Asymptotics for the spectrum of a Floquet-parametric family of homogenization problems associated with a Dirichlet waveguide

D. Gómez, S.A. Nazarov, R. Orive-Illera and M.-E. Pérez-Martínez

1.1 Introduction

In this paper we consider a parametric family of spectral problems for the Laplace operator in a rectangular perforated domain ϖ^ε. The perforations are periodically placed along the ordinate axis at a distance $O(\varepsilon)$ between them, where ε is an small parameter $\varepsilon \ll 1$, see Fig. 1.1 a). We impose Dirichlet conditions on the boundary of the perforation and on the horizontal sides of the rectangle, while we impose *quasi-periodicity* conditions on the lateral sides containing the so-called *Floquet-parameter* $\eta \in [-\pi,\pi]$. This parametric family arises as the model problem of a spectral problem posed in an unbounded strip periodically perforated by a string of holes, which is referred to as *perforation string*, cf. Fig. 1.1 b). For each $\eta \in [-\pi,\pi]$, the spectral problem in the periodicity cell ϖ^ε, is itself a homogenization problem, and we study the asymptotic behavior of the eigenvalues and eigenfunctions as $\varepsilon \to 0$. In this way, we revisit the spectral problem for the Dirichlet-Laplace operator in a perforated waveguide addressed in [NaOrPe19a], providing new results that complement those.

The setting of the perturbation spectral problem is in Section 1.1.1; the homogenized problem is in Section 1.1.2, while the state of the art is in Section 1.1.3. Our

D. Gómez
Universidad de Cantabria, 39005 Santander, Spain
e-mail: gomezdel@unican.es

S.A. Nazarov
St. Petersburg State University, 198504 St. Petersburg, Russia
e-mail: srgnazarov@yahoo.co.uk

R. Orive-Illera
Universidad Autónoma de Madrid, 28049 Madrid, Spain
e-mail: rafael.orive@icmat.es

M.-E. Pérez-Martínez
Universidad de Cantabria, 39005 Santander, Spain
e-mail: meperez@unican.es

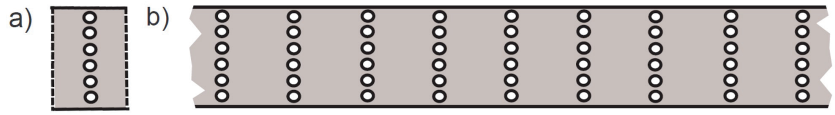

Fig. 1.1 **a)** The perforated domain ϖ^ε. **b)** The perforated strip Π^ε .

aim is to study the asymptotic behavior of the spectrum as $\varepsilon \to 0$ at the same time that we provide precise bounds for convergence rates which are uniform in both parameters ε and η. This is in Section 1.3. Some preliminary results obtained in [NaOrPe19a] and [GoEtAl21a] are stated in Section 1.2.

1.1.1 The parametric family of homogenization spectral problems

Let ω be a domain in the plane $\mathbb{R}^2$ which is bounded by a smooth simple closed curve $\partial\omega$ and has the compact closure $\overline{\omega} = \omega \cup \partial\omega \subset \varpi^0$, where ϖ^0 is the rectangle

$$\varpi^0 = (-1/2, 1/2) \times (0, H). \tag{1.1}$$

We introduce the perforated domain ϖ^ε, see Fig. 1.1 a), obtained from ϖ^0 by removing the family of holes

$$\omega^\varepsilon(k) = \{x : \varepsilon^{-1}(x_1, x_2 - \varepsilon k H) \in \omega\}, \quad k = 0, \dots, N-1,$$

which are distributed periodically along the ordinate x_2-axis. Each hole is homothetic to ω of ratio ε and translation of $\varepsilon\omega = \omega^\varepsilon(0)$. Namely,

$$\varpi^\varepsilon = \varpi^0 \setminus \overline{\omega^\varepsilon} \quad \text{where} \quad \omega^\varepsilon = \bigcup_{k=0}^{N-1} \omega^\varepsilon(k). \tag{1.2}$$

Here, ε is a small positive parameter and N is a big natural number, both related by $N = \varepsilon^{-1}$. The period is εH with $\varepsilon \ll 1$.

In the domain ϖ^ε, we consider the spectral problem defined by the equations

$$-\Delta U^\varepsilon(x;\eta) = \Lambda^\varepsilon(\eta) U^\varepsilon(x;\eta), \quad x \in \varpi^\varepsilon, \tag{1.3}$$

$$U^\varepsilon(x;\eta) = 0, \quad x \in \Gamma^\varepsilon, \tag{1.4}$$

$$U^\varepsilon(1/2, x_2;\eta) = e^{i\eta} U^\varepsilon(-1/2, x_2;\eta), \quad x_2 \in (0,H), \tag{1.5}$$

$$\frac{\partial U^\varepsilon}{\partial x_1}(1/2, x_2;\eta) = e^{i\eta} \frac{\partial U^\varepsilon}{\partial x_1}(-1/2, x_2;\eta), \quad x_2 \in (0,H), \tag{1.6}$$

where

$$\Gamma^\varepsilon = \partial\varpi^\varepsilon \setminus \{\pm 1/2\} \times (0,H),$$

η is the dual variable, namely, *the Floquet-parameter*. $\Lambda^\varepsilon(\eta)$ and $U^\varepsilon(\cdot;\eta)$, respectively, denote the eigenvalues and eigenfunctions which depend on both the perturbation parameter and the Floquet-parameter. Conditions (1.5)-(1.6) are the so-called *quasi-periodicity conditions* on the lateral sides $\{\pm 1/2\} \times (0,H)$ of ϖ^ε.

The variational formulation of the spectral problem (1.3)-(1.6) reads: Find $\Lambda^\varepsilon(\eta)$ and $U^\varepsilon(\cdot;\eta) \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$, $U^\varepsilon(\cdot;\eta) \neq 0$ satisfying

$$(\nabla U^\varepsilon(\cdot;\eta), \nabla V)_{\varpi^\varepsilon} = \Lambda^\varepsilon(\eta)\,(U^\varepsilon(\cdot;\eta), V)_{\varpi^\varepsilon} \qquad \forall V \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon), \tag{1.7}$$

where $H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ denotes the subspace of $H^1(\varpi^\varepsilon)$ of functions which satisfy the quasi-periodicity conditions (1.5)-(1.6) and vanish on Γ^ε, and $(\cdot,\cdot)_{\varpi^\varepsilon}$ denotes the scalar product in $L^2(\varpi^\varepsilon)$.

As is well known (cf. [NaOrPe19a], Ch. 10 in [BiSo80], Ch. 13 in [ReSi78] and Ch. 4 in [SaSa89]) problem (1.7) has a discrete spectrum constituting the monotone unbounded sequence of eigenvalues

$$0 < \Lambda_1^\varepsilon(\eta) \leq \Lambda_2^\varepsilon(\eta) \leq \cdots \leq \Lambda_m^\varepsilon(\eta) \leq \cdots \to \infty, \quad \text{as } m \to \infty, \tag{1.8}$$

which are repeated according to their multiplicities. Also, the corresponding eigenfunctions $\{U_m^\varepsilon(\cdot;\eta)\}_{m=1}^\infty$ are assumed to form an orthonormal basis in $L^2(\varpi^\varepsilon)$. Furthermore, the function

$$\eta \in [-\pi,\pi] \mapsto \Lambda_m^\varepsilon(\eta) \tag{1.9}$$

is continuous and 2π-periodic. This last assertion is due to the fact that problem (1.3)-(1.6) is the model problem associated with a waveguide, which is referred to as the *Dirichlet strip*, and has been recently considered in the literature (cf. (1.20), Fig. 1.1 b), [NaOrPe19a] and [NaOrPe19b]). For the sake of completeness, in order to outline the interest of the problem under consideration (1.3)-(1.6), as well as its properties we introduce briefly this waveguide in Section 1.1.3.

1.1.2 The homogenized problem

For each $\eta \in [-\pi,\pi]$, the homogenized problem of (1.3)-(1.6) reads

$$-\Delta U^0(x;\eta) = \Lambda^0(\eta) U^0(x;\eta), \quad x \in \widetilde{\varpi}^0, \tag{1.10}$$

$$U^0(x;\eta) = 0, \quad x \in \Gamma_{lu0}, \tag{1.11}$$

$$U^0(1/2,x_2;\eta) = e^{i\eta} U^0(-1/2,x_2;\eta), \quad x_2 \in (0,H), \tag{1.12}$$

$$\frac{\partial U^0}{\partial x_1}(1/2,x_2;\eta) = e^{i\eta} \frac{\partial U^0}{\partial x_1}(-1/2,x_2;\eta), \quad x_2 \in (0,H), \tag{1.13}$$

where $\widetilde{\varpi}^0$ and Γ_{lu0} denote

$$\widetilde{\varpi}^0 := (-1/2,0) \times (0,H) \cup (0,1/2) \times (0,H)$$

and

$$\Gamma_{lu0} := \{x : x_1 \in (-1/2, 1/2), x_2 \in \{0, H\}\} \cup \{x : x_1 = 0, x_2 \in (0, H)\} \quad (1.14)$$

respectively, $\Lambda^0(\eta)$ is the spectral parameter and $U^0(\cdot;\eta)$ the corresponding eigenfunction.

The variational formulation of the spectral problem (1.10)-(1.13) reads: Find $\Lambda^0(\eta)$ and $U^0(\cdot;\eta) \in H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu0})$, $U^0(\cdot;\eta) \neq 0$ satisfying

$$\left(\nabla U^0(\cdot;\eta), \nabla V\right)_{\widetilde{\varpi}^0} = \Lambda^0(\eta) \left(U^0(\cdot;\eta), V\right)_{\widetilde{\varpi}^0} \qquad \forall V \in H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu0}), \quad (1.15)$$

where $H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu0})$ denotes the subspace of $H^1(\varpi^0)$ of functions which satisfy the quasi-periodicity conditions (1.12)-(1.13) and vanish on Γ_{lu0}. Similarly to (1.7), problem (1.15) has a discrete spectrum $\{\Lambda^0_m(\eta)\}_{m=1}^{\infty}$ with corresponding eigenfunctions $\{U^0_m(\cdot;\eta)\}_{m=1}^{\infty}$ which form an orthogonal basis in $L^2(\varpi^0)$.

Comparing the homogenization problem (1.3)-(1.6) with other homogenization problems having Dirichlet conditions on the boundary of the perforations, we see that it differs only in the quasi-periodicity boundary conditions on the lateral sides and one can easily guess the homogenized problem (1.10)-(1.13), see for instance [LoEtAl98]. However, in this case, one can show that the eigenvalues coincide with those of the Dirichlet problem

$$\begin{aligned} &-\Delta U^0(x) = \Lambda^0 U^0(x), \quad x \in \upsilon, \quad \upsilon := (0,1) \times (0,H), \\ &U^0(x) = 0, \quad x \in \partial\upsilon, \end{aligned} \quad (1.16)$$

and consequently, do not depend on η (cf. [NaOrPe19a]).

Problem (1.16) has a discrete spectrum which forms the increasing sequence of eigenvalues

$$0 < \Lambda^0_1 < \Lambda^0_2 \leq \cdots \leq \Lambda^0_m \leq \cdots \to \infty, \quad \text{as } m \to \infty, \quad (1.17)$$

repeated according to their multiplicities. In addition, the eigenpairs of (1.16) can be computed explicitly

$$\Lambda^0_{np} = \pi^2 \left(n^2 + \frac{p^2}{H^2}\right), \quad U^0_{np}(x) = \frac{2}{\sqrt{H}} \sin(n\pi x_1) \sin(p\pi x_2/H), \ p,n \in \mathbb{N}. \quad (1.18)$$

Note that the eigenvalues Λ^0_{np} are numerated with two indexes and must be reordered in order to obtain the increasing sequence (1.17); the corresponding eigenfunctions U^0_{np} are normalized in $L^2(\upsilon)$. Also, we note that if H^2 is an irrational number all the eigenvalues are simple.

As noticed in [NaOrPe19a], extending by quasi-periodicity the eigenfunctions $U^0_m(\cdot;\eta)$,

$$u^0_m(x;\eta) = \begin{cases} U^0_m(x;\eta), & x_1 \in (0,1/2), \\ e^{i\eta} U^0_m(x_1 - 1, x_2;\eta), & x_1 \in (1/2,1), \end{cases} \quad (1.19)$$

we obtain a smooth function in the rectangle υ, and the pair $(\Lambda_m^0(\eta),\, u_m^0(\cdot,\eta))$ satisfies (1.16).

The orthogonality of $\{U_m^0(\cdot;\eta)\}_{m=1}^{\infty}$ in $L^2(\varpi^0)$ implies that the functions in (1.19), $\{u_m^0(\cdot;\eta)\}_{m=1}^{\infty}$, form an orthogonal basis in $L^2(\upsilon)$, and this shows that the set $\{\Lambda_m^0(\eta)\}_{m=1}^{\infty}$ coincides with $\{\Lambda_m^0\}_{m=1}^{\infty}$ in the sequence (1.17) for any $\eta \in [-\pi,\pi]$.

Similarly to (1.18), we compute the eigenvalues and eigenfunctions of (1.10)-(1.13):

$$U_{np}^0(x,\eta) = \begin{cases} \frac{2}{\sqrt{H}} \sin(n\pi x_1)\sin(p\pi \frac{x_2}{H}), & x_1 \in (0,1/2), \\ \frac{2e^{-\mathrm{i}\eta}}{\sqrt{H}} \sin(n\pi(x_1+1))\sin(p\pi \frac{x_2}{H}), & x_1 \in (-1/2,0), \end{cases}$$

is the eigenfunction corresponding to $\Lambda_{np}^0 = \pi^2\left(n^2 + \frac{p^2}{H^2}\right)$ with $p,n \in \mathbb{N}$.

1.1.3 The Dirichlet strip and some background

For convenience, we introduce here a problem closely related to (1.3)-(1.6): a Dirichlet problem for the Laplace operator in a strip with periodic dense transversal perforations by identical holes of diameter ε.

Extending ϖ^ε (cf. (1.2) and Fig. 1.1 a)) by periodicity along the x_1 axis, we create the unbounded perforated strip Π^ε (see Fig. 1.1 b)):

$$\Pi^\varepsilon = \mathbb{R}\times(0,H) \setminus \bigcup_{j\in\mathbb{Z}} \bigcup_{k=0}^{N-1} \overline{\omega^\varepsilon(j,k)}$$

where $\omega^\varepsilon(j,k) = \{x : \varepsilon^{-1}(x_1-j, x_2-\varepsilon kH) \in \omega\}$ with $j\in\mathbb{Z}$, $k=0,1,\dots,N-1$. In the waveguide Π^ε, we consider the Dirichlet spectral problem

$$\begin{cases} -\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x\in\Pi^\varepsilon, \\ u^\varepsilon(x) = 0, \quad x \in \partial\Pi^\varepsilon. \end{cases} \tag{1.20}$$

Then, applying the Floquet-Bloch-Gelfand transform

$$u^\varepsilon(x) \to U^\varepsilon(x;\eta) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-\mathrm{i}n\eta} u^\varepsilon(x_1+n, x_2)$$

see, for instance, [Ge50], [ReSi78], [Sk85], [Ku93] and [CoPlVa94], problem (1.20) converts into a η-parametric family of spectral problems in the periodicity cell ϖ^ε, namely, into the parametric family of boundary value problems (1.3)-(1.6), see Fig. 1.1 a).

The spectrum of the operator on the Hilbert space $L^2(\Pi^\varepsilon)$ associated with problem (1.20) is given by

$$\sigma^\varepsilon = \bigcup_{m\in\mathbb{N}} B_m^\varepsilon \tag{1.21}$$

where

$$B_m^\varepsilon = \{\Lambda_m^\varepsilon(\eta) : \eta \in [-\pi,\pi]\}. \tag{1.22}$$

As a consequence of the previously mentioned continuity of $\Lambda_m^\varepsilon(\eta)$, cf. (1.9), the sets B_m^ε are closed, connected and bounded intervals of the real positive axis $\overline{\mathbb{R}_+}$.

Results (1.21) and (1.22) for the spectrum of the boundary value problem (1.20) are well-known in the framework of the Floquet-Bloch-Gelfand theory (see the above references). The segments B_m^ε and B_{m+1}^ε may intersect but also they can be disjoints so that a spectral gap becomes open between them. Recall that a spectral gap is a non empty interval which is free of the spectrum but has both endpoints in the spectrum.

Therefore, studying the asymptotic behavior of the spectrum of (1.3)-(1.6) becomes essential to detect the band gap structure of the spectrum (1.21). In this respect, an extensive asymptotic analysis of the spectral bands (1.22) has been performed in [NaOrPe19a]. In particular, we have obtained asymptotic formulas for the endpoints of the spectral bands (1.22) and show that σ^ε has a long number of short bands of length $O(\varepsilon)$ which alternate with wide gaps of width $O(1)$, while we can guarantee that indeed there are open gaps corresponding with B_m^ε and B_{m+1}^ε only when the limit eigenvalue Λ_m^0 in the sequence (1.17) is simple, cf. Fig. 1.2 (on the right), and this strongly depends on H.

We note that the explicit formulas (1.18) are of great interest to draw the *limit dispersion curves* for different values of H and, after obtaining bounds for discrepancies of the type (1.42) (cf. also (1.28)), they also allow us to draw possible configurations of the perturbed dispersion curves associated with (1.20), cf. Fig. 1.2 (on the left). Recall that these curves are the graphs of $\Lambda_m^\varepsilon(\eta)$, for $\eta\in[-\pi,\pi]$. On account of (1.18), the limiting dispersion curves are independent of η.

We refer to [BaPe18] for a very different perturbed waveguide with limiting dispersion curves independent of the Floquet-parameter and to [GoEtAl21b] and [GoEtAl21c] for the geometry of the waveguide here considered but with Neumann conditions instead of Dirichlet. Also, we refer to [GoEtAl21b] and [GoEtAl21c] for further references and an extensive comparison between the behaviors of the spectral bands when we change Dirichlet by Neumann conditions both in (1.20) and (1.10)-(1.13). As a matter of fact, in the case of the Neumann-strip we find long bands, of order $O(1)$, which are separated from each other by short spectral gaps of order $O(\varepsilon)$. Moreover, it should be mentioned that, as a consequence of the fact that the limiting dispersion curves are not constant in the case of the Neumann-strip, the asymptotic analysis is much more complicated and delicate, in particular, it becomes multiscale in several variables, not only in the geometrical ones, but also in the Floquet-parameter.

Finally, let us observe that opening gaps in [NaOrPe19a] implies a thorough asymptotic analysis to obtain corrector terms of order $O(\varepsilon)$ that improves the uniform bounds (1.42). For the sake of brevity, we avoid defining the correctors here

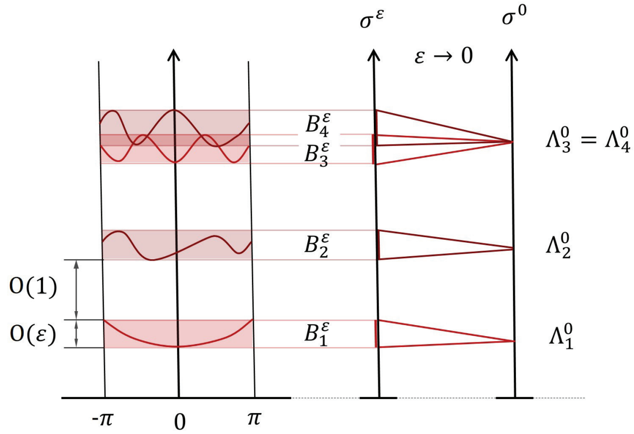

Fig. 1.2 On the left: a sketch of possible dispersion curves in the axis (η, Λ) for the problem in the waveguide Π^ε. On the right: a sketch of the possible distribution of the spectral bands B^ε.

which involves introducing some boundary layer problems and the so-called *polarization matrix*. We refer to [NaOrPe19a] and [NaOrPe19b] in this connection.

1.2 Preliminary results

Let us introduce here some estimates for the eigenvalues of the perturbation problem that improves that in [NaOrPe19a], and a couple of theorems whose proofs are in [NaOrPe19a]. The results of these theorems are improved in Section 1.3.

Lemma 1. *For each fixed m, there are constants $\varepsilon_0 < 1$, $K_m(\eta)$ and C_m such that*

$$0 < K_m(\eta) \le \Lambda_m^\varepsilon(\eta) \le C_m \qquad \forall \eta \in [-\pi, \pi], \quad \varepsilon \le \varepsilon_0. \tag{1.23}$$

Proof. To obtain the lower bound in (1.23) with $K_m(\eta) \equiv C$ independent of m and η, it suffices to consider (1.7) for the eigenpair $(\Lambda_1^\varepsilon(\eta), U_1^\varepsilon(\cdot;\eta))$ and apply the Poincaré inequality in $H^1(\varpi^0)$ once that $U_1^\varepsilon(\cdot;\eta)$ is extended by zero in $\overline{\varpi}^\varepsilon$, cf. (1.1) and (1.2). However, we can also obtain better bounds depending on η that somehow could isolate the branches $\{\Lambda_m^\varepsilon(\eta) : \eta \in [-\pi, \pi]\}$.

Indeed, let us consider $\{\Lambda_m^*(\eta)\}_{m=1}^\infty$ to be the sequence of eigenvalues of the following problem in ϖ^0:

$$
\begin{aligned}
&-\Delta U_m^*(x;\eta) = \Lambda_m^*(\eta) U_m^*(x;\eta), \quad x \in \varpi^0,\\
&U_m^*(x;\eta) = 0, \quad x \in \Gamma_{lu},\\
&U_m^*(1/2,x_2;\eta) = e^{i\eta} U_m^*(-1/2,x_2;\eta), \quad x_2 \in (0,H),\\
&\frac{\partial U_m^*}{\partial x_1}(1/2,x_2;\eta) = e^{i\eta} \frac{\partial U_m^*}{\partial x_1}(-1/2,x_2;\eta), \quad x_2 \in (0,H),
\end{aligned}
\tag{1.24}
$$

where we have denoted by Γ_{lu} lower and upper basis of the rectangle ϖ^0, namely,

$$
\Gamma_{lu} := \{x : x_1 \in (-1/2,1/2), x_2 \in \{0,H\}\}, \tag{1.25}
$$

cf. (1.14) to compare, and by $\{U_m^*(\cdot;\eta)\}_{m=1}^{\infty}$ the eigenfunctions.

Using the minimax principle,

$$
\Lambda_m^*(\eta) = \min_{E_m \subset H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu})} \max_{V \in E_m, V \neq 0} \frac{(\nabla V, \nabla V)_{\varpi^0}}{(V,V)_{\varpi^0}},
$$

where the minimum is computed over the set of subspaces E_m of $H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu})$ with dimension m.

Consider the subspace E_m^ε of $H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ with dimension m, of the eigenfunctions $U_k^\varepsilon(\cdot;\eta)$ of (1.3)-(1.6) associated with the eigenvalues $\Lambda_k^\varepsilon(\eta)$ in the sequence (1.8) with $k \leq m$. These eigenfunctions have been taken to be orthonormal in $L^2(\varpi^\varepsilon)$, and are extended by 0 inside the holes, they are still denoted by $U_m^\varepsilon(\cdot;\eta)$ and orthonormal in $L^2(\varpi^0)$, and we take the particular subspace of dimension m of $H^{1,\eta}_{per}(\varpi^0;\Gamma_{lu})$ to be $E_m^* = \left[U_1^\varepsilon(\cdot;\eta), U_2^\varepsilon(\cdot;\eta), \cdots U_m^\varepsilon(\cdot;\eta)\right]$. Then, we can write:

$$
\Lambda_m^*(\eta) \leq \max_{V \in E_m^*, V \neq 0} \frac{(\nabla V, \nabla V)_{\varpi^0}}{(V,V)_{\varpi^0}} = \max_{V \in E_m^*, \|V\|_{L^2(\varpi^0)}=1} (\nabla V, \nabla V)_{\varpi^0}.
$$

For each $V \in E_m^*$, with $\|V\|_{L^2(\varpi^0)} = 1$, we write $V = \sum_{i=1}^m \alpha_i^\varepsilon(\eta) U_i^\varepsilon(\cdot;\eta)$ for certain constants $\alpha_i^\varepsilon(\eta)$. On account of the above mentioned orthonormality, these constants satisfy

$$
\|V\|^2_{L^2(\varpi^0)} = \sum_{i=1}^m (\alpha_i^\varepsilon(\eta))^2 = 1.
$$

Similarly, because of the extension by zero, the orthonormality, and (1.7), for the gradients, we can write:

$$
\|\nabla V\|^2_{L^2(\varpi^0)} = \sum_{i=1}^m (\alpha_i^\varepsilon(\eta))^2 \|\nabla U_i^\varepsilon(\cdot;\eta)\|^2_{L^2(\varpi^0)} = \sum_{i=1}^m (\alpha_i^\varepsilon(\eta))^2 \Lambda_i^\varepsilon(\eta) \leq \Lambda_m^\varepsilon(\eta),
$$

which gives

$$
\Lambda_m^*(\eta) \leq \Lambda_m^\varepsilon(\eta), \quad \forall \eta \in [-\pi,\pi], \quad m \geq 1.
$$

Therefore, the left hand side of (1.23) holds for $K_m(\eta) = \Lambda_m^*(\eta)$ the eigenvalue of the mixed problem (1.24).

Finally, the precise constant C_m on the right hand sides of (1.23) has been obtained in [NaOrPe19a], related to the m-th eigenvalue of a Dirichlet problem in any fixed rectangle $(\alpha,\beta)\times(0,H)$, with $0<\alpha<\beta<1/2$. $\square$

The first convergence result is given in Theorem 1 below. It shows the somehow expected convergence of the spectrum with conservation of the multiplicity in homogenization theory. Also, the convergence of the corresponding eigenfunctions is stated. The proof in [NaOrPe19a] has been performed adapting standard techniques in homogenization and spectral perturbation theory: see, for instance, Ch. 3 in [OlShYo92] for a general framework and [LoEtAl98] for its application to spectral problems in perforated domains with different boundary conditions.

Theorem 1. *Let us consider the spectral problem* (1.3)-(1.6) *and the sequence of eigenvalues* (1.8)*. Then, for any $\eta\in[-\pi,\pi]$, we have the convergence*

$$\Lambda_m^\varepsilon(\eta)\to\Lambda_m^0, \quad \text{as } \varepsilon\to 0, \tag{1.26}$$

where Λ_m^0 are the set of eigenvalues in the sequence (1.17) *of the Dirichlet problem* (1.16)*. In addition, for each sequence, we can extract a subsequence, still denoted by ε, such that the extension by zero of the eigenfunctions $\{U_m^\varepsilon(\cdot;\eta)\}_{m=1}^\infty$ normalized in $L^2(\varpi^\varepsilon)$, $\{\widehat{U}_m^\varepsilon(\cdot;\eta)\}_{m=1}^\infty$, converge towards the eigenfunctions of* (1.10)-(1.13) *in $L^2(\varpi^0)$, which form an orthonormal basis of $L^2(\varpi^0)$.*

As a consequence of the asymptotic analysis in [NaOrPe19a] we state the following result:

Theorem 2. *Let $m\in\mathbb{N}$ and let Λ_m^0 be an eigenvalue of the Dirichlet problem* (1.16) *in the sequence* (1.17)*. There is at least one eigenvalue $\Lambda_p^\varepsilon(\eta)$ of problem* (1.3)-(1.6)*, with $p=p(\varepsilon,\eta,m)\geq m$, satisfying*

$$|\Lambda_p^\varepsilon(\eta)-\Lambda_m^0|\leq c_m\varepsilon, \quad \forall\varepsilon\leq\varepsilon_m,\ \eta\in[-\pi,\pi], \tag{1.27}$$

where ε_m and c_m are certain positive constants that are independent of η and ε.

The proof of Theorem 2 can be found in [NaOrPe19a], based on a Lemma on almost eigenvalues and eigenfunctions from the spectral perturbation theory, cf. [ViLu57]. It involves the construction of approximations to eigenpairs of the perturbation problem by means of asymptotic expansions from the solutions of the homogenized problem and a boundary layer problem in an unbounded perforated strip, namely, in the "unit periodicity cell" for the homogenization problem (1.3)-(1.6) (cf. also [NaOrPe19b]).

In the next section, we show that the index p provided by Theorem 2 coincides with m, cf. Theorem 4. Although the bound (1.27) with $p=m$ has been used to detect spectral gaps in [NaOrPe19a], we think that the proof in Section 1.3 of this paper may clarify that in [NaOrPe19a].

Remark 1. It should be noted that bounds (1.23) can be improved as follows: For each fixed m, there are positive constants $\varepsilon_0<1$, $\theta<1$, k_m and c_m independent of ε and η, such that

$$\Lambda_m^0 - k_m \varepsilon^{2\theta} \leq \Lambda_m^\varepsilon(\eta) \leq \Lambda_m^0 + c_m \varepsilon \qquad \forall \eta \in [-\pi,\pi], \quad \varepsilon \leq \varepsilon_0. \tag{1.28}$$

The proof of (1.28) can be obtained using the reasoning of [GoEtAl21a] (Section 3) with minor modifications. This implies using the max-min principle, Hardy inequality, the normalization procedure used to obtain the left hand side inequality in (1.23) (applied both to finite dimensional spaces of eigenfunctions of the perturbation and homogenized problem), weighted estimates in Sobolev spaces and some cut-off functions vanishing in ε-neighborhoods of the perforation string. This result allows a simplification of the proof of Theorem 3 related to the eigenvalues. However, the bounds (1.28) are associated with the homogenization of perforated domains along lines with Dirichlet boundary conditions in the perforations (see [GoEtAl21a] and [GoEtAl21c] for other boundary conditions) and the suitable bounds cannot be obtained in many problems of perturbed waveguides, see [GoEtAl21a], [GoEtAl21b] and [GoEtAl21c] to compare. In contrast, the technique developed in Theorem 3 can be applied to many problems even when the limit dispersion curves depend on η, cf. [GoEtAl21b].

Also, it shoud be emphasized that the result in Theorem 4 improves the bound (1.28) providing the precise value of $\theta = 1/2$. □

1.3 Convergence and convergence rates for eigenvalues

A first approach to the asymptotics for eigenpairs of (1.3)-(1.6) is given by Theorem 1, when the parameter η is fixed. Theorem 3 below also allows a certain perturbation of this parameter and therefore improves the result in Theorem 1.

Theorem 3. *Let us consider the spectral problem* (1.3)-(1.6) *and the sequence of eigenvalues* (1.8). *Then, for each sequence* $\{(\varepsilon_r,\eta_r)\}_{r=1}^\infty$ *such that* $\varepsilon_r \to 0$ *and* $\eta_r \to \widehat{\eta} \in [-\pi,\pi]$*, as* $r \to \infty$*, we have the convergence*

$$\Lambda_m^{\varepsilon_r}(\eta_r) \to \Lambda_m^0, \quad \text{as } r \to \infty, \tag{1.29}$$

where Λ_m^0 *are the set of eigenvalues of the Dirichlet problem* (1.16) *in the sequence* (1.17). *In addition, we can extract a subsequence, still denoted by* ε_r*, such that the extension by zero of the eigenfunctions* $\{U_m^{\varepsilon_r}(\cdot;\eta_r)\}_{m=1}^\infty$ *normalized in* $L^2(\varpi^{\varepsilon_r})$*,* $\{\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r)\}_{m=1}^\infty$*, converge towards the eigenfunctions of* (1.10)-(1.13) *in* $L^2(\varpi^0)$*, which form an orthonormal basis of* $L^2(\varpi^0)$.

Proof. Let us consider $\Lambda_m^{\varepsilon_r}(\eta_r)$ and $U_m^{\varepsilon_r}(\cdot;\eta_r) \in H_{per}^{1,\eta_r}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r})$ the eigenpair of (1.7). Namely, for fixed (η_r,ε_r) and $m = 1,2,\cdots$, they satisfy

$$(\nabla U_m^{\varepsilon_r}(\cdot;\eta_r),\nabla V)_{\varpi^{\varepsilon_r}} = \Lambda_m^{\varepsilon_r}(\eta_r)\,(U_m^{\varepsilon_r}(\cdot;\eta_r),V)_{\varpi^{\varepsilon_r}}, \quad V \in H_{per}^{1,\eta}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r}), \tag{1.30}$$

Taking $V = U_m^{\varepsilon_r}(\cdot;\eta_r)$, (1.30) reads

$$\|\nabla U_m^{\varepsilon_r}(\cdot;\eta_r)\|_{L^2(\varpi^{\varepsilon_r})}^2 = \Lambda_m^{\varepsilon_r}(\eta_r)\|U_m^{\varepsilon_r}(\cdot;\eta_r)\|_{L^2(\varpi^{\varepsilon_r})}^2.$$

Let us extend the eigenfunctions by zero inside the holes. Then, using (1.23), the normalization $\|U_m^{\varepsilon_r}(\cdot;\eta_r)\|_{L^2(\varpi^{\varepsilon_r})} = 1$ and the Poincaré inequality, for each m we get a uniform bound for the eigenvalues and eigenfunctions in $H^1(\varpi^0)$. Indeed, the inequalities

$$\min_{\eta\in[-\pi,\pi]} K_m(\eta) \le \Lambda_m^{\varepsilon_r}(\eta_r) \le C_m \quad \text{and} \quad \|U_m^{\varepsilon_r}(\cdot;\eta_r)\|_{H^1(\varpi^{\varepsilon_r})} \le c_m, \tag{1.31}$$

hold for constants c_m, C_m which do not depend on ε_r and η_r.

Hence, for each fixed m, we can extract a subsequence of ε_r and η_r, still denoted by r such that

$$(\eta_r,\varepsilon_r) \to (\widehat{\eta},0), \text{ as } r\to\infty, \tag{1.32}$$

and

$$\Lambda_m^{\varepsilon_r}(\eta_r) \to \widehat{\Lambda}_m^0, \qquad \widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r) \rightharpoonup \widehat{U}_m^0 \text{ in } H^1(\varpi^0) - weak, \text{ as } r\to\infty, \tag{1.33}$$

for a certain positive $\widehat{\Lambda}_m^0$ and a certain function $\widehat{U}_m^0 \in H^1(\varpi^0)$ which vanishes on the lower and upper bases of ϖ^0, namely on Γ_{lu}, cf. (1.4) and (1.25). Let us prove that $\widehat{U}_m^0$ also vanishes along the line $\{x_1 = 0\} \cap \varpi^0$.

Indeed, we use the Poincaré inequality on the domains $\varpi^0 \setminus \overline{\varpi}$ and ϖ^0, cf. (1.1),

$$\|U\|_{L^2(\varpi^0\setminus\overline{\varpi})} \le C\|\nabla U\|_{L^2(\varpi^0\setminus\overline{\varpi})} \quad \forall U \in H^1(\varpi^0\setminus\overline{\varpi};\Gamma_{lu}\cup\partial\omega),$$

and

$$\|U\|_{L^2(\varpi^0)} \le C\|\nabla U\|_{L^2(\varpi^0)} \quad \forall U \in H^1(\varpi^0;\Gamma_{lu}).$$

We deduce

$$\varepsilon_r^{-1}\left\|U_m^{\varepsilon_r}(\cdot;\eta_r)\right\|^2_{L^2(\{|x_1|\le\varepsilon_r/2\}\cap\varpi^0)} \le C\varepsilon_r\left\|\nabla U_m^{\varepsilon_r}(\cdot;\eta_r)\right\|^2_{L^2(\{|x_1|\le\varepsilon_r/2\}\cap\varpi^0)}, \tag{1.34}$$

where C is a constant independent of r and m. Now, taking limits in (1.34) as $r\to\infty$, or equivalently as $\varepsilon_r \to 0$, we get $\widehat{U}_m^0 = 0$ on $\{x_1 = 0\}\cap\varpi^0$ (cf., e.g., [MaKh06] and (1.31)) as it has been announced.

Therefore, the limit function in (1.33) satisfies $\widehat{U}_m^0 \in H^1(\varpi^0;\Gamma_{lu0})$; cf. (1.14). Let us prove that it also satisfies the quasi-periodicity conditions on the lateral sides of ϖ^0:

$$\widehat{U}_m^0(1/2,x_2) = e^{i\widehat{\eta}}\widehat{U}_m^0(-1/2,x_2) \text{ and } \frac{\partial\widehat{U}_m^0}{\partial x_1}(1/2,x_2) = e^{i\widehat{\eta}}\frac{\partial\widehat{U}_m^0}{\partial x_1}(-1/2,x_2). \tag{1.35}$$

To do this, notice that the change $V_m^{\varepsilon_r}(\cdot;\eta_r) = U_m^{\varepsilon_r}(\cdot;\eta_r)e^{-\mathrm{i}\eta_r x_1}$ converts the Laplacian into the differential operator

$$-\big(\frac{\partial}{\partial x_1}+\mathrm{i}\eta_r\big)\big(\frac{\partial}{\partial x_1}+\mathrm{i}\eta_r\big) - \frac{\partial^2}{\partial x_2^2},$$

and the η_r quasi-periodicity condition for $U_m^{\varepsilon_r}(\cdot;\eta_r)$ becomes a periodicity condition for $V_m^{\varepsilon_r}(\cdot;\eta_r) \in H^1_{per}(\varpi^0;\Gamma_{lu})$. Consequently, since the convergence (1.33) holds, we also have a bound for $\widehat{V}_m^{\varepsilon_r} \in H^1_{per}(\varpi^0;\Gamma_{lu})$ which holds uniformly in η_r and ε_r, and consequently a convergence of $\widehat{V}_m^{\varepsilon_r}(\cdot;\eta_r)$ ($V_m^{\varepsilon_r}(\cdot;\eta_r)$ extended by zero inside the holes) towards a function $\widehat{V}_m^0(\cdot;\eta_r) \in H^1_{per}(\varpi^0;\Gamma_{lu0})$ holds in the weak topology of $H^1(\varpi^0;\Gamma_{lu0})$. Then, we obtain $\widehat{V}_m^0 = \widehat{U}_m^0 e^{-\mathrm{i}\widehat{\eta}x_1}$, as a consequence of the convergence

$$\|\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r)e^{-\mathrm{i}\eta_r x_1} - \widehat{U}_m^0 e^{-\mathrm{i}\widehat{\eta}x_1}\|_{L^2(\varpi^0)} \to 0 \text{ as } r \to \infty.$$

To verify the last convergence it suffices to consider

$$\begin{aligned}&\|\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r)e^{-\mathrm{i}\eta_r x_1} - \widehat{U}_m^0 e^{-\mathrm{i}\widehat{\eta}x_1}\|_{L^2(\varpi^0)}\\ &\qquad \le \|\big(\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r) - \widehat{U}_m^0\big)e^{-\mathrm{i}\eta_r x_1}\|_{L^2(\varpi^0)} + \|\widehat{U}_m^0\big(e^{-\mathrm{i}\eta_r x_1} - e^{-\mathrm{i}\widehat{\eta}x_1}\big)\|_{L^2(\varpi^0)},\end{aligned}$$

the convergence (1.33), the smoothness of the exponential function and the convergence η_r towards $\widehat{\eta}$ as $r \to \infty$.

Thus, we have $\widehat{U}_m^0 = \widehat{V}_m^0 e^{\mathrm{i}\widehat{\eta}x_1}$, with $\widehat{V}_m^0 \in H^1_{per}(\varpi^0)$, and this already implies (1.35). Consequently, we have shown that $\widehat{U}_m^0 \in H^{1,\widehat{\eta}}_{per}(\varpi^0;\Gamma_{lu0})$ and depends on $\widehat{\eta}$. Also, the normalization of the eigenfunctions $\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r)$ in $L^2(\varpi^0)$ and the convergence (1.33) provides $\widehat{U}_m^0 \neq 0$.

In addition, by taking limits in the variational formulation (1.30) for the test functions $V \in \mathscr{C}_0^\infty((-1/2,0)\times(0,H))$ and for $V \in \mathscr{C}_0^\infty((0,1/2)\times(0,H))$, we obtain the partial differential equation

$$-\Delta\widehat{U}_m^0 = \widehat{\Lambda}_m^0\widehat{U}_m^0 \quad \text{ for } x \in \widetilde{\varpi}^0. \tag{1.36}$$

All of this together, allows us to identify $(\widehat{\Lambda}_m^0, \widehat{U}_m^0)$ with an eigenpair of the boundary value problem (1.10)-(1.13), cf. also (1.15).

Note that the extracted subsequence and limits, cf. (1.32) and (1.33), may depend on m. However, using a diagonalization argument, for each sequence of r, we can extract another subsequence of r, still denoted by r but independent of m, such that (1.33) holds $\forall m \in \mathbb{N}$. Hence, by construction, we have obtained an increasing sequence of eigenvalues of (1.10)-(1.13)

$$0 < \widehat{\Lambda}_1^0 \le \widehat{\Lambda}_2^0 \le \cdots \le \widehat{\Lambda}_m^0 \le \cdots. \tag{1.37}$$

In what follows we prove that the sequence $\{\widehat{\Lambda}_m^0\}_{m=1}^\infty$ converges towards infinity as $m \to \infty$ while the whole sequence coincides with that in (1.17).

Indeed, from the orthonormality of $U_m^{\varepsilon_r}(\cdot;\eta_r)$ in $L^2(\varpi^{\varepsilon_r})$, we get the orthonormality of $\widehat{U}_m^0 := \widehat{U}_m^0(\cdot;\widehat{\eta})$ in $L^2(\varpi^0)$ just writing

$$(\widehat{U}_m^{\varepsilon_r}(\cdot;\eta_r), \widehat{U}_p^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^0} = \delta_{m,n}, \quad \forall m,n \in \mathbb{N},$$

and taking limits as $r \to \infty$. This confirms that the sequence (1.37) converge towards infinity as $m \to \infty$.

Let us prove that the sequence (1.37) coincides with that in (1.17). Since for each (ε_r, η_r) we have a spectral problem with the corresponding spectrum (1.8) and the eigenfunctions forming an orthonormal basis of $L^2(\varpi^{\varepsilon_r})$, we can follow the idea of Section 3.1 in [OlShYo92] or Section III.9.1 in [At84] to show the convergence of the whole sequence of eigenvalues $\{\Lambda_m^{\varepsilon_r}(\eta_r)\}_{m=1}^{\infty}$ towards those of (1.10)-(1.13) with conservation of the multiplicity, and that the set $\{\widehat{U}_m^0\}_{m=1}^{\infty}$ forms a basis of $L^2(\varpi^0)$. The fact that the eigenvalues $\widehat{\Lambda}_m^0$ do not depend on $\widehat{\eta}$ is due to the identification performed by means of the change (1.19). However, since we are dealing with a double perturbation, the technique must be adapted and, for the sake of completeness, we provide here the whole proof.

We proceed by contradiction, assuming that there is some Λ^* eigenvalue of (1.10)-(1.13) in the sequence (1.17) which is not in the sequence (1.37). Therefore, for some $m \in \mathbb{N}$:

$$\Lambda^* < \widehat{\Lambda}_{m+1}^0.$$

Let $U^*(\cdot;\widehat{\eta}) \in H_{per}^{1,\eta}(\varpi^0;\Gamma_{lu0})$ be a corresponding eigenfunction that is orthogonal to the constructed sequence of eigenfunctions $\{\widehat{U}_l^0(\cdot;\widehat{\eta})\}_{l=1}^{\infty}$. Then, we consider the function $U_*^{\varepsilon_r}(\cdot;\eta_r) \in H_{per}^{1,\eta_r}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r})$, solution of the problem

$$(\nabla U_*^{\varepsilon_r}(\cdot;\eta_r), \nabla V)_{\varpi^{\varepsilon_r}} = \Lambda^*\,(U^*(\cdot;\widehat{\eta}), V)_{\varpi^{\varepsilon_r}} \qquad \forall V \in H_{per}^{1,\eta_r}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r}).$$

Applying the Poincaré inequality, we obtain that the extension by zero of $U_*^{\varepsilon_r}(\cdot;\eta_r)$ inside the holes, $\{\widehat{U}_*^{\varepsilon_r}(\cdot;\eta_r)\}_r$ constitutes a sequence uniformly bounded in $H^1(\varpi^0)$. Therefore, up to a subsequence, still denoted by r,

$$\widehat{U}_*^{\varepsilon_r}(\cdot;\eta_r) \rightharpoonup U^*(\cdot;\widehat{\eta}) \text{ in } H^1(\varpi^0) - weak, \text{ as } r \to \infty. \tag{1.38}$$

Note that to show the convergence (1.38) we need to rewrite the argument above, cf. (1.30)-(1.36), with minor modifications.

From $U_*^{\varepsilon_r}(\cdot;\eta_r)$ we construct a new function $W_*^{\varepsilon_r}(\cdot;\eta_r)$ orthogonal to the set $\{U_l^{\varepsilon_r}(\cdot;\eta_r)\}_{l=1}^{m}$ in the space $L^2(\varpi^{\varepsilon_r})$ as follows:

$$W_*^{\varepsilon_r}(\cdot;\eta_r) = U_*^{\varepsilon_r}(\cdot;\eta_r) - \sum_{l=1}^{m} (U_*^{\varepsilon_r}(\cdot;\eta_r), U_l^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}} U_l^{\varepsilon_r}(\cdot;\eta_r).$$

In addition, from the above convergence for eigenfunctions, (1.38), the orthogonality of the limit eigenfunctions in $L^2(\varpi^0)$, and the assumption performed on the orthogonality of $U^*(\cdot;\widehat{\eta})$ to the limit eigenfunctions, we can write

$$(U_*^{\varepsilon_r}(\cdot;\eta_r), U_l^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}} \to 0, \quad \text{as } r \to \infty, \quad l = 1, 2, \cdots, m, \tag{1.39}$$

$$\widehat{W}_*^{\varepsilon_r}(\cdot;\eta_r) \rightharpoonup U^*(\cdot;\widehat{\eta}) \text{ in } H^1(\varpi^0) - weak, \text{ as } r \to \infty, \tag{1.40}$$

$\widehat{W}_*^{\varepsilon_r}(\cdot;\eta_r)$ being the extension by zero on the holes of $W_*^{\varepsilon_r}(\cdot;\eta_r)$, and

$$(\nabla W_*^{\varepsilon_r}(\cdot;\eta_r), \nabla W_*^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}} \to \Lambda^*(U^*(\cdot;\widehat{\eta}), U^*(\cdot;\widehat{\eta}))_{\varpi^0}, \quad \text{as } r \to \infty. \tag{1.41}$$

Then, since for each ε_r, we have constructed a function $W_*^{\varepsilon_r}(\cdot;\eta_r) \in \{V \in H^{1,\eta_r}_{per}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r}); (V,(U_l^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}} = 0, l = 1,2,\cdots,m\}$, we can apply the Rayleigh principle, see, for instance Section I.7 in [SaSa89],

$$\Lambda^{\varepsilon_r}_{m+1}(\eta_r) = \inf_V \frac{(\nabla V, \nabla V)_{\varpi^{\varepsilon_r}}}{(V,V)_{\varpi^{\varepsilon_r}}},$$

where the infimum is computed over the elements of the space

$$\{V \in H^{1,\eta_r}_{per}(\varpi^{\varepsilon_r};\Gamma^{\varepsilon_r}) : (V, U_l^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}} = 0,\ l = 1,2,\cdots,m\}.$$

Consequently

$$\Lambda^{\varepsilon_r}_{m+1}(\eta_r) \leq \frac{(\nabla W_*^{\varepsilon_r}(\cdot;\eta_r), \nabla W_*^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}}}{(W_*^{\varepsilon_r}(\cdot;\eta_r), W_*^{\varepsilon_r}(\cdot;\eta_r))_{\varpi^{\varepsilon_r}}},$$

and taking limits as $r \to \infty$, from (1.33) and (1.39)-(1.41) we ready get

$$\widehat{\Lambda}^0_{m+1} \leq \Lambda^*$$

which contradicts our assumption, and we have proved that all the eigenvalues of the homogenized problem in (1.17) are in the sequence $\{\widehat{\Lambda}_0^m\}_{m=1}^\infty$.

Also, this confirms the fact that the set of limiting eigenfunctions $\{\widehat{U}^0_m(\cdot;\widehat{\eta})\}_{m=1}^\infty$ in (1.33) forms an orthogonal basis in $L^2(\varpi^0)$ and the set of limiting eigenvalues (1.37) and (1.17) coincide and are independent on the Floquet-parameter. Therefore, the theorem is proved. □

Theorem 4. *Let $m \in \mathbb{N}$, let Λ^0_m be an eigenvalue of the Dirichlet problem* (1.16) *in the sequence* (1.17). *There exist positive ε_m and c_m independent of η and ε such that, for any $\varepsilon \in (0,\varepsilon_m]$, the eigenvalue $\Lambda^\varepsilon_m(\eta)$ of problem* (1.3)-(1.6) *in the sequence* (1.8) *meets the estimate*

$$|\Lambda^\varepsilon_m(\eta) - \Lambda^0_m| \leq c_m\varepsilon, \quad \forall \varepsilon \leq \varepsilon_m,\ \eta \in [-\pi,\pi]. \tag{1.42}$$

Proof. Let us recall Theorem 2 which provides (1.27) for a certain $p(\varepsilon,\eta,m) \geq m$. Here without any restriction, we can assume that $\Lambda^0_{m+1} > \Lambda^0_m$, otherwise $p(\varepsilon,\eta,m) \geq m+1$ also. Let us show that $p(\varepsilon,\eta,m) = m$ and consequently the result of the statement holds. We proceed by contradiction, denying (1.42).

This implies that there is η^* such that the estimate (1.42) does not hold. That is, for this η^* we can find a $\varepsilon_{\eta^*} \leq \varepsilon_m$ for which $p(\varepsilon_{\eta^*},\eta^*,m) \geq m+1$ (and, obviously, strictly greater than $m+1$ depending on whether the multiplicity of Λ^0_m be greater than 1). First of all, we observe that the numbers ε_{η^*} that we can find must range in a finite set $\{\varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \cdots \varepsilon_{\eta^*,k_{\eta^*}}\}$, because, otherwise, we can take a subsequence $\{\varepsilon_{\eta^*,l}\}_{l=1}^\infty$, $\varepsilon_{\eta^*,l} \to 0$ as $l \to \infty$, for which $p(\varepsilon_{\eta^*,l},\eta^*,m) \geq m+1$. Then, from (1.27) we write

$$\Lambda^{\varepsilon_{\eta^*,l}}_{m+1}(\eta^*) \leq \Lambda^{\varepsilon_{\eta^*,l}}_{p(\varepsilon_{\eta^*,l},\eta^*,m)}(\eta^*) \leq \Lambda^0_m + c_m\varepsilon_{\eta^*,l},$$

and taking limits, as $l \to \infty$, we get a contradiction, see the convergence (1.26), for fixed η^*:

$$\Lambda_{m+1}^0 \leq \Lambda_m^0. \tag{1.43}$$

Note that the limit is independent of η.

Consequently, for each η^* such that (1.42) does not hold, we associate the finite set $\{\varepsilon_{\eta^*,l}\}_{l=1}^{k_{\eta^*}}$ for which $p(\varepsilon_{\eta^*,l}, \eta^*, m) \geq m+1$. In addition, we note that if there is only one η^* for which (1.42) does not hold, taking $\varepsilon_m^* = \min(\varepsilon_m, \varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \cdots \varepsilon_{\eta^*,k_{\eta^*}})$, the inequality (1.42) holds for $\varepsilon \leq \varepsilon_m^*$, and the same occurs if there is only a finite number of η^* for which (1.42) does not hold.

Therefore, we deduce that there is at least one subsequence $\{\eta_r^*\}_{r=1}^\infty$ that converge towards some $\widehat{\eta} \in [-\pi, \pi]$ as $r \to \infty$ such that (1.42) is not satisfied for $\varepsilon_{\eta_r^*,1}, \varepsilon_{\eta_r^*,2}, \cdots \varepsilon_{\eta_r^*,k_{\eta_r^*}}$, $r = 1, 2, \cdots$ while (1.27) holds. Without any restriction we can assume that there is also a subsequence of $\varepsilon_{\eta_r^*}$ converging towards zero as $r \to \infty$. Indeed, let us explain the last assertion in further detail. For the set $\mathscr{J} := \{\eta^* \in [-\pi, \pi] : \text{ (1.42) is not satisfied } \} \subset [-\pi, \pi]$, we consider the associated set of parameters constructed above: $\mathscr{E} := \{\varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \cdots \varepsilon_{\eta^*,k_{\eta^*}}\}_{\eta^* \in \mathscr{J}}$. Either $\mathscr{E}$ has a lower bound $\varepsilon_m^{**} > 0$ or we can extract a sequence $\{\varepsilon_{\eta_r^*}\}_{r=1}^\infty$ converging towards zero as $r \to \infty$, each one associated to a certain value $\eta_r^* \in \mathscr{J}$. In the first case, (1.42) holds for $\varepsilon \leq \varepsilon_m^* := \min(\varepsilon_m^{**}, \varepsilon_m)$ and the proof is ended. In the second case, since the sequence $\{\eta_r^*\}_{r=1}^\infty$ is bounded from above and from below, we can construct a subsequence, still denoted by r, such that

$$(\eta_r^*, \varepsilon_{\eta_r^*}) \to (\widehat{\eta}, 0) \text{ as } r \to \infty.$$

To show that this last assertion leads us to a contradiction, we note that from (1.27) we can write that the corresponding sequence of eigenvalues satisfy

$$\Lambda_{m+1}^{\varepsilon_{\eta_r^*}}(\eta_r^*) \leq \Lambda_{p(\varepsilon_{\eta_r^*}, \eta_r^*, m)}^{\varepsilon_{\eta_r^*}}(\eta_r^*) \leq \Lambda_m^0 + c_m \varepsilon_{\eta_r^*}.$$

Taking limits as $r \to \infty$, from the convergence (1.29) we get again the contradiction (1.43). Therefore, the result of the theorem holds true. □

Acknowledgements

The work has been partially supported by MICINN through PGC2018-098178-B-I00, PID2020-114703GB-I00 and Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000904-S).

References

[At84] Attouch, H.: Variational Convergence for Functions and Operators. Applicable Math. Series, Pitman, London (1984).

[BaPe18] Bakharev, F.L., and Pérez, E.: Spectral gaps for the Dirichlet-Laplacian in a 3-D waveguide periodically perturbed by a family of concentrated masses, *Math. Nachr.* **291**:4, 556-575 (2018).

[BiSo80] Birman, M.Sh., and Solomjak, M.Z.: *Spectral Theory of Selfadjoint Operators in Hilbert Spaces*. [Translated from the 1980 Russian original by S. Khrushchëv and V. Peller], Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht (1987).

[CoPlVa94] Conca, C., Planchard, J., and Vanninathan, M.: *Fluids and Periodic Structures*. RAM: Research in Applied Mathematics, 38. John Wiley & Sons, Ltd., Chichester; Masson, Paris (1995).

[Ge50] Gelfand, I. M.: Expansion in characteristic functions of an equation with periodic coefficients (Russian). *Doklady Akad. Nauk SSSR*, **73**, 1117-1120 (1950).

[GoEtAl21a] Gómez, D., Nazarov, S.A., Orive-Illera, R., and Pérez-Martínez, M.-E.: Remark on justification of asymptotics of spectra of cylindrical waveguides with periodic singular perturbations of boundary and coefficients. *J. Math. Sci. (N.Y.)* **257**: 5, 597-623 (2021); in russian, *Problems in mathematical analysis* **111**, 43–66 (2021).

[GoEtAl21b] Gómez, D., Nazarov, S.A., Orive-Illera, R., and Pérez-Martínez, M.-E.: Spectral gaps in a periodically perforated Neumann strip. Submitted (2021).

[GoEtAl21c] Gómez, D., Nazarov, S.A., Orive-Illera, R., and Pérez-Martínez, M.-E.: Asymptotic stability of the spectrum of a parametric family of homogenization problems associated with a perforated waveguide. In preparation.

[Ku93] Kuchment, P.: *Floquet Theory for Partial Differential Equations*. Birkhäuser Verlag, Basel (1993).

[LoEtAl98] Lobo, M., Oleinik, O.A., Pérez, M.E., and Shaposhnikova, T.A.: On homogenization of solutions of boundary value problems in domains perforated along manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** 3-4, 611-629 (1998) .

[MaKh06] Marchenko, V.A., and Khruslov, E.Y.: *Homogenization of Partial Differential Equations*. Progress in Mathematical Physics, 46. Birkhäuser, Boston (2006).

[NaOrPe19a] Nazarov, S.A., Orive-Illera, R., and Pérez-Martínez, M.-E.: Asymptotic structure of the spectrum in a Dirichlet-strip with double periodic perforations. *Netw. Heterog. Media* **14**, no. 4, 733-757 (2019).

[NaOrPe19b] Nazarov, S.A., Orive-Illera, R., and Pérez-Martínez, M.-E.: On the polarization matrix for a perforated strip. In *Integral Methods in Science and Engineering: Analytic Treatment and Numerical Approximations,* Birkhäuser, New York (2019), pp. 267-281.

[OlShYo92] Oleinik, O.A., Shamaev, A.S., and Yosifia, G.A.: *Mathematical Problems in Elasticity and Homogenization*. North-Holland, Amsterdam (1992).

[ReSi78] Reed, M., and Simon, B.: *Methods of Modern Mathematical Physics. IV. Analysis of operators*. Academic Press, New York-London (1978).

[SaSa89] Sanchez-Hubert, J., and Sanchez-Palencia, E.: *Vibration and Coupling of Continuous Systems. Asymptotic Methods*. Springer-Verlag, Berlin (1989).

[Sk85] Skriganov, M.M.: Geometric and Arithmetic Methods in the Spectral Theory of Multidimensional Periodic Operators. A translation of *Trudy Mat. Inst. Steklov*, **171** (1985); *Proc. Steklov Inst. Math.*, **171**:2 (1987).

[ViLu57] Vishik, M.I., and Lusternik, L.A.: Regular degeneration and boundary layer for linear differential equations with small parameter. *[Russian] Usp. Mat. Nauk* **12**(5), 3-122 (1957). *Amer. Math. Soc. Transl. (2)* **20**, 239–364 (1962).