of the sheet] Let’s see, here are the four children.
> Interviewer: OK, [S1 takes two more blocks] and what are these?
> S1: The tables [takes two more blocks]
> Interviewer: OK, these are four children. At which table are these four children sitting?
> S1: Ah! That means there are four chairs.
> Interviewer: Where?
> S1: At the two tables.
> Interviewer: So how many children are there in total?
> S1: Four [moving the four blocks]
> Interviewer: In total or at each table?
> S1: In each… in total. In total.

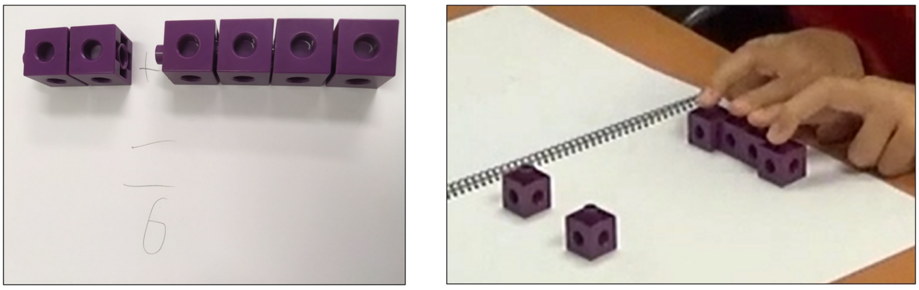

Figure 8. Incorrect additive relationships strategy by S1 when solving problem P1(S).

S1's solution shows that despite changing his initial reasoning, he kept the four blocks representing the children. Also, in a moment of enthusiasm, he noted, as an important aspect, that there were four chairs. We also see his confusion when the interviewer asked him if he was referring to the total or to each table, which could indicate difficulties understanding the key words of the problem.

In the case of problem P2(S), S1 took 10 blocks and arranged them in a row. He then drew ten children, one under each block (see Figure 9, left).

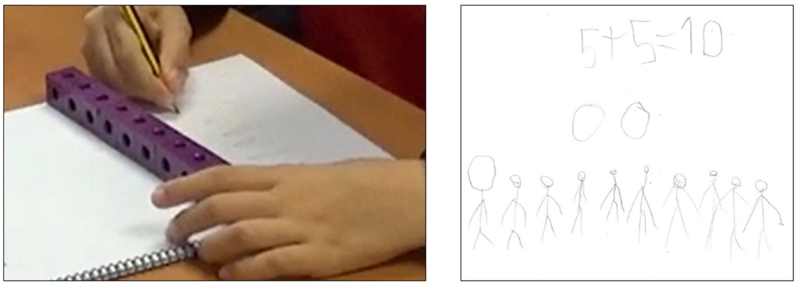

Figure 9. Modeling strategy used by S1 to solve problem P2(S).

As he drew, S1 reasoned:

S1: And this one is going to have a huge head [referring to the first child in the row starting from the left, whose head he draws last]

Interviewer: OK, are those the ten children?

S1: Yes.

Interviewer: OK, how many are sitting at each table?

S1: Hmm... well, ten.

Interviewer: Ten in total, right?

S1: Oh, yes.
Interviewer: And how many tables are there?
S1: [moves the blocks] Two [draws two representing the two tables]
Interviewer: So how many are sitting at each table?
S1: [automatically] Five.
Interviewer: OK, and how do you know there are five sitting at each table?
S1: Because there are five chairs.

The interviewer then asked him to write the answer, and S1 wrote above the two circles: “5+5=10” (see Figure 9, right). We interpret that S1 understood the problem, which he tackled through modeling, combining manipulative representation (blocks) with specific drawings (children and tables). Finally, he expressed the solution as a repeated sum using symbolic representation. We note once again the reference to the chairs as part of the solution, which he used when asked by the interviewer to explain his answer.

Since he correctly solved problem P2(S), S1 was also given problem P2(L). On this occasion, after reading the problem, he said:

S1: There are 27 of us in my class.
Interviewer: You are 27. OK, and what if there was one more student in your class? How many would there be if there was one more child?
S1: 28.
Interviewer: 28, just like here. So, if a new child joins your class, there will be 28 of you.
S1: Right.
Interviewer: Now suppose there are 4 tables in your classroom. Then, how many children would sit at each table?
S1: In my table there’s always a team.
Interviewer: OK, in this class there are teams too.

S1 then drew 27 squares, added a rectangle underneath and said, “and here is the teacher’s table.” When asked by the interviewer how many children were at each table, he answered, “4 tables for 28 children,” and solved the multiplication algorithm (see Figure 10).

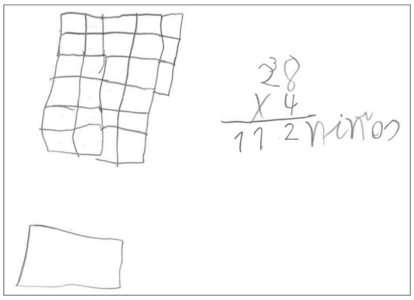

Figure 10. S1's incorrect solution to problem P2(L) using an inverse operation strategy

In problem P3(S), S1 made use of the manipulatives to model the situation. He took 15 blocks, put them together in a row, and said, "These are the 15 toys." Next, he set aside three blocks, reasoning: "15 toys. If I take 3 away, there are 12 toys left. The answer is 12."

Discussion

This research provides a detailed analysis of the strategies used by 10 students diagnosed with autism when solving multiplicative structure problems, and supplements other studies focusing on mathematical problem solving with students with autism (e. g. Polo-Blanco *et al*, 2019; Polo-Blanco *et al*, 2021). In terms of the operation involved, the least difficult problems for our study's participants were the multiplication and partitive division problems, with an identical success rate in the small-number problems (solved correctly by five out of 10 participants). Although it is not straightforward to compare results due to the small sample in this work, we note that the higher success rate in the multiplication problems is consistent with the results obtained in the study by Ivars and Fernández (2016) involving students ages 6 to 9. The measurement division problems were also more difficult than the partitive division problems in similar studies with TD students of similar ages (Bell *et al.*, 1984; Ivars and Fernández, 2016). As in the work by Mulligan and Mitchelmore (1997), the size of the numbers was also an important variable in the solution process, with this success rate diminishing in the associated large-number problems.

In terms of the strategies used, low-level strategies as modeling were generally employed, with little use of multiplication and division strategies. Three students constantly used counting strategies or operation strategies, and they were successful in every case. The other seven students resorted to modeling or used incorrect strategies in all the problems. This is consistent with other studies focusing on mathematical word problem solving by autistic students (Bae *et al.*, 2015; Polo-Blanco *et al.*, 2019) and could be associated with characteristics of the autism, such as language comprehension or executive function deficits (Happé *et al.*, 2006). These deficits can make it difficult for them to understand the problem situation, or to select and execute a successful solution strategy (Bae *et al.*, 2015). The results are also in line with others focusing on strategy use on multiplication problems by students with difficulties (Zhang *et al.*, 2016), in the sense that the students were quite consistent in their choice of strategies when solving all problems However, the results are in contrast with similar studies in TD children that reveal a variety of strategies and progression to more complex strategies as they advance in age towards the grades of our study participants (Ivars and Fernández, 2016; Mulligan and Mitchelmore, 1997). For example, in the study by Ivars and Fernández (2016), the students used incorrect strategies (most notably that of additive relationships), and modeling strategies only in the first two years (ages 6 to 8). These strategies disappeared in subsequent grades, with a shift towards operations that was not observed in our study participants.

The detailed description of student S1 allowed us to analyze in depth the reasoning employed to solve the various problems. For example, we observed how his difficulties understanding language, which are typical of the disorder (APA, 2013), interfered with his understanding of the problems. Specifically, this was clearly displayed by how S1 interpreted some of the interviewer's questions literally (e.g., "how do you know that five of them can sit?", "Because there are five chairs"), or the association he made between the problems and situations familiar to him ("28 students? There are 27 students in my class"). This could be related to the characteristic which involves literal thinking, identified in people with autism (Happé, 1993). This fact could have distracted the student from the mathematical aspects of the problems and hampered him to find the solutions (Polo-Blanco *et al.*, 2019; 2021-a). On the occasion mentioned above, the interviewer's guiding phrases (e.g., "if there was one more, it would be 28, like here") and her insistence proved essential to helping the student connect with the situation in the problem and provided a deeper understanding of the answers given by the student.

The results of this work agree with those of other studies that show that mathematics is a problematic subject for autistic individuals, and have important implications for teaching students with autism. For instance, investigating the strategies used by students provides teachers with more informal information which is especially useful for designing an instructional sequence, taking into account at what stage of this learning process the student is. In this regard, teachers could consider the propensity to model, either through manipulatives or drawings, that several students manifested in order to help them represent the problem situation and improve their understanding (for example, by proposing sharing scenarios using modeling to help them assign meaning to division). Additionally, grading strategies by levels provides a guide that can help teachers plan their teaching: once the students understand the problem by using modeling strategies, they can be steered toward counting strategies before concluding the process by resorting to number facts and algorithms.

In addition, the strategies and representations used by students could be incorporated as part of instructional methodologies for learning mathematical problem solving. Specifically, two well-known evidence-based practices for students with learning difficulties are: Conceptual Model-based Mathematics Problem Solving (COMPS) and Schema-Based Instruction (SBI) methodology. The COMPS methodology, developed by Xin (2012), relies on the use of schematic diagrams that emphasize the representations of mathematical relations using models. SBI uses schematic diagrams that combine visual and heuristic representations of the solution. In view of the results of this work, the representations of the problem data used in students' strategies could be incorporated as part of these methodologies, for instance through the use of manipulative material on a schematic diagram in SBI, thus helping them to relate their own representation with the operation to be performed to solve the problem. Other adaptations to the SBI and COMPS methodology have previously been successfully implemented with students with ASD (Root *et al.*, 2017; Cox and Root, 2020; Polo-Blanco *et al.*, 2021-b; 2022; García-Moya *et al.*, in press). Furthermore, they can be implemented in a group and also reinforce instruction on 1-1 tuition.

As an area of future research, and in keeping with the work of Polo-Blanco *et al.* (2021-a), it would be beneficial to analyze the extent to which contextualization in familiar situations could help students with autism gain a better understanding of the problem and identify the operation required to solve it. Moreover, this study leaves room for wider research. For instance, the problem-solving processes involving other operations within a larger sample of students could follow a similar approach to what is presented in this paper. Finally, we need to continue delving into the difficulties experienced by students with autism in order to establish instructional guidelines that address their needs and help them improve their learning of this noteworthy subject.

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