Asymptotics for spectral problems with rapidly alternating boundary conditions on a strainer Winkler foundation

Delfina Gómez · Sergei A. Nazarov · María-Eugenia Pérez-Martínez

Abstract We consider a spectral homogenization problem for the linear elasticity system posed in a domain Ω of the upper half-space $\mathbb{R}^{3+}$, a part of its boundary Σ being in contact with the plane $\{x_3 = 0\}$. We assume that the surface Σ is traction-free out of small regions T^ε, where we impose Winkler-Robin boundary conditions. This condition links stresses and displacements by means of a symmetric and positive definite matrix-function $M(x)$ and a reaction parameter $\beta(\varepsilon)$ that can be very large when $\varepsilon \to 0$. The size of the regions T^ε is $O(r_\varepsilon)$, where $r_\varepsilon \ll \varepsilon$, and they are placed at a distance ε between them. We provide all the possible spectral homogenized problems depending on the relations between ε, r_ε and $\beta(\varepsilon)$, while we address the convergence, as $\varepsilon \to 0$, of the eigenpairs in the critical cases where some *strange terms* arise on the homogenized Robin boundary conditions on Σ. *New capacity matrices* are introduced to define these strange terms.

Keywords boundary homogenization · spectral perturbations · elasticity · Winkler foundation · capacity matrices · critical relations

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1 Introduction

In this paper, we address the asymptotic behavior of a spectral problem associated with the vibrations of a deformable elastic solid $\Omega \subset \mathbb{R}^{3+} = \{x : x_3 > 0\}$ whose boundary $\partial\Omega$ has a part clamped to an absolutely rigid profile Γ_Ω and the other part $\Sigma \subset \{x : x_3 = 0\}$ in contact with a strainer Winkler foundation which can be modeled by a series of *small springs* periodically placed along Σ, *the reaction regions* T^ε. On these small regions, the boundary conditions are of Winkler-Robin type, also so-called of *spring type*, while outside, they are traction-free. The small regions T^ε have diameter $O(r_\varepsilon)$ and are at a distance ε between them, where ε measures the period of the structure. Here ε and r_ε are two small parameters $r_\varepsilon \ll \varepsilon \ll 1$; see Figure 1.

The elastic coefficients of the small springs are defined through the so-called *Robin reaction matrix*, which we denote by $\beta(\varepsilon)M(x)$. Matrix $M(x)$ depends on the point where the reaction regions T^ε are placed, while the parameter $\beta(\varepsilon)$, which is referred to as *the reaction parameter*, can range from very small to very large. Each T^ε is assumed to be a domain of the plane $\mathbb{R}^2$ homothetic to a fixed domain T, with a Lipschitz boundary. Analyzing the different relations between the three parameters of the problem, ε, r_ε and $\beta(\varepsilon)$, it is crucial to detect several behaviors of vibrations of the structure. We study the asymptotic behavior of the eigenvalues and eigenfunctions, when $\varepsilon \to 0$; this also involves asymptotics for solutions of associated stationary problems.

The stationary problem, for an isotropic homogeneous media, and a surface Σ which is stuck to the plane along the regions T^ε, has been studied in [21] and [6], where they provide a *critical size* of the stuck regions $O(\varepsilon^2)$ (cf. (1) with $r_0 > 0$), which is somehow *classical* in the literature of applied mathematics. For this size, the asymptotic behavior of the solution is intermediate between the extreme cases. Namely, for $r_\varepsilon \gg \varepsilon^2$ the stuck regions are large enough and the body behaves as if the whole Σ is stuck to the plane, for $r_\varepsilon \ll \varepsilon^2$ the stuck regions are very small and the surface behaves as if it were traction-free, while for $r_\varepsilon = O(\varepsilon^2)$ a Winkler-Robin boundary condition is asymptotically imposed as an intermediate condition between Dirichlet and Neumann. It contains the so-called *strange term*, and links stresses and displacements, the elastic coefficients of this spring being given by a constant matrix: the so-called *capacity matrix* (cf. (25) and (26) for $W^{l,\hat{x}} \equiv W^l$).

Here, we deal with a different problem, and obtain the above-mentioned homogenized problems only for a particular relation between the parameters, in the case of the isotropic media (cf. Remark 2). As a matter of fact, in addition to the critical size, it appears a *critical relation for parameters* (cf. (3) with $\beta^* > 0$) which also provides asymptotic behavior of solutions different from extreme cases. Now, several kinds of elastic capacity matrices arise, which are obtained from the microstructure of the problem and depend on the macroscopic variable. This dependence is due to both, the nonhomogeneous media filling Ω and the nonconstant *Robin matrix* M. A formal study of the problem, based on asymptotic expansions, has been addressed in [14] describing convergence as an open problem that we broach here. For the sake of completeness, we provide all the homogenized problems depending on the relations between ε, r_ε and $\beta(\varepsilon)$ (cf. Section 3).

Notice that other different boundary homogenization problems in linear elasticity have been studied in the literature. Let us mention [29] and [16], which treat stationary homogenization problems for the elasticity system in a perforated media along a plane, the size r_ε of the perforations in the plane being $r_\varepsilon = O(\varepsilon)$. Also, [5] considers a cylindrical body, the regions where the displacements vanish being thin bands which are rolled around the body. For the case of a certain non periodical distribution of the regions T^ε, for extreme cases, let us mention [30]. For a strongly oscillating boundary, see [26].

Other papers investigating homogenization problems for the elasticity operator, with the same geometrical configuration here considered, are [17] (for $r_\varepsilon \ll \varepsilon$) and [13] (for $r_\varepsilon = O(\varepsilon)$). Both deal with spectral problems with alternating boundary conditions of Steklov type and, consequently, they strongly differ from our problem. Also the results are very different.

All these works belong to a large class of boundary homogenization problems for several operators, which have been studied for a long time: in this respect, we refer to [14] for an extensive annotated bibliography on vector and scalar problems. Below, we mention just some of the pioneering works in the literature, either because of the geometry or the key words here used. See [23], [36] and [9] for critical sizes and strange terms in scalar problems. See [24], [20] and [8] for different "sieve" scalar models. For the Stokes fluid problem in a perforated domain along a plane, we mention the works [2] and [12] where, also, a so-called *Stokes capacity matrix* appears on the transmission condition on Σ when $r_\varepsilon = O(\varepsilon^2)$; the large parameter $\beta(\varepsilon)$ appears in [12] related to the adsorption process. See [35] for various effects on the perforated walls in fluid models. See [10] for critical parameters in a fluid homogenization problem. We mention [15] and [33] in connection with the homogenization of spectral problems, for the Laplacian, with large parameters on the boundary conditions; the technique cannot be extended to the vectorial case here considered. [14] and the present work represent the first spectral boundary homogenization models with large parameters in elasticity theory; [14] contains the formal procedure which differs completely from the technique here used for justifications.

Let us introduce parameters r_0, β^0 and β^* which play an important role in the description of the homogenized problems. They are defined through three limits:

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon^2} = r_0, \tag{1}$$

$$\lim_{\varepsilon\to 0} r_\varepsilon \beta(\varepsilon) = \beta^0, \tag{2}$$

and

$$\lim_{\varepsilon\to 0} \frac{\beta(\varepsilon) r_\varepsilon^2}{\varepsilon^2} = \beta^*. \tag{3}$$

In the case where $r_0 > 0$ we deal with the *classical critical size* of the reaction regions T^ε mentioned above. (2) provides a relation between sizes of reaction regions and the reaction parameter which is important in determining the *local*

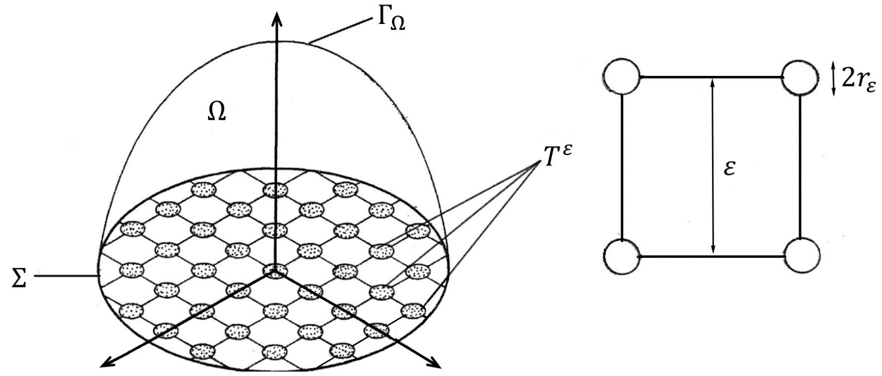

Fig. 1 Geometrical configuration of the problem

problems. The case where $\beta^* > 0$ is referred to as the *critical relation between parameters* (*critical reaction*, in short). It occurs when the total area of the reaction regions $O(\varepsilon^{-2} r_\varepsilon^2)$ multiplied by the reaction parameter $\beta(\varepsilon)$ is of order 1.

The most critical situation happens when $r_0 > 0$ and $\beta^0 > 0$ which also amounts to $r_0 > 0$ and $\beta^* > 0$, cf. the intersecting lines in Figure 2. In this case, the strange term has a character completely different from that obtained in the literature. It contains a so-called *extended capacity matrix* $\mathcal{C}^e(x)$, cf. (20), which depends on the Robin matrix $M(x)$ in a non trivial way. It also contains the parameter β^0.

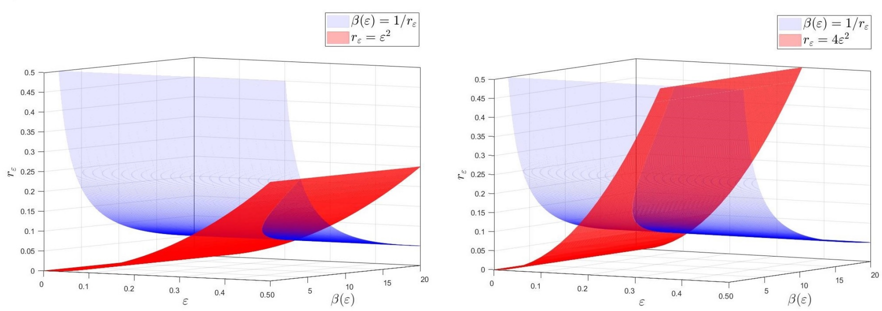

Fig. 2 Two examples of critical relations between parameters

The rest of the critical relations between parameters, for which a spring type boundary condition intermediate between Dirichlet and Neumann is obtained, deal with $r_0 > 0$ and $\beta^0 = +\infty$ or $\beta^* > 0$ and $r_0 = +\infty$. The first one, $r_0 > 0$ and $\beta^0 = +\infty$ (also $\beta^* = +\infty$), asymptotically amounts to regions T^ε stuck to the plane because of the large reaction parameter and, consequently, the spring boundary condition ignores $M(x)$. It contains a new *capacity matrix*

$\mathcal{C}(x)$, which depends on the macroscopic variable x but only due to the nonhomogeneous media filling Ω. The second relation $\beta^* > 0$ and $r_0 = +\infty$, always keeping $r_\varepsilon = O(\beta(\varepsilon)^{-1/2}\varepsilon)$, provides an averaged spring type condition on Σ where the Robin reaction matrix is $M(x)$ multiplied by the average constant $\beta^*|T|$, cf. (27). Let us refer to [28] for other extended capacity matrices in very different problems.

For the sake of brevity, throughout the paper, we address the convergence in the two cases where the strange terms arise, namely $r_0 > 0$ and $\beta^0 > 0$ or $\beta^0 = +\infty$ (cf. Remark 1). In both cases, the local problems providing microscopic information are elasticity problems posed $\mathbb{R}^{3+}$, cf. Figure 3, with the macroscopic variable appearing as a parameter, the corresponding media being homogeneous, but anisotropic, while a nonhomogeneous Winkler-Robin boundary condition (a Dirichlet one, respectively) appears on the unit reaction region T. These problems appear for the first time in homogenization theory and, hence, their correct setting in suitable Hilbert spaces and the smoothness properties of solutions in the above-mentioned parameter are new in the literature (cf. Sections 4 and 7.1). Note that constructing test functions to pass to the limit in the variational formulation of homogenization problems relies on the solutions of these problems (cf. Sections 5-6 and 7.2-7.3).

On the other hand, it should be emphasized that the usual techniques for scalar problems (cf. [3], [24], [20] and [33]) based on results of convergence of measures on manifolds and comparison of measures do not work for the elasticity system under consideration, and therefore, we use a technique based on projections over spaces of finite elements (cf. [23], [32] and [22] in this connection).

Finally, the structure of the paper is as follows. Section 2 contains the setting of the spectral homogenization problem. The corresponding stationary problem and some preliminary results are collected in Section 2.1. Section 3 presents the list of homogenized problems both stationary and spectral problems. It also describes the corresponding stationary local problems which allow us to define the strange terms. Throughout Sections 4-7, we show the convergence. Further specifying, for $r_0 > 0$ and $\beta^0 > 0$, Section 4 contains the setting of the parametric family of local problems in the suitable Hilbert spaces, as well as certain smoothness properties of solutions in the macroscopic variable (the parameter). Section 5 deals with the construction of test functions, and Section 6 addresses the convergence of solutions and spectra. Section 7 contains proofs for $r_0 > 0$ and $\beta^0 = +\infty$.

2 The setting of the problem

Let Ω be a bounded domain of $\mathbb{R}^3$ situated in the upper half-space $\mathbb{R}^{3+} = \{x \in \mathbb{R}^3 : x_3 > 0\}$, with a Lipschitz boundary $\partial\Omega$. Let Σ be the part of the boundary in contact with the plane $\{x_3 = 0\}$ which is assumed to be nonempty and let Γ_Ω be the rest of the boundary of Ω: $\partial\Omega = \overline{\Gamma}_\Omega \cup \overline{\Sigma}$. Let T denote a bounded domain of the plane $\{x_3 = 0\}$ with a Lipschitz boundary.

Without any restriction we can assume that both Σ and T contain the origin of coordinates while $|\Sigma|$ and $|T|$ stand for their surface measures.

Let ε be a small parameter, $\varepsilon \ll 1$, and r_ε be another parameter such that $r_\varepsilon \ll \varepsilon$. For $\mathrm{k} = (k_1, k_2) \in \mathbb{Z}^2$, we denote by $\widetilde{x}^\varepsilon_{\mathrm{k}}$ the point of the plane $\{x_3 = 0\}$ with coordinates $\widetilde{x}^\varepsilon_{\mathrm{k}} = (k_1\varepsilon, k_2\varepsilon, 0)$, and by $T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}$ the homothetic domain of T of ratio r_ε after translation to the point $\widetilde{x}^\varepsilon_{\mathrm{k}}$:

$$T^\varepsilon_{\widetilde{x}_{\mathrm{k}}} = \widetilde{x}^\varepsilon_{\mathrm{k}} + r_\varepsilon T\,.$$

If there is no ambiguity, we shall write $\widetilde{x}_{\mathrm{k}}$ instead of $\widetilde{x}^\varepsilon_{\mathrm{k}}$, and T^ε instead of $T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}$.

In this way, for a fixed ε, we have constructed a grid of squares in the plane $\{x_3 = 0\}$ whose vertices are inside the regions T^ε (cf. Figure 1). Let $\mathcal{J}^\varepsilon$ denote $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 \,:\, T^\varepsilon_{\widetilde{x}_{\mathrm{k}}} \subset \Sigma\}$, while N_ε stands for the number of elements of $\mathcal{J}^\varepsilon$:

$$N_\varepsilon \approxeq \frac{|\Sigma|}{\varepsilon^2} = O(\varepsilon^{-2}). \tag{4}$$

Finally, if no confusion arises, $\bigcup T^\varepsilon$ implies the union of all the T^ε contained in Σ:

$$\bigcup T^\varepsilon \equiv \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}\,.$$

In what follows $x = (x_1, x_2, x_3)$ denotes the usual cartesian coordinates, while by $\hat{x} = (x_1, x_2)$ we refer to the two first components of $x \in \mathbb{R}^3$. Also, we use the summation convention over repeated indexes.

Under the basis that the domain Ω is filled by an elastic material, for $i, j, k, l = 1, 2, 3$, we denote by $a_{ijkl}(x)$ the elastic coefficients of the material, which are assumed to be $C(\overline{\Omega})$ functions and satisfy the standard symmetry and coercivity properties (see, e.g., [31] and [37]):

$$\begin{aligned} &a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x), \quad i,j,k,l = 1,2,3, \quad \forall x \in \overline{\Omega} \quad \text{and} \\ &\exists \gamma_1 > 0 \,:\, a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \gamma_1 \xi_{ij}\xi_{ij} \;\; \forall \xi \text{ symmetric matrix}, \ \xi_{ij} \in \mathbb{R}, \ \ \forall x \in \overline{\Omega}, \end{aligned} \tag{5}$$

$\xi = (\xi_{ij})_{i,j=1,2,3}$. For a given displacement vector $u(x) = (u_1(x), u_2(x), u_3(x))$ we use the standard notations for stress and strain tensors $\sigma(u)$ and $e(u)$; namely, we denote by $(\sigma_{ij}(u))_{i,j=1,2,3}$ the stress tensor which is related to the strain tensor $(e_{ij}(u))_{i,j=1,2,3}$ by the Hooke's law

$$\sigma_{ij}(u) = a_{ijkl}(x) e_{kl}(u), \qquad \text{with} \qquad e_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big). \tag{6}$$

In connection with the boundary conditions on T^ε, let us introduce a 3×3-symmetric matrix M, with $M_{ij} \in C(\overline{\Sigma})$, $i, j = 1, 2, 3$, and positive definite, namely,

$$\exists \gamma_2 > 0 \,:\, \bar{\alpha} M(x_1, x_2, 0)\bar{\alpha}^\top \geq \gamma_2 \bar{\alpha}\bar{\alpha}^\top \quad \forall \bar{\alpha} \in \mathbb{R}^3 \quad \forall (x_1, x_2, 0) \in \overline{\Sigma}, \tag{7}$$

where $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\top$ indicates transposition.

Let us consider the spectral problem

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^{\varepsilon}}{\partial x_j} = \lambda^{\varepsilon} u_i^{\varepsilon} & \text{in } \Omega\,, \\ u^{\varepsilon} = 0 & \text{on } \Gamma_{\Omega}\,, \\ \sigma_{ij}^{\varepsilon} n_j = 0 & \text{on } \Sigma \setminus \bigcup T^{\varepsilon}, \\ \sigma_{ij}^{\varepsilon} n_j + \beta(\varepsilon) M_{ij} u_j^{\varepsilon} = 0 & \text{on } \bigcup T^{\varepsilon}\,, \end{cases} \qquad i = 1,2,3, \tag{8}$$

where λ^{ε} is the spectral parameter, and $u^{\varepsilon} = (u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon})$ the corresponding eigenvector. u^{ε} is related to stress and strain tensors by (6). In particular, in (8), we have set

$$\sigma_{ij}^{\varepsilon} := \sigma_{ij}(u^{\varepsilon}) = a_{ijkl} e_{kl}(u^{\varepsilon}), \tag{9}$$

while $\bar{n}$ represents the unit outer normal to Ω along Σ, namely, $\bar{n} = (0, 0, -1)$. The parameter $\beta(\varepsilon)$ arising in the equations on T^{ε} is positive and can range from very large to very small; in particular, it can be of order one.

2.1 Some background

Let us denote by $\mathbf{V}$ the space obtained by completion of $\{v \in (C^1(\overline{\Omega}))^3 : v = 0 \text{ on } \Gamma_{\Omega}\}$ in the norm generated by the scalar product, *an elastic pseudo-energy bilinear form*

$$(u, v)_{\mathbf{V}} = \int_{\Omega} e_{ij}(u) e_{ij}(v)\, dx\,. \tag{10}$$

For fixed $\varepsilon > 0$, the weak formulation of problem (8) reads: find $\lambda^{\varepsilon} \in \mathbb{R}$, $u^{\varepsilon} \in \mathbf{V}$, $u^{\varepsilon} \neq 0$, satisfying

$$\int_{\Omega} \sigma_{ij}(u^{\varepsilon}) e_{ij}(v)\, dx + \beta(\varepsilon) \int_{\bigcup T^{\varepsilon}} M_{ij} u_i^{\varepsilon} v_j\, d\hat{x} = \lambda^{\varepsilon} \int_{\Omega} u_i^{\varepsilon} v_i\, dx \qquad \forall v \in \mathbf{V}. \tag{11}$$

On account of (5) and (7), the left hand side of (11) defines a bilinear, symmetric continuous and coercive form on $\mathbf{V} \subset (L^2(\Omega))^3$. Consequently, (11) has the discrete spectrum:

$$0 < \lambda_1^{\varepsilon} \leq \lambda_2^{\varepsilon} \leq \cdots \leq \lambda_n^{\varepsilon} \leq \cdots \xrightarrow{n \to \infty} +\infty, \tag{12}$$

where we have adopted the convention of repeated eigenvalues according to their multiplicities. The corresponding vector eigenfunctions form a basis in $\mathbf{V}$ and $(L^2(\Omega))^3$, and we assume that they are subject to the orthonormalization condition

$$\left(u^{n,\varepsilon}, u^{n,\varepsilon}\right)_{(L^2(\Omega))^3} = \delta_{n,m}. \tag{13}$$

Based on the minimax principle we obtain the uniform bound:

$$0 < C \leq \lambda_n^{\varepsilon} \leq C_n \qquad \forall \varepsilon > 0, \tag{14}$$

where C and C_n are constants independent of ε. For the sake of completeness, we outline this proof, cf. [14] for further details.

The left hand side of (14) is obtained using the Poincaré and Korn inequalities (cf., e.g., [31] and [37]), (5) and (7) . Indeed, we have

$$\lambda_n^\varepsilon \geq \lambda_1^\varepsilon = \frac{\int\limits_\Omega \sigma_{ij}(u^{1,\varepsilon})e_{ij}(u^{1,\varepsilon})\,dx + \beta(\varepsilon)\int\limits_{\bigcup T^\varepsilon} M_{ij}u_i^{1,\varepsilon}u_j^{1,\varepsilon}\,d\hat{x}}{\int\limits_\Omega u_i^{1,\varepsilon}u_i^{1,\varepsilon}\,dx,} \geq C.$$

For the right hand side, we write

$$\lambda_n^\varepsilon = \min_{E_n\subset\mathbf{V}}\ \max_{v\in E_n, v\not\equiv 0} \frac{\int\limits_\Omega \sigma_{ij}(v)e_{ij}(v)\,dx + \beta(\varepsilon)\int\limits_{\bigcup T^\varepsilon} M_{ij}v_iv_j\,d\hat{x}}{\int\limits_\Omega v_iv_i dx}$$

$$\leq \max_{v\in E_n^*, v\not\equiv 0} \frac{\int\limits_\Omega \sigma_{ij}(v)e_{ij}(v)\,dx}{\int\limits_\Omega v_iv_i dx} = \lambda_n^0,$$

where the minimum has been taken over the set of all the subspaces E_n of $\mathbf{V}$ of dimension n. For the last inequality, we have taken the particular space E_n^* generated by the eigenvectors $[u^{1,0}, u^{2,0}, \cdots, u^{n,0}]$ corresponding to the eigenvalues $\{\lambda_1^0, \lambda_2^0, \cdots \lambda_n^0\}$ of the Dirichlet problem

$$\int\limits_\Omega \sigma_{ij}(u^0)e_{ij}(v)\,dx = \lambda^0 \int\limits_\Omega u_i^0 v_i\,dx \quad \forall v \in (H_0^1(\Omega))^3 \tag{15}$$

(cf. also (29), (32)-(33)). Therefore, (14) holds true.

In this paper, we address the asymptotic behavior of $(\lambda^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$, depending on the different values of r_0, β^0 and β^* in (1), (2) and (3) respectively. The proof of the convergence (cf. Theorems 4 and 8) is based on a general result on spectral perturbation theory (cf. Section III.1 of [31] and Section III.9.1 in [3]). In order to be self-contained, we introduce below a simplified version of such a result, cf. Lemma 1.

On account of this result, (13) and (14), showing the convergence for the eigenpairs of (11) amounts to showing the convergence of solutions of associated stationary problems. Hence, it proves useful to introduce here the stationary homogenization problem:

Find $u^\varepsilon \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(v)\,dx + \beta(\varepsilon)\int_{\bigcup T^\varepsilon} M_{ij}u_i^\varepsilon v_j\,d\hat{x} = \int_\Omega f_iv_i\,dx \qquad \forall v \in \mathbf{V}, \tag{16}$$

where $f = (f_1, f_2, f_3) \in (L^2(\Omega))^3$ represent given forces acting on the body.

Because of the Korn and Poincaré inequalities, (5) and (7), the unique solution of (16) satisfies

$$\|u^\varepsilon\|_{\mathbf{V}} \leq C, \tag{17}$$

with C a constant independent of ε, cf. (10). Therefore, for each sequence of $\{u^\varepsilon\}_{\varepsilon>0}$ we can extract a subsequence, still denoted by ε, such that

$$u^\varepsilon \xrightarrow{\varepsilon\to 0} u^0 \quad \text{weakly in } (H^1(\Omega))^3, \tag{18}$$

for some $u^0 \in \mathbf{V} \subset \{v \in (H^1(\Omega))^3 \,:\, v = 0 \text{ on } \Gamma_\Omega\}$ (cf., e.g., [4], [31] and [37]). As usual in homogenization, we aim to identify u^0 with the solution of a homogenized problem. In Section 3, we provide the list of possible stationary homogenized problems depending on the relations between the parameters ε, r_ε and $\beta(\varepsilon)$.

The following result links convergence of stationary and spectral problems; we refer to Lemma 1.6 in Section III.1 of [31] for the proof.

Lemma 1 *Let $\mathbf{H}$ be a separable Hilbert space with the norm $\|\cdot\|$. Let $\mathcal{L}(\mathbf{H})$ denote the space of continuous linear operators on $\mathbf{H}$. Let $\mathcal{A}^\varepsilon$, $\mathcal{A}^0 \in \mathcal{L}(\mathbf{H})$ and $\mathcal{W}$ be a subspace of $\mathbf{H}$ such that $Im\,\mathcal{A}^0 = \{v \,|\, v = \mathcal{A}^0 u \,:\, u \in \mathbf{H}\} \subset \mathcal{W}$. We assume that the following properties are satisfied:*

- *i1). $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ are positive, compact and self-adjoint operators on $\mathbf{H}$ and $\|\mathcal{A}^\varepsilon\|_{\mathcal{L}(\mathbf{H})} \le \mathbf{c}$, where $\mathbf{c}$ denotes a constant independent of ε.*
- *i2). For any $f \in \mathcal{W}$, $\|\mathcal{A}^\varepsilon f - \mathcal{A}^0 f\| \to 0$ as $\varepsilon \to 0$.*
- *i3). The family of operators $\mathcal{A}^\varepsilon$ is uniformly compact, that is, for any sequence $f^\varepsilon \in \mathbf{H}$ such that $\sup_\varepsilon \|f^\varepsilon\| \le \mathbf{c}$, we can extract a subsequence $f^{\varepsilon'}$ satisfying $\|\mathcal{A}^{\varepsilon'} f^{\varepsilon'} - w^0\|_{\varepsilon'} \to 0$, as $\varepsilon' \to 0$, for a certain $w^0 \in \mathcal{W}$.*

Let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ ($\{\mu_i^0\}_{i=1}^\infty$, respectively) be the sequence of the eigenvalues of $\mathcal{A}^\varepsilon$ ($\mathcal{A}^0$, respectively) with the usual convention of repeated eigenvalues. Let $\{w_i^\varepsilon\}_{i=1}^\infty$ and ($\{w_i^0\}_{i=1}^\infty$, respectively) be the corresponding eigenvectors which are assumed to form an orthonormal basis in $\mathbf{H}$.

Then, for each fixed k, $\mu_k^\varepsilon \to \mu_k^0$, as $\varepsilon \to 0$. In addition, for each sequence, still denoted by ε, we can extract a subsequence $\varepsilon' \to 0$ such that

$$\|\mathcal{A}^{\varepsilon'} w_k^{\varepsilon'} - w_k^*\| \to 0 \quad \text{as} \quad \varepsilon' \to 0\,,$$

where w_k^* is an eigenvector of $\mathcal{A}^0$ corresponding to μ_k^0 and the set $\{w_i^*\}_{i=1}^\infty$ forms an orthogonal basis of $\mathbf{H}$.

3 The homogenized problems

In this section, for the sake of completeness, we state all the stationary homogenized problems. They can be obtained as in [14], using the technique of matched asymptotic expansions, with minor modifications. We also state the local problems that allow us to describe the strange terms in the boundary conditions.

P1). In the most critical situation where $\beta^0 > 0$ and $r_0 > 0$, the homogenized problem reads: Find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij,x}(u^0) e_{ij,x}(v)\,dx + r_0 \int_\Sigma \mathcal{C}^e_{ij}(\hat{x}) u^0_i v_j\,d\hat{x} = \int_\Omega f_i v_i\,dx \quad \forall v \in \mathbf{V}, \tag{19}$$

where, for $\hat{x} \in \Sigma$, the matrix $\mathcal{C}^e = (\mathcal{C}^e_{ij})_{i,j=1,2,3}$ is defined as

$$\mathcal{C}^e_{ij}(\hat{x}) = \int_T \sigma^{\hat{x}}_{i3,y}(W^{j,M,\hat{x}})d\hat{y}, \tag{20}$$

$W^{l,M,\hat{x}}$ being the solution of the $M(\hat{x})$-*dependent local problem*

$$\begin{cases} -\dfrac{\partial \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})}{\partial y_j} = 0 & \text{in } \mathbb{R}^{3+}, \\ \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})n_j = 0 & \text{on } \{y_3 = 0\} \setminus T, \\ \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})n_j - \beta^0 M_{ij}(\hat{x})(\mathrm{e}^l_j - W^{l,M,\hat{x}}_j) = 0 & \text{on } T, \\ W^{l,M,\hat{x}}(y) \longrightarrow 0 & \text{as } |y| \to \infty,\ y_3 > 0, \end{cases} \tag{21}$$

$i = 1,2,3$. Above, and in what follows, $y = (y_1, y_2, y_3)$ are auxiliary variables in $\mathbb{R}^3$ (cf. (23)), and lower indexes x or y in the components of the stress and strain tensors mean the variable for derivation. The upper index $\hat{x}$ is a parameter which refers to the elastic homogeneous media with constant elastic coefficients $a_{ijkl}(\hat{x})$. Namely,

$$\sigma^{\hat{x}}_{ij,y}(V) = a_{ijkl}(\hat{x}) e_{kl,y}(V). \tag{22}$$

Also, e^l stands for the unit vector in the y_l-direction, while $l = 1,2,3$. Macroscopic and local variables, as usual, are related by

$$y = \frac{x - \widetilde{x}_{\mathrm{k}}}{r_\varepsilon}. \tag{23}$$

According to Proposition 2, using the Korn and Poincaré inequalities we deduce that there exists a unique solution of (19) in the space $\mathbf{V}$.

P2). For the critical size $r_0 > 0$, when $\beta^0 = +\infty$, the homogenized problem reads: Find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij,x}(u^0) e_{ij,x}(v)\,dx + r_0 \int_\Sigma \mathcal{C}_{ij}(\hat{x}) u^0_i v_j\,d\hat{x} = \int_\Omega f_i v_i\,dx \quad \forall v \in \mathbf{V}, \tag{24}$$

where the matrix $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1,2,3}$ is defined as

$$\mathcal{C}_{ij}(\hat{x}) = -\big\langle \sigma_{i3,y}(W^{j,\hat{x}}), 1 \big\rangle_{H^{-1/2}(T) \times H^{1/2}(T)}, \tag{25}$$

$W^{l,\hat{x}}$ being the solution of the $\hat{x}$-*dependent local problem*

$$\begin{cases} -\dfrac{\partial\sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}})}{\partial y_j} = 0 & \text{in } \mathbb{R}^{3+} \\ \sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}})n_j = 0 & \text{on } \{y_3 = 0\} \setminus T \\ W^{l,\hat{x}}(y) = \mathrm{e}^l & \text{on } T \\ W^{l,\hat{x}}(y) \longrightarrow 0 & \text{as } |y| \to \infty,\, y_3 > 0 \end{cases}, \qquad i = 1,2,3, \tag{26}$$

$\sigma^{\hat{x}}_{ij,y}$ and e^l in (25) and (26) are defined as in the previous item, cf. (22). As in item $P1$), problem (24) has a unique solution (cf. Proposition 6).

P3). For the critical relation where $\beta^* > 0$ with $r_0 = +\infty$, the homogenized problem reads: Find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij,x}(v)\,dx + \beta^*|T| \int_\Sigma M_{ij}(\hat{x})u^0_i v_j\,d\hat{x} = \int_\Omega f_i v_i\,dx \quad \forall v \in \mathbf{V}. \tag{27}$$

P4). For the extreme cases where $\beta^* = 0$ or $r_0 = 0$, the homogenized problem is the mixed boundary value problem: Find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij,x}(v)\,dx = \int_\Omega f_i v_i\,dx \quad \forall v \in \mathbf{V}. \tag{28}$$

P5). For the extreme cases where $r_0 = +\infty$ and, $\beta^0 > 0$, or $\beta^0 = +\infty$, or $\beta^0 = 0$ and $\beta^* = +\infty$, the homogenized problem is the Dirichlet problem: Find $u^0 \in (H^1_0(\Omega))^3$ satisfying

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij}(v)\,dx = \int_\Omega f_i v_i\,dx \quad \forall v \in (H^1_0(\Omega))^3. \tag{29}$$

The existence and uniqueness of solution of (29) and (28) are classical while that of (27) holds as that of (16).

In Sections 6 and 7, we show the convergence of the solutions and the corresponding spectra in the two critical cases $P1$) and $P2$) (cf. Remark 2 for the rest). Hence, for convenience, we introduce here the associated spectral problems

$$\begin{cases} -\dfrac{\partial\sigma_{ij,x}(u^0)}{\partial x_j} = \lambda^0 u^0_i & \text{in } \Omega \\ u^0 = 0 & \text{on } \Gamma_\Omega \\ \sigma_{ij,x}(u^0)n_j + r_0\mathcal{C}^e_{ij}u^0_j = 0 & \text{on } \Sigma \end{cases}, \qquad i = 1,2,3, \tag{30}$$

when $r_0 > 0$ and $\beta^0 > 0$, and

$$\begin{cases} -\dfrac{\partial\sigma_{ij,x}(u^0)}{\partial x_j} = \lambda^0 u^0_i & \text{in } \Omega \\ u^0 = 0 & \text{on } \Gamma_\Omega \\ \sigma_{ij,x}(u^0)n_j + r_0\mathcal{C}_{ij}u^0_j = 0 & \text{on } \Sigma \end{cases}, \qquad i = 1,2,3, \tag{31}$$

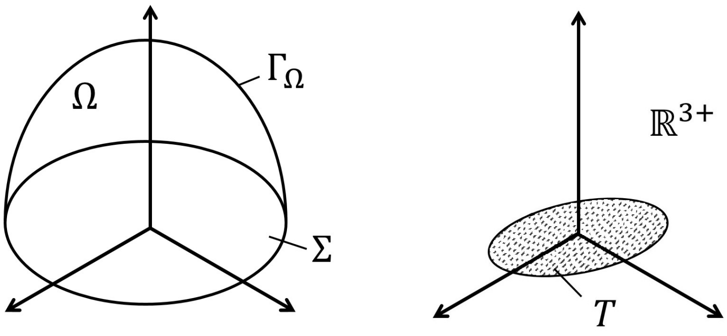

Fig. 3 The domains of setting for homogenized and local problems

when $r_0 > 0$ and $\beta^0 = +\infty$. We recall the different definition of the elastic capacity matrices $\mathcal{C}^e = (\mathcal{C}^e_{ij})_{i,j=1,2,3}$ and $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1,2,3}$ appearing in (30) and (31). They depend on the macroscopic variable. However, this dependence for $\mathcal{C}^e_{ij}$ is due to both, the nonhomogeneous media and the nonconstant Robin matrix M (cf. (20) and (21)), while that for $\mathcal{C}_{ij}$ ignores M (cf. (25) and (26)).

The variational formulation of (30) and (31) reads: Find $\lambda^0 \in \mathbb{R}$, $u^0 \in \mathbf{V}$, $u^0 \neq 0$, satisfying

$$\int_\Omega \sigma_{ij,x}(u^0) e_{ij,x}(v)\, dx + r_0 \int_\Sigma \mathcal{B}_{ij} u^0_i v_j \, d\hat{x} = \lambda^0 \int_\Omega u^0_i v_i \, dx \quad \forall v \in \mathbf{V}, \tag{32}$$

where $\mathcal{B} = \mathcal{C}^e$ when dealing with (30) and $\mathcal{B} = \mathcal{C}$ when dealing with (31). We denote the discrete spectrum by

$$0 < \lambda^0_1 \leq \lambda^0_2 \leq \cdots \leq \lambda^0_n \leq \cdots \xrightarrow{n\to\infty} +\infty, \tag{33}$$

where we have adopted the convention of repeated index. Also, we can choose the corresponding vector eigenfunctions $\{u^{n,0}\}_{n=1}^{\infty}$ to form an orthonormal basis in $(L^2(\Omega))^3$.

From the definitions of $\mathcal{C}^e(\hat{x})$ and $\mathcal{C}(\hat{x})$, cf. (20) and (25), it is self-evident that the discreteness of the spectrum of problems (30) and (31) is linked to the setting of problems (21) and (26), as well as to the properties of their respective solutions. All this is addressed in Sections 4 and 7.1.

Note that (32) is also the spectral problem associated with (28) when $r_0 = 0$, and that associated with (27) when we replace matrix $r_0\mathcal{B}$ by the *averaged Robin reaction matrix* $\beta^*|T|M$, $\beta^* > 0$ in (3). The spectral Dirichlet problem associated with (29) is (15).

4 The parametric family of local problems

In this section, we describe certain properties of the solutions of the parametric family of local problems (21) which are necessary for the correct setting of the homogenized problem (19), cf. (20). They are also necessary to obtain appropriate estimates for test functions, cf. Section 5.

Let $(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ be the space of functions which are the restrictions to $\overline{\mathbb{R}^{3+}}$ of the elements of $(\mathcal{D}(\mathbb{R}^3))^3$. Consider the space $\mathcal{V}$, completion of $(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ with respect to the norm

$$\|U\|_{\mathcal{V}} = \Big(\sum_{i,j=1}^{3} \|e_{ij,y}(U)\|^2_{L^2(\mathbb{R}^{3+})} + \sum_{i=1}^{3} \|U_i\|^2_{L^2(T)} \Big)^{1/2}. \tag{34}$$

Due to Korn's inequality in bounded Lipschitz domains of $\mathbb{R}^{3+}$ whose boundary contains $\overline{T}$, the continuous embedding $\mathcal{V} \subset (H^1_{loc}(\mathbb{R}^{3+}))^3$ holds.

For each fixed $l = 1,2,3$ and $\hat{x} \in \overline{\Sigma}$, problem (21) has the variational formulation: Find $W^{l,M,\hat{x}} \in \mathcal{V}$ satisfying

$$\begin{aligned} \int_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}}) e_{ij,y}(V)\, dy + \beta^0 M_{ij}(\hat{x}) \int_T W_i^{l,M,\hat{x}} V_j \, d\hat{y} \\ = \beta^0 M_{il}(\hat{x}) \int_T V_i \, d\hat{y} \quad \forall V \in \mathcal{V}. \end{aligned} \tag{35}$$

Indeed, it is simple to verify that (35) is the weak formulation of $(21)_1$-$(21)_3$, while the condition at infinity $(21)_4$ for the solution of (35) is obtained as a consequence of the following theorem which provides the precise convergence rate at infinity.

Theorem 1 *For each $\hat{x} \in \overline{\Sigma}$ and $l = 1,2,3$, problem (35) has a unique solution $W^{l,M,\hat{x}} \in \mathcal{V}$ and it can be represented in terms of the Green matrix-function $G^{\hat{x}}(y)$ as follows:*

$$W_i^{l,M,\hat{x}}(y_1,y_2,y_3) = \int_T \sigma_j^{l,\hat{x}}(\xi_1,\xi_2)\, G^{\hat{x}}_{ij}(y_1-\xi_1, y_2-\xi_2, y_3)\, d\xi_1 d\xi_2, \quad i = 1,2,3. \tag{36}$$

Here,

$$\sigma^{l,\hat{x}} = (\sigma_1^{l,\hat{x}}, \sigma_2^{l,\hat{x}}, \sigma_3^{l,\hat{x}}) := (\sigma^{\hat{x}}_{13,y}(W^{l,M,\hat{x}}), \sigma^{\hat{x}}_{23,y}(W^{l,M,\hat{x}}), \sigma^{\hat{x}}_{33,y}(W^{l,M,\hat{x}}).$$

and $G^{\hat{x}} = (G^{\hat{x}}_{ij})_{i,j=1,2,3}$ is a symmetric tensor which depends on the elastic constants of the media $a_{ijkp}(\hat{x})$ and admits the representation

$$G^{\hat{x}}(y) = |y|^{\Lambda} \Phi^{\hat{x}}(\omega) \quad \text{with} \quad \Lambda = -1, \quad y \in \mathbb{R}^{3+}, \quad \omega \in \overline{S^2_+}, \tag{37}$$

where $\Phi^{\hat{x}}$ is a symmetric matrix whose elements are smooth functions on the unit semi-sphere in $\mathbb{R}^{3+}$, $\overline{S^2_+} = \{y \in \mathbb{R}^{3+} : |y| = 1\} \ni \omega$. In addition, $\Phi^{\hat{x}}$ depends continously on the parameter $\hat{x} \in \overline{\Sigma}$, in such a way that for $i,j = 1,2,3$,

$$\max_{\hat{x}\in\overline{\Sigma},\, \omega\in\overline{S^2_+}} |\Phi^{\hat{x}}_{ij}(\omega)| \le C \quad \text{and} \quad \max_{\hat{x}\in\overline{\Sigma},\, \omega\in\overline{S^2_+}} \left|\nabla_\omega \Phi^{\hat{x}}_{ij}(\omega)\right| \le C, \tag{38}$$

where ∇_ω is the gradient-operator on the sphere, and C a certain constant.

Proof In order to show the existence and uniqueness of solution of (35), we denote by $a_{\hat{x}}(.,.)$ the bilinear, symmetric, continuous and coercive form on $\mathcal{V}$:

$$a_{\hat{x}}(U,V) = \int_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(U) e_{ij,y}(V)\, dy + \beta^0 M_{ij}(\hat{x}) \int_T U_i V_j \, d\hat{y} \quad \forall U, V \in \mathcal{V}.$$

On account of (5) and (7), cf. (22), $a_{\hat{x}}(.,.)$ defines a norm in $\mathcal{V}$ equivalent to $\|\cdot\|_{\mathcal{V}}$. Also, we consider the linear continuous functional on $\mathcal{V}$

$$F_{l,\hat{x}}(U) = \beta^0 M_{il}(\hat{x}) \int_T U_i \, d\hat{y} \quad \forall U \in \mathcal{V}.$$

Then, the Riesz representation theorem ensures that there exists a unique function $W^{l,M,\hat{x}} \in \mathcal{V}$ satisfying $a_{\hat{x}}(W^{l,M,\hat{x}}, V) = F_{l,\hat{x}}(V) \quad \forall V \in \mathcal{V}$. This is nothing but (35).

The representation (36) for the solution of (21) can be derived as that for Bussinesq-Cerutti tensor in the case of an isotropic media, but without explicit computations of the components of the Green matrix-function (37): see, e.g., [19] for the Bussinesq-Cerutti tensor; see also [1] and [18] for the Mindlin tensor and other related tensors. Since the half-space is a cone with generator the semi-sphere, for anisotropic media, the representation (36) is supported by general results in [25]. Indeed, explicit formulas for the Green matrix-function and accompanying tensors are known in the case of isotropy and, for their existence and main properties in anisotropic media, we refer to Sections 2 and 5 of [25] where more general boundary value problems for elliptic systems in conical domains are considered.

To conclude on the structure (37) of the matrix function $G^{\hat{x}}$ and its continuous dependence on the parameter $\hat{x}$ as well as the representation formula (36), it suffices to mention some basic facts: first, the columns of the amplitude part in (37) are eigenfunctions of a pencil $\mathfrak{A}(\Lambda)$ of differential operators on the unit sphere and its equator corresponding to the eigenvalue $\Lambda = -1 \in \mathbb{C}$. Second, according to the usual Green's formula for the elasticity equations in $\mathbb{R}^{3+}$, the adjoint pencil for $\mathfrak{A}(\Lambda)$ is nothing but $\mathfrak{A}(1-\bar{\Lambda})$ so that $\mathfrak{A}(-1)^* = \mathfrak{A}(0)$. Third, the eigenspace of $\mathfrak{A}(0)$ consists of constant vector functions because any solution $|y|^{\Lambda}\Psi(\omega)$ with $\Lambda = 0$ is a translational rigid motion. Finally, the eigenvalue $\Lambda = 0$ and, the eigenvalue $\Lambda = -1$, are algebraically simple and have geometrical multiplicity 3 the number of translations in $\mathbb{R}^3$. The above-listed information provides all desired properties of the Green matrix-function on the basis of the theory of non-selfadjoint operators [11], see also the monographs [27] and [18].

To show that the function with components defined by the right-hand side of (36) belongs to $\mathcal{V}$, we follow the technique based on a density argument in Theorem 4.1 of [21], with minor modifications, taking into account that $\sigma^{l,\hat{x}}$ denotes the normal component of the stress tensor, corresponding to $W^{l,M,\hat{x}} \in \mathcal{V}$, on the plane $\{y_3 = 0\}$, which has a compact support on $\overline{T}$. Consequently, (36) is the solution of (35), and this concludes with the proof of the theorem.

Proposition 1 *For $l = 1, 2, 3$, the solution $W^{l,M,\hat{x}}$ depends continuously on $\hat{x} \in \overline{\Sigma}$ in the norm of $\mathcal{V}$. In addition, for $l, p, i, j = 1, 2, 3$, the functions*

$$\int\limits_{\mathbb{R}^{3+}} e_{ij,y}(W^{l,M,\hat{x}}) e_{ij,y}(W^{p,M,\hat{x}}) dy, \quad \int\limits_{T} W_i^{l,M,\hat{x}} W_j^{p,M,\hat{x}} d\hat{y} \quad \text{and} \quad \int\limits_{T} \sigma^{\hat{x}}_{i3,y}(W^{l,M,\hat{x}}) d\hat{y}$$

are continuous in $\overline{\Sigma}$, too.

Proof Let us show that for each $\eta > 0$, there is $\delta_\eta > 0$ such that if $\hat{x}, \widehat{x'} \in \Sigma$ satisfy $|\hat{x} - \widehat{x'}| < \delta_\eta$ then $\|W^{l,M,\hat{x}} - W^{l,M,\widehat{x'}}\|_{\mathcal{V}} \leq \eta$.

Let us consider the integral identity (35) with $V = W^{l,M,\hat{x}} - W^{l,M,\widehat{x'}}$ as well as the same identity for $W^{l,M,\widehat{x'}}$ and $V = W^{l,M,\widehat{x'}} - W^{l,M,\hat{x}}$. Then, summing up, performing straightforward computations, and using (5), (7) and (34), we derive

$$\begin{aligned}
&\big\|W^{\widehat{x},\widehat{x'}}\big\|^2_{\mathcal{V}} := \big\|W^{l,M,\hat{x}} - W^{l,M,\widehat{x'}}\big\|^2_{\mathcal{V}} \\
&\leq C\beta^0 \max_{i,j=1,2,3} |M_{ij}(\hat{x}) - M_{ij}(\widehat{x'})| \, \big\|W^{\widehat{x},\widehat{x'}}\big\|_{L^2(T)} \big(\big\|W^{l,M,\widehat{x'}}\big\|_{L^2(T)} + |T|^{\frac{1}{2}}\big) \\
&+ \max_{i,j,k,p=1,2,3} |a_{ijkp}(\hat{x}) - a_{ijkp}(\widehat{x'})| \, \big\|e_{ij,y}(W^{\widehat{x},\widehat{x'}})\big\|_{L^2(\mathbb{R}^{3+})} \big\|e_{kp,y}(W^{l,M,\widehat{x'}})\big\|_{L^2(\mathbb{R}^{3+})}.
\end{aligned}$$

Now, we take into account the continuity of a_{ijkl} and M_{ij}, and the inequalities

$$\big\|W^{l,M,\hat{x}}\big\|_{L^2(T)} \leq C \quad \text{and} \quad \sum_{i,j=1}^{3} \big\|e_{ij,y}(W^{l,M,\hat{x}})\big\|_{L^2(\mathbb{R}^{3+})} \leq C \quad \forall \hat{x} \in \overline{\Sigma}, \tag{39}$$

where also C is a constant independent of $\hat{x}$ (cf. identity (35) with $V = W^{l,M,\hat{x}}$). In this way, we can choose δ_η such that

$$\|W^{l,M,\hat{x}} - W^{l,M,\widehat{x'}}\|_{L^2(T)} \leq \eta \quad \text{and} \quad \|W^{l,M,\hat{x}} - W^{l,M,\widehat{x'}}\|_{\mathcal{V}} \leq \eta.$$

The continuity of the scalar products $\big\langle e_{ij,y}(W^{l,M,\hat{x}}) \, , \, e_{ij,y}(W^{p,M,\hat{x}})\big\rangle_{L^2(\mathbb{R}^{3+})}$ and $\big\langle W_i^{l,M,\hat{x}} \, , \, W_j^{p,M,\hat{x}}\big\rangle_{L^2(T)}$, hold because of the estimates (39) and the continuity of $W^{l,M,\hat{x}}$ in the norms of $\mathcal{V}$ and $L^2(T)$: we choose δ_η such that for $|\hat{x} - \hat{x}'| \leq \delta_\eta$,

$$\Big|\big\langle e_{ij,y}(W^{l,M,\hat{x}}), e_{ij,y}(W^{p,M,\hat{x}})\big\rangle_{L^2(\mathbb{R}^{3+})} - \big\langle e_{ij,y}(W^{l,M,\widehat{x'}}), e_{ij,y}(W^{p,M,\widehat{x'}})\big\rangle_{L^2(\mathbb{R}^{3+})}\Big| \leq \eta$$

and

$$\Big|\langle W^{l,M,\hat{x}}, \, W^{p,M,\hat{x}}\big\rangle_{L^2(T)} - \langle W^{l,M,\widehat{x'}}, \, W^{p,M,\widehat{x'}})\rangle_{L^2(T)}\Big| \leq \eta.$$

Finally, according to the equation on T in (21), we write

$$\begin{aligned}
&\int\limits_T \sigma^{\hat{x}}_{i3,y}(W^{l,M,\hat{x}}) d\hat{y} - \int\limits_T \sigma^{\widehat{x'}}_{i3,y}(W^{l,M,\widehat{x'}}) d\hat{y} \\
&= \beta^0 M_{ij}(\hat{x}) \int\limits_T (W_j^{l,M,\widehat{x'}} - W_j^{l,M,\hat{x}}) d\hat{y} + \beta^0 (M_{ij}(\hat{x}) - M_{ij}(\widehat{x'})) \int\limits_T (\mathrm{e}_j^l - W_j^{l,M,\widehat{x'}}) d\hat{y}.
\end{aligned}$$

Again, from (39), the continuity of M_{ij} and that of $W_j^{l,M,\hat{x}}$ in the topology of $L^2(T)$, the continuity of $\langle \sigma^{\hat{x}}_{i3,y}(W^{l,M,\hat{x}}), 1\rangle_{L^2(T)}$ also holds and the proposition is proved.

Corollary 1 *For $l = 1, 2, 3$ there is a positive constant C independent of $\hat{x}$, such that for $y \in \mathbb{R}^{3+}$, with $|y|$ large enough, we have*

$$|W_i^{l,M,\hat{x}}(y)| \leq C\frac{1}{|y|} \quad \text{and} \quad \Big|\frac{\partial W_i^{l,M,\hat{x}}}{\partial y_j}(y)\Big| \leq C\frac{1}{|y|^2}, \quad i, j = 1, 2, 3. \tag{40}$$

Proof Bounds (40) with the constant $C = C_{\hat{x}}$ depending on the parameter $\hat{x}$ are a consequence of (36), (37) and (38). Using the equation on T in (21) and (39), we derive that $C_{\hat{x}}$ is uniformly bounded for $\hat{x} \in \overline{\Sigma}$, and the proposition is proved.

Proposition 2 *For each fixed $\hat{x} \in \Sigma$, the matrix $\mathcal{C}^e(\hat{x})$ defined by* (20) *is symmetric and positive definite and depends continuously on $\hat{x} \in \Sigma$.*

Proof For each fixed l, we multiply the elasticity equations in (21) by $W^{i,M,\hat{x}}$ and integrate over the half-ball $B(0, R) \cap \mathbb{R}^{3+}$. Then, we apply the Green formula and take limits as $R \to \infty$. On account of (40) and the boundary conditions in (21), we obtain the chain of equalities

$$\int_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{pj,y}(W^{l,M,\hat{x}})e_{pj,y}(W^{i,M,\hat{x}})\,dy + \beta^0 M_{pj}(\hat{x})\int_T (\mathrm{e}^l - W^{l,M,\hat{x}})_j(\mathrm{e}^i - W^{i,M,\hat{x}})_p\,d\hat{y}$$
$$= \beta^0 M_{ij}(\hat{x})\int_T (\mathrm{e}^l - W^{l,M,\hat{x}})_j\,d\hat{y} = \int_T \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})n_j d\hat{y} = \mathcal{C}^e_{il}(\hat{x}). \tag{41}$$

In this way, the symmetry of $\mathcal{C}^e$ comes from that of M and $(5)_1$ while the positivity is due to $(5)_2$ and (7). Indeed, it is simple to verify that $\exists\gamma > 0$, such that $\forall\bar{\alpha} \in \mathbb{R}^3$, $\bar{\alpha} \neq 0$,

$$\bar{\alpha}\mathcal{C}^e\bar{\alpha}^\top \geq \gamma\Big(\sum_{i,j=1}^{3}\big\|e_{ij,y}(\alpha_l W^{l,M,\hat{x}})\big\|^2_{L^2(\mathbb{R}^{3+})} + \sum_{i=1}^{3}\big\|\alpha_l(\mathrm{e}^l - W^{l,M,\hat{x}})_i\big\|^2_{L^2(T)}\Big).$$

Finally, from Proposition 1 and (41), it follows that the elements $\mathcal{C}^e_{ij}$, for $i, j = 1, 2, 3$, are continuous functions in $\overline{\Sigma}$. Hence, the proposition is proved.

5 Test functions for critical size and critical reaction.

In this section, based on the solution of (21), we introduce some functions which prove to be essential to define the test functions for obtaining the convergence of the solution of (16) towards that of the homogenized problem (19), when $r_0 > 0$ and $\beta^0 > 0$.

Let us consider $\varphi \in C^\infty[0,1]$, $0 \le \varphi \le 1$, $\varphi = 1$ in $[0,1/8]$ and $\mathrm{Supp}(\varphi) \subset [0,1/4]$. We define the function

$$\varphi^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}\right), \\ \varphi\Big(\dfrac{|x-\widetilde{x}_\mathrm{k}| - r_\varepsilon}{\varepsilon}\Big) & \text{for } x \in \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}, \quad \mathrm{k} \in \mathcal{J}^\varepsilon \\ 0 & \text{for } x \in \Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right). \end{cases} \tag{42}$$

where $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 : T^\varepsilon_{\widetilde{x}_\mathrm{k}} \subset \Sigma\}$, $B^+(\widetilde{x}_\mathrm{k}, r) = B(\widetilde{x}_\mathrm{k}, r) \cap \{x_3 > 0\}$ is the half-ball of radius r centered at the point $\widetilde{x}_\mathrm{k}$, and $\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}$ the half-annulus (cf. Figure 4)

$$\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}} = B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right) \setminus B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}\right).$$

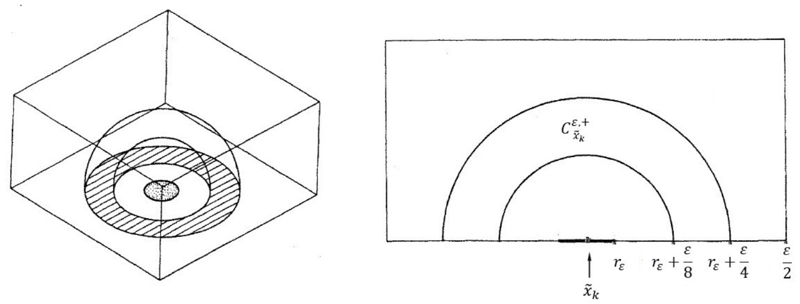

Fig. 4 The half-annulus centered at $\widetilde{x}_\mathrm{k}$.

For $l = 1,2,3$, and $\mathrm{k} \in \mathcal{J}^\varepsilon$, we construct the functions $W^{l,\mathrm{k},\varepsilon}(x)$, $\widetilde{W}^{l,\mathrm{k},\varepsilon}(x)$ and $\widetilde{W}^{l,\varepsilon}(x)$ using the solutions $W^{l,M,\widetilde{x}_\mathrm{k}}$ of the local problems (21), as follows:

$$W^{l,\mathrm{k},\varepsilon}(x) = W^{l,M,\widetilde{x}_\mathrm{k}}\left(\frac{x-\widetilde{x}_\mathrm{k}}{r_\varepsilon}\right)\varphi^\varepsilon(x) \quad \text{for} \quad x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right), \tag{43}$$

$$\widetilde{W}^{l,\mathrm{k},\varepsilon}(x) = \mathrm{e}^l - W^{l,\mathrm{k},\varepsilon}(x) \quad \text{for} \quad x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right).$$

The last one is extended by e^l in $\Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right)$. Finally, we set

$$\widetilde{W}^{l,\varepsilon}(x) = \begin{cases} \widetilde{W}^{l,\mathrm{k},\varepsilon}(x) & \text{for } x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right), \quad \mathrm{k} \in \mathcal{J}^\varepsilon, \\ \mathrm{e}^l & \text{for } x \in \Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right). \end{cases} \tag{44}$$

Below, C denotes a positive constant independent of ε and $\widetilde{x}_\mathrm{k}$, with $\mathrm{k} \in \mathcal{J}^\varepsilon$. Also, Ω_1 denotes any Lipschitz domain, $\Omega_1 \subseteq \Omega$ with $\overline{\Sigma}_1 := \partial\Omega_1 \cap \overline{\Sigma} \neq \emptyset$.

Proposition 3 *There is a constant C such that, for $x \in \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}$, and ε sufficiently small, the inequalities*

$$\left|\frac{\partial \varphi^\varepsilon}{\partial x_j}(x)\right| \leq C\frac{1}{\varepsilon}, \quad j=1,2,3, \tag{45}$$

$$\left|W_i^{l,M,\widetilde{x}_{\mathrm{k}}}\Big(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial W_i^{l,M,\widetilde{x}_{\mathrm{k}}}}{\partial x_j}\Big(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon^2}, \quad i,j,l=1,2,3, \tag{46}$$

and

$$\left|W_i^{l,\mathrm{k},\varepsilon}(x)\right| \leq C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial W_i^{l,\mathrm{k},\varepsilon}}{\partial x_j}(x)\right| \leq C\frac{r_\varepsilon}{\varepsilon^2}, \quad i,j,l=1,2,3, \tag{47}$$

are satisfied. In addition, for $l,p=1,2,3$, and $\Omega_1 \subseteq \Omega$, we have

$$\|\widetilde{W}^{l,\varepsilon}\|_{(H^1(\Omega))^3} \leq C \quad \text{and} \quad \widetilde{W}^{l,\varepsilon} \xrightarrow{\varepsilon\to 0} \mathrm{e}^l \quad \text{weakly in } (H^1(\Omega))^3, \tag{48}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_1} \sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})dx = r_0\int_{\Sigma_1}\int_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})e_{ij,y}(W^{p,M,\hat{x}})dy\,d\hat{x} \tag{49}$$

and

$$\lim_{\varepsilon\to 0}\beta(\varepsilon)\int_{\Sigma_1\cap\bigcup T^\varepsilon} M_{ij}\widetilde{W}_i^{l,\varepsilon}\widetilde{W}_j^{p,\varepsilon}d\hat{x} = r_0\beta^0\int_{\Sigma_1} M_{ij}(\hat{x})\int_T(\mathrm{e}^l - W^{l,M,\hat{x}})_i(\mathrm{e}^p - W^{p,M,\hat{x}})_j d\hat{y}\,d\hat{x}. \tag{50}$$

Proof Estimate (45) is a consequence of the definition (42), while estimates (46) are a consequence of (40). From (45) and (46), estimates (47) are also satisfied. In order to show (48), we evaluate

$$\begin{aligned}\left\|e_{ij,x}(\widetilde{W}^{l,\varepsilon})\right\|^2_{L^2(\Omega)} &\leq \sum_{\widetilde{x}_{\mathrm{k}}}\left\|e_{ij,x}(W^{l,\mathrm{k},\varepsilon})\right\|^2_{L^2(\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}})} + \sum_{\widetilde{x}_{\mathrm{k}}}\left\|e_{ij,x}(W^{l,\mathrm{k},\varepsilon})\right\|^2_{L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{8}))}\\ &\leq C\frac{r_\varepsilon^2}{\varepsilon^4}\sum_{\widetilde{x}_{\mathrm{k}}}\int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}} dx + r_\varepsilon\sum_{\widetilde{x}_{\mathrm{k}}}\left\|e_{ij,y}(W^{l,\mathrm{k},\varepsilon})\right\|^2_{L^2(B^+(0,1+\varepsilon/(r_\varepsilon 8)))} \leq C,\end{aligned} \tag{51}$$

where we have employed (47), (23), (39), (4) and $r_0>0$ in (1).

Now, we show that the convergence in (48) holds in the topology of $(L^2(\Omega))^3$ by applying the Poincaré inequality on each half-ball and the Korn inequality in Ω. Indeed, using (51), we readily obtain

$$\begin{aligned}\left\|\widetilde{W}^{l,\varepsilon}-\mathrm{e}^l\right\|^2_{(L^2(\Omega))^3} &\leq \sum_{\widetilde{x}_{\mathrm{k}}}\left\|W^{l,\mathrm{k},\varepsilon}\right\|^2_{(L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{4})))^3}\\ &\leq C\varepsilon^2\sum_{i=1}^3\sum_{\widetilde{x}_{\mathrm{k}}}\left\|\nabla_x W_i^{l,\mathrm{k},\varepsilon}\right\|^2_{(L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{4})))^3}\\ &\leq C\varepsilon^2\sum_{i=1}^3\left\|\nabla_x(\widetilde{W}_i^{l,\varepsilon}-\mathrm{e}^l)\right\|^2_{(L^2(\Omega))^3} \leq C\varepsilon^2\left\|e_{ij,x}(\widetilde{W}^{l,\varepsilon}-\mathrm{e}^l)\right\|^2_{L^2(\Omega)} \leq C\varepsilon^2.\end{aligned}$$

Therefore, also (48) is proved.

In order to verify (49), we proceed in a similar way as in (51):

$$\int_{\Omega_1} \sigma_{ij,x}(\widetilde{W}^{l,\varepsilon}) e_{ij,x}(\widetilde{W}^{p,\varepsilon})\, dx = \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} \int_{B^+(\widetilde{x}_{\rm k}, r_\varepsilon + \frac{\varepsilon}{8})} \sigma_{ij,x}(W^{l,{\rm k},\varepsilon}) e_{ij,x}(W^{p,{\rm k},\varepsilon}) dx + o(\varepsilon)$$

$$= \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} \int_{B^+(\widetilde{x}_{\rm k}, r_\varepsilon + \frac{\varepsilon}{8})} a_{ijkq}(x) e_{kq,x}(W^{l,{\rm k},\varepsilon}) e_{ij,x}(W^{p,{\rm k},\varepsilon}) dx + o(\varepsilon)$$

$$= \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} r_\varepsilon \int_{B^+(0,1+\frac{\varepsilon}{8r_\varepsilon})} a_{ijkq}(\widetilde{x}_{\rm k}) e_{kq,y}(W^{l,M,\widetilde{x}_{\rm k}}) e_{ij,y}(W^{p,M,\widetilde{x}_{\rm k}}) dy + o(1)$$

$$= \frac{r_\varepsilon}{\varepsilon^2} \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} \varepsilon^2 \int_{\mathbb{R}^{3+}} \sigma^{\widetilde{x}_{\rm k}}_{ij,y}(W^{l,M,\widetilde{x}_{\rm k}}) e_{ij,y}(W^{p,M,\widetilde{x}_{\rm k}}) dy + o(1),$$

where $o(\varepsilon)$ stands for a function bounded by $C\varepsilon$, and $o(1)$ is any infinitesimal as $\varepsilon \to 0$. For these formulas, we have considered (47), (23), (39), (4), the continuity of $a_{ijkp}(x)$ on $x \in \overline{\Omega}$, $r_0 > 0$ in (1), the fact that the sum of all the terms in which $d(\widetilde{x}_{\rm k}, \partial\Sigma_1) \le r_\varepsilon + \frac{\varepsilon}{4}$ is also $o(\varepsilon)$, and the inequality

$$\left| \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} r_\varepsilon \int_{\{|y|>1+\frac{\varepsilon}{8r_\varepsilon},\, y_3>0\}} \sigma^{\widetilde{x}_{\rm k}}_{ij,y}(W^{l,M,\widetilde{x}_{\rm k}}) e_{ij,y}(W^{p,M,\widetilde{x}_{\rm k}}) dy \right|$$

$$\le C \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} r_\varepsilon \int_{1+\frac{\varepsilon}{8r_\varepsilon}}^{\infty} \frac{1}{|y|^2} dy \le o(\varepsilon),$$

which is based on estimates (40). Now, taking into account Proposition 1, the last chain of equalities leads us to (49).

Let us obtain (50). In each integral on $T^\varepsilon_{\widetilde{x}_{\rm k}}$ we introduce the change (23) and obtain

$$\beta(\varepsilon) \int_{\Sigma_1 \cap \bigcup T^\varepsilon} M_{ij} \widetilde{W}^{l,\varepsilon}_i \widetilde{W}^{p,\varepsilon}_j d\hat{x}$$

$$= \beta(\varepsilon) \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} r_\varepsilon^2 \int_T M_{ij}(\widetilde{x}_{\rm k} + \hat{y} r_\varepsilon)({\rm e}^l - W^{l,M,\widetilde{x}_{\rm k}})_i ({\rm e}^p - W^{p,M,\widetilde{x}_{\rm k}})_j d\hat{y}$$

$$= \beta(\varepsilon) r_\varepsilon^2 \varepsilon^{-2} \sum_{\widetilde{x}_{\rm k}\in\Sigma_1} \varepsilon^2 M_{ij}(\widetilde{x}_{\rm k}) \int_T ({\rm e}^l - W^{l,M,\widetilde{x}_{\rm k}})_i ({\rm e}^p - W^{p,M,\widetilde{x}_{\rm k}})_j d\hat{y} + o(1),$$

where we have used (3), (2) and (1), (4), (39), the continuity of M and Proposition 1. Due to the same argument, the last integral converges towards the right hand side of (50) and, therefore, the proposition is proved.

Proposition 4 *For $\Omega_1 \subseteq \Omega$, $\Phi \in C(\overline{\Omega}_1)$ and $l, p = 1, 2, 3$, we have*

$$\lim_{\varepsilon\to 0}\int_{\Omega_1}\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})\Phi dx = r_0\int_{\Sigma_1}\Phi(\hat{x})\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})e_{ij,y}(W^{p,M,\hat{x}})dy\, d\hat{x}, \tag{52}$$

$$\lim_{\varepsilon\to 0}\beta(\varepsilon)\int_{\Sigma_1\cap\bigcup T^\varepsilon}M_{ij}\widetilde{W}^{l,\varepsilon}_i\widetilde{W}^{l,\varepsilon}_j\Phi d\hat{x} = r_0\beta^0\int_{\Sigma_1}\Phi(\hat{x})M_{ij}(\hat{x})\int_T(\mathrm{e}^l - W^{l,M,\hat{x}})_i(\mathrm{e}^p - W^{p,M,\hat{x}})_j d\hat{y}\, d\hat{x} \tag{53}$$

and

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_1}\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})\Phi dx + \beta(\varepsilon)\int_{\Sigma_1\cap\bigcup T^\varepsilon}M_{ij}\widetilde{W}^{l,\varepsilon}_i\widetilde{W}^{p,\varepsilon}_j\Phi d\hat{x}\right) = r_0\int_{\Sigma_1}\mathcal{C}^e_{pl}(\hat{x})\Phi(\hat{x})\, d\hat{x}. \tag{54}$$

Proof Let us first verify convergence (52)-(54) for any stepwise function Φ in Ω_1. Without any restriction we can assume that $\Omega_1 \equiv \Omega$.

Let $\{Q_m\}_{m=1}^M$ be a partition of Ω: Q_m a Lipshitz domain, $Q_m \subset \Omega$, $Q_m \cap Q_{m'} = \emptyset$ when $m \neq m'$ and $\overline{\Omega} = \bigcup_{m=1}^M \overline{Q}_m$. Also, $\overline{\Sigma} = \bigcup_{m=1}^M \overline{S}_m$ where $\overline{S}_m = \partial Q_m \cap \overline{\Sigma}$. Let $\Phi(x) = \sum_{m=1}^M \alpha_m \chi_{Q_m}(x)$, with χ_{Q_m} the characteristic function of Q_m and α_m a constant.

Then, we apply the convergence (49) on each Q_m. We obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega}\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})\Phi(x)dx = \lim_{\varepsilon\to 0}\sum_{m=1}^M\alpha_m\int_{Q_m}\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})dx$$
$$= r_0\int_{\Sigma}\Phi(\hat{x})\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{ij,y}(W^{l,M,\hat{x}})e_{ij,y}(W^{p,M,\hat{x}})dy\, d\hat{x},$$

and (52) is proved.

Let us show (53). We write

$$\beta(\varepsilon)\int_{\bigcup T^\varepsilon}M_{ij}\widetilde{W}^{l,\varepsilon}_i\widetilde{W}^{l,\varepsilon}_j\Phi\, d\hat{x} = \beta(\varepsilon)\sum_{m=1}^M\alpha_m\int_{S_m\cap\bigcup T^\varepsilon}M_{ij}\widetilde{W}^{l,\varepsilon}_i\widetilde{W}^{l,\varepsilon}_j\, d\hat{x}$$

and apply the convergence (50) to each S_m so that (53) also holds. Hence, (54) is a consequence of (52) and (53) and definition of $\mathcal{C}^e$, cf. (41).

Next, we verify (52), (53) and (54) for $\Phi \in C(\overline{\Omega}_1)$.

For each $k, l = 1, 2, 3$, we consider the Radon measures $\mu^\varepsilon_{kl,\Omega_1}$ and μ_{kl,Σ_1} defined by

$$\mu^\varepsilon_{kl,\Omega_1}(\Phi) = \int_{\Omega_1}\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})\Phi(x)dx$$

and

$$\mu^{\varepsilon}_{kl,\Sigma_1}(\Phi) = \beta(\varepsilon) \int\limits_{\Sigma_1 \cap \bigcup T^{\varepsilon}} M_{ij} \widetilde{W}^{l,\varepsilon}_i \widetilde{W}^{p,\varepsilon}_j \Phi d\hat{x}.$$

As a consequence of (48), (23), (39), (4), and the finite limits (1) and (2), we obtain the uniform bounds (independent of ε) in the topology of $\mathcal{M}(\overline{\Omega}_1)$

$$\|\mu^{\varepsilon}_{kl,\Omega_1}\|_{\mathcal{M}(\overline{\Omega}_1)} \leq C \quad \text{and} \quad \|\mu^{\varepsilon}_{kl,\Sigma_1}\|_{\mathcal{M}(\overline{\Omega}_1)} \leq C.$$

Consequently, for each sequence there are subsequences of measures $\mu^{\varepsilon}_{kl,\Omega_1}$ and $\mu^{\varepsilon}_{kl,\Sigma_1}$(still denoted by ε), and some $\widehat{\mu}_{kl,\Omega_1}$, $\widehat{\mu}_{kl,\Sigma_1} \in \mathcal{M}(\overline{\Omega}_1)$, such that the convergences

$$\mu^{\varepsilon}_{kl,\Omega_1} \xrightarrow{\varepsilon\to 0} \widehat{\mu}_{kl,\Omega_1} \quad \text{and} \quad \mu^{\varepsilon}_{kl,\Sigma_1} \xrightarrow{\varepsilon\to 0} \widehat{\mu}_{kl,\Omega_1}$$

occur in the weak* topology $\sigma(\mathcal{M}(\overline{\Omega}_1), C(\overline{\Omega}_1))$ (see, e.g., Chapter 4 of [4] in connection with the space of the Radon measures $\mathcal{M}(\overline{\Omega}_1)$). From the above convergence for stepwise functions, we identify $\widehat{\mu}_{kl,\Omega_1}$ and $\widehat{\mu}_{kl,\Sigma_1}$ with the measures defined by the right-hand side of (52) and (53), respectively. Thus, the proposition is proved.

6 Convergence for critical size and critical reaction.

Throughout the section we set $r_0 > 0$ and $\beta^0 > 0$ in (1) and (2). In Sections 6.1 and 6.2, we show the convergence of the solutions of the stationary problems, cf. (16) and (19). In Section 6.3, we derive the convergence of the eigenvalues of (8) towards those of (30) with conservation of multiplicity. The main results are stated in Theorems 3 and 4.

6.1 The convergence for solutions of stationary problems

Since the solution u^{ε} of (16) already converges towards some u^0 in the weak topology of $(H^1(\Omega))^3$, cf. (18), in this section, we identify u^0 with the solution of (19).

For $\phi \in (C^1(\overline{\Omega}))^3$, $\phi = 0$ on Γ_{Ω}, and $\widetilde{W}^{l,\varepsilon}$ defined by (44), we consider the vector-function $\phi_l(x)\widetilde{W}^{l,\varepsilon}(x)$. On account of (48), we have that $\phi_l \widetilde{W}^{l,\varepsilon} \to \phi$ weakly in $(H^1(\Omega))^3$, as $\varepsilon \to 0$. Also, we verify

$$e_{ij,x}(\phi_l \widetilde{W}^{l,\varepsilon}) = e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l + \frac{1}{2}\Big(\frac{\partial \phi_l}{\partial x_j}\widetilde{W}^{l,\varepsilon}_i + \frac{\partial \phi_l}{\partial x_i}\widetilde{W}^{l,\varepsilon}_j\Big). \tag{55}$$

We insert the *test function* $v(x) = \phi_l(x)\widetilde{W}^{l,\varepsilon}(x)$ into (16). We write

$$\int\limits_{\Omega} \sigma_{ij,x}(u^{\varepsilon}) e_{ij,x}(\phi_l \widetilde{W}^{l,\varepsilon})\, dx + \beta(\varepsilon) \int\limits_{\bigcup T^{\varepsilon}} M_{ij} u^{\varepsilon}_i \phi_l \widetilde{W}^{l,\varepsilon}_j \, d\hat{x} = \int\limits_{\Omega} f_i \phi_l \widetilde{W}^{l,\varepsilon}_i \, dx, \tag{56}$$

and considering (48) and (18), the limit passage as $\varepsilon \to 0$ in (56) gives

$$\int_{\Omega} \sigma_{ij,x}(u^0) e_{ij,x}(\phi)\, dx - \int_{\Omega} f_i \phi_i\, dx$$
$$= - \lim_{\varepsilon \to 0} \Big(\int_{\Omega} \sigma_{ij,x}(u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} u_i^\varepsilon \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x} \Big). \quad (57)$$

Next, we show that the limit on (57) is given by $r_0 \int_{\Sigma} \mathcal{C}_{ij}^e u_i^0 \phi_j d\hat{x}$. In order to do it, we use the following theorem which is proved in Section 6.2.

Theorem 2 *For any $u^0 \in \mathbf{V}$ which is the weak limit in $(H^1(\Omega))^3$ of a subsequence of u^ε, still denoted by ε, cf. (18), we construct a sequence $\widetilde{u}^\varepsilon \in \mathbf{V}$ such that*

$$\widetilde{u}^\varepsilon \to u^0 \text{ weakly in } (H^1(\Omega))^3, \quad \text{as } \varepsilon \to 0, \quad (58)$$

$$\beta(\varepsilon) \int_{\bigcup T^\varepsilon} (u_p^\varepsilon - \widetilde{u}_p^\varepsilon)^2 d\hat{x} \le C, \quad p = 1, 2, 3, \quad (59)$$

and, for any $\phi \in (C^1(\overline{\Omega}))^3$, with $\phi = 0$ on Γ_Ω, the following convergences occur:

$$\lim_{\varepsilon \to 0} \Big(\int_{\Omega} \sigma_{ij,x}(\widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} \widetilde{u}_i^\varepsilon \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x} \Big) = r_0 \int_{\Sigma} \mathcal{C}_{ij}^e u_i^0 \phi_j d\hat{x}, \quad (60)$$

and

$$\lim_{\varepsilon \to 0} \Big(\int_{\Omega} \sigma_{ij,x}(\widetilde{u}^\varepsilon - u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} (\widetilde{u}_i^\varepsilon - u_i^\varepsilon) \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x} \Big) = 0. \quad (61)$$

So that, accepting (60) and (61), we write

$$\int_{\Omega} \sigma_{ij,x}(u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} u_i^\varepsilon \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x}$$
$$= \int_{\Omega} \sigma_{ij,x}(\widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} \widetilde{u}_i^\varepsilon \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x}$$
$$+ \int_{\Omega} \sigma_{ij,x}(u^\varepsilon - \widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} M_{ij} (u_i^\varepsilon - \widetilde{u}_i^\varepsilon) \phi_l \widetilde{W}_j^{l,\varepsilon}\, d\hat{x}.$$

and the limit as $\varepsilon \to 0$ gives that u^0 satisfies

$$\int_{\Omega} \sigma_{ij,x}(u^0) e_{ij,x}(\phi)\, dx + r_0 \int_{\Sigma} \mathcal{C}_{ij}^e u_i^0 \phi_j\, d\hat{x} = \int_{\Omega} f_i \phi_i\, dx \quad \forall \phi \in (C^1(\overline{\Omega}))^3,\ \phi\big|_{\Gamma_\Omega} = 0.$$

By a density argument, we get that u^0 is the unique solution of (19), and we have proved the following result.

Theorem 3 *The solution u^ε of* (16) *converges weakly in* $(H^1(\Omega))^3$*, as* $\varepsilon \to 0$*, towards the solution u^0 of* (19).

6.2 The auxiliary functions: proof of Theorem 2

In this section, we prove Theorem 2. We follow an idea in [32] based on projections over spaces of finite elements; see also [22] for a scalar problem. We divide the proof in several steps. To make the reading easier, we prove (61) first, under the basis of the existence of $\widetilde{u}^\varepsilon$ satisfying (58), (60) and (59).

6.2.1 Proof of (61)

Assuming (58)-(60), d^ε denotes the difference $d^\varepsilon := u^\varepsilon - \widetilde{u}^\varepsilon$ which satisfies

$$d^\varepsilon \xrightarrow{\varepsilon\to 0} 0 \text{ weakly in } (H^1(\Omega))^3. \tag{62}$$

In (55) we replace $\widetilde{W}^{l,\varepsilon}$ by d^ε, so that for the first integral in (61), we have

$$\mathbf{I}_\varepsilon := -\int_\Omega \sigma_{ij,x}(d^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l \, dx = -\int_\Omega \sigma_{ij,x}(d^\varepsilon \phi_l) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) dx + o(1).$$

Hence, using the definition (44), we write

$$\begin{aligned}
\mathbf{I}_\varepsilon &= \sum_{\widetilde{x}_k} \int_{B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4})} \sigma_{ij,x}(d^\varepsilon \phi_l) e_{ij,x}(\tau_x W^{l,M,\widetilde{x}_k} \varphi^\varepsilon) dx + o(1) \\
&= \sum_{\widetilde{x}_k} \int_{B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4})} \sigma_{ij,x}(d^\varepsilon \phi_l \varphi^\varepsilon) e_{ij,x}(\tau_x W^{l,M,\widetilde{x}_k}) dx + o(1)
\end{aligned}$$

where τ_x denotes the change $y \mapsto x$, cf. (23), and in the last term $o(1)$ we have gathered the terms of the integrals in the half-annuli $\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}$, cf. (42), which are sums of

$$\sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} \sigma_{ij,x}(d^\varepsilon \phi_l) \tau_x W_p^{l,M,\widetilde{x}_k} \frac{\partial \varphi^\varepsilon}{\partial x_q} dx \quad \text{and} \quad \sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} a_{ijkl} d_p^\varepsilon \phi_l e_{ij}(\tau_x W_p^{l,M,\widetilde{x}_k}) \frac{\partial \varphi^\varepsilon}{\partial x_q} dx,$$

for $i, j, p, q, l, k = 1, 2, 3$. Let us show that indeed, all these terms vanish in the limit as $\varepsilon \to 0$. Using estimates (45), (46), (47) and (4) the first sums above are bounded by

$$C\varepsilon^{-1} \|d^\varepsilon\|_{\mathbf{V}} \sum_{p=1}^{3} \Big(\sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} |\tau_x W_p^{l,M,\widetilde{x}_k}|^2 dx \Big)^{1/2} \le C\varepsilon^{1/2}.$$

For the second sums, we use the same estimates as well as (62) so that they are bounded by

$$C\varepsilon^{-1}\sum_{p=1}^{3}\Big(\int\limits_{\bigcup \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_{\rm k}}}(d^{\varepsilon}_{p})^{2}dx\Big)^{\frac{1}{2}}\cdot\sum_{i,j=1}^{3}\Big(\sum_{\widetilde{x}_{\rm k}}\int\limits_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_{\rm k}}}\Big|e_{ij,x}(\tau_x W^{l,M,\widetilde{x}_{\rm k}}_{p})\Big|^{2}dx\Big)^{\frac{1}{2}}\leq C\sum_{p=1}^{3}(\mathbf{L}_{\varepsilon,p})^{\frac{1}{2}}$$

where (see, e.g., Lemma 2.4 in [23])

$$\mathbf{L}_{\varepsilon,p}:=\lim_{\varepsilon\to 0}\varepsilon^{-1}\int\limits_{0<x_3<\varepsilon}(d^{\varepsilon}_{p})^{2}dx=0,\quad p=1,2,3. \tag{63}$$

All together, along with (5), (22) and (23), give

$$\begin{aligned}\mathbf{I}_{\varepsilon}&=\sum_{\widetilde{x}_{\rm k}}\int\limits_{B^{+}(\widetilde{x}_{\rm k},r_{\varepsilon}+\frac{\varepsilon}{4})}\sigma_{ij,x}(\tau_x W^{l,M,\widetilde{x}_{\rm k}})e_{ij,x}(d^{\varepsilon}\phi_l\varphi^{\varepsilon})dx+o(1)\\&=r_{\varepsilon}\sum_{\widetilde{x}_{\rm k}}\int\limits_{B^{+}(0,1+\frac{\varepsilon}{4r_{\varepsilon}})}\sigma^{\widetilde{x}_{\rm k}}_{ij,y}(W^{l,M,\widetilde{x}_{\rm k}})e_{ij,y}(d^{\varepsilon}\phi_l\varphi^{\varepsilon})dy+o(1),\end{aligned}$$

where, to obtain the last $o(1)$, we have used the continuity of the elastic coefficients, estimates (39) and (45) and convergences (62) and (63).

Thus, considering (42) for φ^{ε}, and applying the Green formula, cf. (21), we get

$$\begin{aligned}\mathbf{I}_{\varepsilon}&=r_{\varepsilon}\sum_{\widetilde{x}_{\rm k}}\int\limits_{T}\sigma^{\widetilde{x}_{\rm k}}_{i3,y}(W^{l,M,\widetilde{x}_{\rm k}})\tau_y(d^{\varepsilon}_{i}\phi_l)d\hat{y}+o(1)\\&=-r_{\varepsilon}\sum_{\widetilde{x}_{\rm k}}\int\limits_{T}\beta^{0}M_{ij}(\widetilde{x}_{\rm k})(\mathrm{e}^{l}_{j}-W^{l,M,\widetilde{x}_{\rm k}}_{j})\tau_y(d^{\varepsilon}_{i}\phi_l)d\hat{y}+o(1).\end{aligned}$$

Using this, (23) and (44) yield

$$\begin{aligned}&\int\limits_{\Omega}\sigma_{ij,x}(d^{\varepsilon})e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx+\beta(\varepsilon)\int\limits_{\bigcup T^{\varepsilon}}M_{ij}d^{\varepsilon}_{i}\phi_l\widetilde{W}^{l,\varepsilon}_{j}\,d\hat{x}\\&=-r_{\varepsilon}\sum_{\widetilde{x}_{\rm k}}\int\limits_{T}\beta^{0}M_{ij}(\widetilde{x}_{\rm k})(\mathrm{e}^{l}_{j}-W^{l,M,\widetilde{x}_{\rm k}}_{j})\tau_y(d^{\varepsilon}_{i}\phi_l)d\hat{y}\\&\quad+\beta(\varepsilon)r^{2}_{\varepsilon}\sum_{\widetilde{x}_{\rm k}}\int\limits_{T}M_{ij}(\widetilde{x}_{\rm k}+r^{\varepsilon}\hat{y})\tau_y(d^{\varepsilon}_{i}\phi_l)(\mathrm{e}^{l}_{j}-W^{l,M,\widetilde{x}_{\rm k}}_{j})\,d\hat{y}+o(1)=o(1),\end{aligned}$$

where, the term $o(1)$ in the last equality has been obtained by means of straightforward computations, taking into account (39), (62), (59), (4), the continuity of M and (2). Consequently, the convergence (61) holds true.

6.2.2 The construction of $\widetilde{u}^\varepsilon$ satisfying (58)-(60).

Since $\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})$ takes values different from zero only in a neighborhood of Σ, and $u^\varepsilon = 0$ on Γ_Ω, there is no loss of generality for the proof to assume that the domain Ω is a polyhedron and the boundary Γ_Ω can be written as a finite union of plane faces. For each fixed $h > 0$, we create a regular triangulation $\{\triangle_{h_q}\}_{q=1}^{M_h}$ of the domain Ω composed of tetrahedrons of diameter h (see, e.g., [7] and [34])

$$\overline{\Omega} = \bigcup_{q=1}^{M_h} \overline{\triangle}_{h_q}. \tag{64}$$

Let $\Pi_h u$ denote the projection of the element $u \in H^1(\Omega)$, with $u = 0$ on Γ_Ω, on the subspace $\mathcal{Y}^h$ of the continuous functions over $\overline{\Omega}$ which are affine functions on each tetrahedron $\triangle_{h_q}$ and take the value 0 on Γ_Ω. As it is well known, for any $u \in H^1(\Omega)$, with $u = 0$ on Γ_Ω,

$$u^h := \Pi_h(u) \to u \text{ in } H^1(\Omega), \quad \text{as } h \to 0. \tag{65}$$

We divide the rest of the proof into four steps.

First step: a first approach to the construction of $\widetilde{u}^\varepsilon$ satisfying (58).

For $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ the solution of (16), for u^0 the limit in (18), and for $l = 1, 2, 3$, let $u_l^{\varepsilon h}$ and u_l^{0h} denote the projections on $\mathcal{Y}^h$ of u_l^ε and u_l^0 respectively. We set

$$\widetilde{u}^{\varepsilon h} = u_l^{\varepsilon h} \widetilde{W}^{l,\varepsilon}. \tag{66}$$

On each $\triangle_{h_q}$ we introduce the polynomial

$$\begin{aligned} u_l^{\varepsilon h}\big|_{\triangle_{h_q}} &= z_{l,r}(\varepsilon, h_q) x_r + \alpha_l(\varepsilon, h_q) \\ u_l^{0h}\big|_{\triangle_{h_q}} &= z_{l,r}(h_q) x_r + \alpha_l(h_q), \end{aligned} \tag{67}$$

whose coefficients $z_{l,r}(\varepsilon, h_q)$, $\alpha_l(\varepsilon, h_q)$, $z_{l,r}(h_q)$, $\alpha_l(h_q)$ are real numbers, and $r = 1, 2, 3$. On account of (18), for any fixed $h > 0$, these coefficients satisfy

$$z_{l,r}(\varepsilon, h_q) \xrightarrow{\varepsilon \to 0} z_{l,r}(h_q) \text{ and } \alpha_l(\varepsilon, h_q) \xrightarrow{\varepsilon \to 0} \alpha_l(h_q), \ q = 1, 2 \cdots, M_h. \tag{68}$$

From (48), (66), (67) and (68), we deduce the convergence

$$\widetilde{u}^{\varepsilon h} \xrightarrow{\varepsilon \to 0} u^{0h} \text{ weakly in } (H^1(\Omega))^3. \tag{69}$$

Second step: a first approach to the convergence (60).

For any $\Phi \in C^1(\overline{\Omega})$, $\Phi = 0$ on Γ_Ω, we set

$$\begin{cases} \mathbf{I}_{\varepsilon,k,h} = \int\limits_{\Omega} \sigma_{ij,x}(\widetilde{u}^{\varepsilon h}) e_{ij,x}(\widetilde{W}^{k,\varepsilon}) \Phi \, dx + \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} M_{ij} \widetilde{u}_i^{\varepsilon h} \Phi \widetilde{W}_j^{k,\varepsilon} \, d\hat{x}, \\ \mathbf{I}_{k,h} = r_0 \int\limits_{\Sigma} \mathcal{C}^e_{lk} u_l^{0h} \Phi \, d\hat{x}, \\ \mathbf{I}_k = r_0 \int\limits_{\Sigma} \mathcal{C}^e_{lk} u_l^0 \Phi \, d\hat{x}. \end{cases} , k = 1, 2, 3.$$

We decompose the integrals above in Ω into the sum of integrals over $\triangle_{h_q}$, while the integrals on Σ into the sum of integrals over $\partial \triangle_{h_q} \cap \Sigma$ covering Σ, cf. (64), and we use (66)-(69), (55) for ϕ the affine function in (67), and Proposition 4. For fixed $h > 0$, and $k = 1, 2, 3$, we define

$$\mathbf{A}_{lr\varepsilon h_q k} = \int\limits_{\triangle_{h_q}} x_r \sigma_{ij,x}(\widetilde{W}^{l,\varepsilon}) e_{ij,x}(\widetilde{W}^{k,\varepsilon}) \Phi \, dx + \beta(\varepsilon) \int\limits_{\partial \triangle_{h_q} \cap \bigcup T^\varepsilon} \hat{x}_r M_{ij} \widetilde{W}_i^{l,\varepsilon} \widetilde{W}_j^{k,\varepsilon} \Phi \, d\hat{x}$$

and

$$\mathbf{B}_{lr\varepsilon h_q k} = \int\limits_{\triangle_{h_q}} \sigma_{ij,x}(\widetilde{W}^{l,\varepsilon}) e_{ij,x}(\widetilde{W}^{k,\varepsilon}) \Phi \, dx + \beta(\varepsilon) \int\limits_{\partial \triangle_{h_q} \cap \bigcup T^\varepsilon} M_{ij} \widetilde{W}_i^{l,\varepsilon} \widetilde{W}_j^{k,\varepsilon} \Phi \, d\hat{x},$$

and we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathbf{I}_{\varepsilon,k,h} &= \sum_{q=1}^{M_h} \lim_{\varepsilon \to 0} \Big(z_{l,r}(\varepsilon, h_q) \mathbf{A}_{lr\varepsilon h_q k} + \alpha_l(\varepsilon, h_q) \mathbf{B}_{lr\varepsilon h_q k} \Big) \\ &= \sum_{q=1}^{M_h} \Big(z_{l,r}(h_q) r_0 \int\limits_{\Sigma \cap \partial \triangle_{h_q}} x_r \mathcal{C}^e_{lk} \Phi \, d\hat{x} + \alpha_l(h_q) r_0 \int\limits_{\Sigma \cap \partial \triangle_{h_q}} \mathcal{C}^e_{lk} \Phi \, d\hat{x} \Big) \\ &= r_0 \int\limits_{\Sigma} \mathcal{C}^e_{lk} u_l^{0h} \Phi \, d\hat{x} = \mathbf{I}_{k,h}. \end{aligned} \tag{70}$$

In addition, on account of (65), and the trace embedding theorem, we obtain

$$\lim_{h \to 0} u^{0h} = u^0 \text{ in } (H^1(\Omega))^3 \quad \text{and} \quad \lim_{h \to 0} \mathbf{I}_{k,h} = \mathbf{I}_k. \tag{71}$$

Third step: a first approach to the estimate (59).
Similarly, to the previous step, we define

$$\begin{cases} \mathbf{J}_{\varepsilon,k,h} = \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} (\widetilde{u}_k^{\varepsilon h})^2 d\hat{x}, \\ \mathbf{J}_{k,h} = \beta^0 r_0 \int\limits_{\Sigma} (u^{0h_l})^2 \int\limits_{T} (\mathrm{e}_k^l - W_k^{l,M,\hat{x}}(y))^2 \, d\hat{y} \, d\hat{x} \\ \mathbf{J}_k = \beta^0 r_0 \int\limits_{\Sigma} (u_l^0)^2 \int\limits_{T} (\mathrm{e}_k^l - W_k^{l,M,\hat{x}}(y))^2 \, d\hat{y} \, d\hat{x} \end{cases} , \quad k = 1, 2, 3, \tag{72}$$

and obtain

$$\lim_{\varepsilon\to 0}\mathbf{J}_{\varepsilon,k,h} = \lim_{\varepsilon\to 0}\beta(\varepsilon)\sum_{q=1}^{M_h}\int_{\partial\triangle_{h_q}\cap\bigcup T^\varepsilon}(z_{l,r}(\varepsilon,h_q)\hat{x}_r+\alpha_l(\varepsilon,h_q))^2\ (\widetilde{W}_k^{l,\varepsilon})^2 d\hat{x}$$

$$=\lim_{\varepsilon\to 0}\sum_{q=1}^{M_h}\beta(\varepsilon)\sum_{\widetilde{x}_{\mathrm{k}}\in\Sigma^\varepsilon_{h_q}} r_\varepsilon^2\int_T(z_{l,r}(\varepsilon,h_q)((\widetilde{x}_{\mathrm{k}})_r+\hat{y}_r r_\varepsilon)+\alpha_l(\varepsilon,h_q))^2(\mathrm{e}_k^l-W_k^{l,M,\widetilde{x}_{\mathrm{k}}}(\hat{y}))^2 d\hat{y}$$

$$=\lim_{\varepsilon\to 0}\sum_{q=1}^{M_h}\beta(\varepsilon)r_\varepsilon\sum_{\widetilde{x}_{\mathrm{k}}\in\Sigma^\varepsilon_{h_q}} r_\varepsilon\,(z_{l,r}(\varepsilon,h_q)(\widetilde{x}_{\mathrm{k}})_r+\alpha_l(\varepsilon,h_q))^2\int_T(\mathrm{e}_k^l-W_k^{l,M,\widetilde{x}_{\mathrm{k}}}(\hat{y}))^2 d\hat{y}$$

$$=\lim_{\varepsilon\to 0}\sum_{q=1}^{M_h}\beta(\varepsilon)r_\varepsilon r_\varepsilon\varepsilon^{-2}\sum_{\widetilde{x}_{\mathrm{k}}\in\Sigma^\varepsilon_{h_q}}\varepsilon^2\,(z_{l,r}(h_q)(\widetilde{x}_{\mathrm{k}})_r+\alpha_l(h_q))^2\int_T(\mathrm{e}_k^l-W_k^{l,M,\widetilde{x}_{\mathrm{k}}}(y))^2 d\hat{y},$$

where $\Sigma^\varepsilon_{h_q}=\partial\triangle_{h_q}\cap\bigcup T^\varepsilon$. That is,

$$\lim_{\varepsilon\to 0}\mathbf{J}_{\varepsilon,k,h}=\sum_{q=1}^{M_h}\beta^0 r_0\int_{\partial\triangle_{h_q}\cap\Sigma_1}(z_{l,r}(h_q)\hat{x}_r+\alpha_l(h_q))^2\int_T(\mathrm{e}_k^l-W_k^{l,M,\hat{x}}(\hat{y}))^2 d\hat{y}\,d\hat{x}$$

$$=\beta^0 r_0\int_\Sigma(u_l^{0h})^2\int_T(\mathrm{e}_k^l-W_k^{l,M,\hat{x}}(\hat{y}))^2 d\hat{y}d\hat{x}=\mathbf{J}_{k,h}.$$

In the limits above, we have used (67), the change of variable (23), the inequality $|\widehat{y}|\le C$ on each integral, (68), (2), (1) and Proposition 1. Now, by the trace embedding theorem, $\mathbf{J}_{k,h}$ converges towards $\mathbf{J}_k$, as $h\to 0$, see (72). Thus, in short, we have proved

$$\lim_{\varepsilon\to 0}\mathbf{J}_{\varepsilon,k,h}=\mathbf{J}_{k,h}\quad\text{and}\quad\lim_{h\to 0}\mathbf{J}_{k,h}=\mathbf{J}_k,\quad k=1,2,3. \tag{73}$$

Fourth step: the function $\widetilde{u}^\varepsilon$ satisfies (59) *and* (60).

Gathering the above convergence results, as $\varepsilon\to 0$, and as $h\to 0$ (cf. (69), (70), (71) and (73)), we get the following convergence in the the topology of $(L^2(\Omega))^3\times\mathbb{R}^2$:

$$(\widetilde{u}^{\varepsilon h},\,\mathbf{I}_{\varepsilon,k,h}\,,\mathbf{J}_{\varepsilon,k,h})\to(u^{0h},\,\mathbf{I}_{k,h},\,\mathbf{J}_{k,h})\ \text{ in } (L^2(\Omega))^3\times\mathbb{R}^2,\ \text{ as } \varepsilon\to 0,$$
$$(\widetilde{u}^{0h},\,\mathbf{I}_{k,h},\,\mathbf{J}_{k,h})\to(u^0,\,\mathbf{I}_k,\,\mathbf{J}_k)\ \text{ in } (L^2(\Omega))^3\times\mathbb{R}^2,\ \text{ as } h\to 0.$$

Then, we apply a result on convergence for double indexed subsequences, see, e.g., Corollary 1.18 in Section I.2 of [3], to extract a sequence $h(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that

$$(\widetilde{u}^{\varepsilon h(\varepsilon)},\,\mathbf{I}_{\varepsilon,k,h(\varepsilon)},\,\mathbf{J}_{\varepsilon,k,h(\varepsilon)})\to(u^0,\,\mathbf{I}_k,\,\mathbf{J}_k)\ \text{ in }\ (L^2(\Omega))^3\times\mathbb{R}^2,\ \text{ as } \varepsilon\to 0. \tag{74}$$

Now, denoting $\widetilde{u}^\varepsilon=\widetilde{u}^{\varepsilon h(\varepsilon)}$, (74) gives that $\widetilde{u}^\varepsilon\to u^0$ weakly in $(H^1(\Omega))^3$ as $\varepsilon\to 0$ and (62). Taking $\Phi=\phi_k$, where $\phi=(\phi_1,\phi_2,\phi_3)\in(C^1(\overline{\Omega}))^3$, $\phi=0$ on Γ_Ω, converts $\mathbf{I}_{\varepsilon,k,h(\varepsilon)}$ into the sum for $k=1,2,3$, and we derive (60).

Also, taking into account (74), and replacing h by $h(\varepsilon)$ in (72) leads to

$$\mathbf{J}_{\varepsilon,k,h(\varepsilon)} = \beta(\varepsilon) \int_{\bigcup T^\varepsilon} (\widetilde{u}_k^\varepsilon)^2 d\hat{x} \leq C.$$

This, along with (16), (18), the continuity of M, (7), and the Cauchy-Buniakovsky-Schwarz inequality, provides (59). Hence, Theorem 2 is proved.

6.3 The spectral convergence

In this section, based on Lemma 1, we derive the convergence for the eigenpairs of (11), when $r_0 > 0$ and $\beta^0 > 0$ in (1) and (2).

Theorem 4 *For each k, $k = 1, 2, 3 \ldots$, λ_k^ε in* (12) *and λ_k^0 in* (33) *satisfy*

$$\lambda_k^\varepsilon \to \lambda_k^0, \quad \text{as } \varepsilon \to 0,$$

where $\{\lambda_k^0\}_{k=1}^\infty$ are the eigenvalues of (32) *with $\mathcal{B}(\hat{x}) = \mathcal{C}^e(\hat{x})$. In addition, for each infinitesimal sequence ε, we can extract a subsequence, still denoted by ε, such that the corresponding eigenfunctions $u^{\varepsilon,k}$ converge towards $u^{0,k}$ in $(L^2(\Omega))^3$, where $u^{0,k}$ is an eigenfunction of* (32) *corresponding to λ_k^0, and the set $\{u^{0,k}\}_{k=1}^\infty$ forms an orthogonal basis of $(L^2(\Omega))^3$.*

Proof Let us introduce the operators $\mathcal{A}^\varepsilon$, $\mathcal{A}^0 : (L^2(\Omega))^3 \to (L^2(\Omega))^3$. For $f \in (L^2(\Omega))^3$, we set $\mathcal{A}^\varepsilon f = u^\varepsilon$, where $u^\varepsilon \in \mathbf{V}$ is the unique solution of (16). Similarly, we set $\mathcal{A}^0 f = u^0$, where $u^0 \in \mathbf{V}$ is the unique solution of (19). So that the eigenpairs of $\mathcal{A}^\varepsilon$ are $\{\big((\lambda_k^\varepsilon)^{-1}, u^{\varepsilon,k}\big)\}_{k=1}^\infty$ with $\{(\lambda_k^\varepsilon, u^{\varepsilon,k})\}_{k=1}^\infty$ the eigenpairs of (11), and, the eigenpairs of $\mathcal{A}^0$ are $\{\big((\lambda_k^0)^{-1}, u^{0,k}\big)\}_{k=1}^\infty$ with $\{(\lambda_k^0, u^{0,k})\}_{k=1}^\infty$ the eigenpairs of (32).

We define $\mathcal{W} = \mathbf{V}$, and considering Theorem 3, properties i1) and i2) in Lemma 1 becomes self-evident. To prove property i3), we consider $f^\varepsilon \in (L^2(\Omega))^3$ uniformly bounded in $(L^2(\Omega))^3$, and hence, we find a subsequence $\varepsilon' \to 0$ and a certain $f \in (L^2(\Omega))^3$ such that $f^{\varepsilon'} \to f$ weakly in $(L^2(\Omega))^3$. We replace f by $f^{\varepsilon'}$ in (16), and since (17) also holds, we rewrite the proof of Theorem 3 with minor modifications, to show the convergence of solutions $u^{\varepsilon'}$ towards u^0 weakly in $(H^1(\Omega))^3$, as $\varepsilon' \to 0$, and property i3) is also verified.

Consequently, the convergence of the eigenvalues and the corresponding eigenfunctions in the statement of the theorem holds from Lemma 1.

7 The other critical case

In this section, we address the convergence of solutions of the stationary problem (16) and the spectral problem (11), as $\varepsilon \to 0$, when $r_0 > 0$ and $\beta^0 = +\infty$ in (1) and (2). The main results are Theorems 6 and 8.

We follow the scheme in Sections 4-6 with the suitable modifications. Section 7.1 presents properties of the solutions of the $\hat{x}$-dependent family of local problems (26). The convergence for the stationary problem is in Section 7.2, while the spectral convergence is in Section 7.3.

Now, the stationary homogenized problem reads (24), where the matrix $\mathcal{C}(\hat{x}) = (\mathcal{C}_{ij}(\hat{x}))_{i,j=1,2,3}$ is defined by (25) with $W^{l,\hat{x}}$ the solution of (26). The spectral homogenized problem is (32) with $\mathcal{B} = \mathcal{C}$, cf. (31).

7.1 Abstract framework for the stationary local problem (26)

Below, we derive the properties of $\hat{x}$-dependent solutions $W^{l,\hat{x}}$ and those of matrix $\mathcal{C}(\hat{x})$.

Let $(\mathcal{D}_1(\overline{\mathbb{R}^{3+}}))^3$ denote the space of functions in $(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ which vanish in a neighbourhood of $\overline{T}$. Let $\mathfrak{V}$ and $\mathfrak{V}_1$ be the spaces obtained by completion of $(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ and $(\mathcal{D}_1(\overline{\mathbb{R}^{3+}}))^3$, respectively, with respect to the norm

$$\|U\|_{\mathfrak{V}} = \Big(\sum_{i,j=1}^{3} \|e_{ij,y}(U)\|^2_{L^2(\mathbb{R}^{3+})} \Big)^{1/2}.$$

Due to Korn's inequality in bounded Lipschitz domains, the continuous embedding $\mathfrak{V}_1 \subset (H^1_{loc}(\mathbb{R}^{3+}))^3$ holds, and the elements of $\mathfrak{V}_1$ have null traces on T.

For each $l = 1, 2, 3$, we take a function

$$\Psi^l \in (\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3, \quad \Psi^l = \mathrm{e}^l \text{ in a neighboorhood of } T.$$

Then, the variational formulation of $(26)_1$-$(26)_3$ reads: Find $W^{l,\hat{x}} \in \Psi^l + \mathfrak{V}_1$ satisfying

$$\int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}}) e_{ij,y}(V)\, dy = 0 \quad \forall V \in \mathfrak{V}_1. \tag{75}$$

Problem (75) has a unique solution which is independent of Ψ^l (see, e.g., Section 4 in [21]). The condition at infinity $(26)_4$ is a consequence of Theorem 5; see [21], [6] and [14] for an isotropic media. Also, $\sigma_{i3}(W^{l,\hat{x}})\big|_{y_3=0}$ is a distribution having compact support contained in $\overline{T}$ and belongs to $H^{-1/2}(T)$. Thus, applying the Green formula, we write

$$\int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{pj,y}(W^{l,\hat{x}}) e_{pj,y}(V)\, dy = \big\langle \sigma^{\hat{x}}_{pj,y} n_j(W^{l,\hat{x}}), V_i \big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \qquad \forall V \in (\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3. \tag{76}$$

By a density argument, we have (76) for any $V \in \mathfrak{V}$, and consequently, for $V = W^{p,\hat{x}}$:

$$\int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{pj,y}(W^{l,\hat{x}}) e_{pj,y}(W^{i,\hat{x}})\, dy = -\big\langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}), \mathrm{e}^i_p \big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} = \mathcal{C}_{il}(\hat{x}). \tag{77}$$

The following propositions deal with results analogous to Proposition 1 and 2 about the continuous dependence of $W^{l,\hat{x}}$ on $\hat{x} \in \overline{\Sigma}$ as well as other related functions. In their statements and proofs C denotes a positive constant independent of $\hat{x}$.

Proposition 5 *For $l = 1, 2, 3$, the solution $W^{l,\hat{x}}$ of (75) depends continuously on $\hat{x} \in \overline{\Sigma}$ in the topology of $\mathfrak{V}$. In addition, for $l, p, i, j = 1, 2, 3$, the functions*

$$\int\limits_{\mathbb{R}^{3+}} e_{ij,y}(W^{l,\hat{x}}) e_{ij,y}(W^{p,\hat{x}}) dy, \qquad \text{and} \qquad \left\langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}), \mathrm{e}^i_p \right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}$$

depend continuously on $\hat{x} \in \overline{\Sigma}$, and

$$\left\| \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}) \right\|_{H^{-1/2}(T)} \leq C. \tag{78}$$

Proof Let us show that for each $\eta > 0$, theres is $\delta_\eta > 0$ such that if $\hat{x}, \widehat{x'} \in \Sigma$ satisfy $|\hat{x} - \widehat{x'}| < \delta_\eta$, then $\|W^{l,\hat{x}} - W^{l,\widehat{x'}}\|_{\mathfrak{V}} \leq \eta$.

First, we obtain bounds for the norm of $W^{l,\hat{x}}$ in $\mathfrak{V}$ which are independent of $\hat{x}$. To this end, we consider (75) taking $V \equiv V^{l,\hat{x}} := W^{l,\hat{x}} - \Psi^l \in \mathfrak{V}_1$, and we obtain

$$\int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(V^{l,\hat{x}}) e_{ij,y}(V^{l,\hat{x}})\, dy = - \int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(\Psi^l) e_{ij,y}(V^{l,\hat{x}})\, dy.$$

Applying the Cauchy-Buniakovsky-Schwarz inequality, (22) and the continuity of the elastic coefficients provides the uniform bound for $\|V^{l,\hat{x}}\|_{\mathfrak{V}}$. Consequently, cf. (77), we obtain

$$\sum_{i,j=1}^{3} \left\| e_{ij,y}(W^{l,\hat{x}}) \right\|_{L^2(\mathbb{R}^{3+})} \leq C \quad \text{and} \quad \left| \left\langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}),\, \mathrm{e}^i_p \right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \right| \leq C \qquad \forall \hat{x} \in \overline{\Sigma}. \tag{79}$$

Next, we take $V = W^{l,\hat{x}} - W^{l,\widehat{x'}}$ in (75), and similarly, in the formulation (75) for $W^{l,\widehat{x'}}$, we take $V = W^{l,\hat{x}} - W^{l,\widehat{x'}}$. By subtracting the second identity from the first one, we obtain:

$$\int\limits_{\mathbb{R}^{3+}} \sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}} - W^{l,\widehat{x'}}) e_{ij,y}(W^{l,\hat{x}} - W^{l,\widehat{x'}})\, dy = \int\limits_{\mathbb{R}^{3+}} (a_{ijkp}(\widehat{x'}) - a_{ijkp}(\hat{x})) e_{kp,y}(W^{l,\widehat{x'}}) e_{ij,y}(W^{l,\hat{x}} - W^{l,\widehat{x'}})\, dy.$$

Now, using (5), (79), we choose $\delta_\eta > 0$ such that $\|W^{l,\hat{x}} - W^{l,\widehat{x'}}\|_{\mathfrak{V}} \leq \eta$.

From this and (79), the continuity of $\left\langle e_{ij,y}(W^{l,\hat{x}}),\, e_{ij,y}(W^{p,\hat{x}})\right\rangle_{L^2(\mathbb{R}^{3+})}$ holds: indeed, we choose $\delta_\eta > 0$ such that

$$\left| \left\langle e_{ij,y}(W^{l,\hat{x}})\, ,\, e_{ij,y}(W^{p,\hat{x}})\right\rangle_{L^2(\mathbb{R}^{3+})} - \left\langle e_{ij,y}(W^{l,\widehat{x'}})\, ,\, e_{ij,y}(W^{p,\widehat{x'}})\right\rangle_{L^2(\mathbb{R}^{3+})} \right| \leq \eta.$$

Finally, the continuity on $\overline{\Sigma}$ of the term $\left\langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}), \mathrm{e}^i_p\right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}$ is a consequence of (77), cf. (5) and (22).

As for the estimate (78), we use the trace embedding theorem for the space

$$\left\{ U \in (H^1(B(0,R_0)\cap \mathbb{R}^{3+}))^3 : \frac{\partial \sigma^{\hat{x}}_{ij,y}(U)}{\partial y_j} \in (L^2(B(0,R_0)\cap \mathbb{R}^{3+}))^3 \right\},$$

with a fixed R_0 such that $\overline{T} \subset B(0,R_0)$. Indeed, because of $(26)_1$ and (79), we write

$$\left\| \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}) \right\|_{H^{-1/2}(T)} \leq C \sum_{i,j=1}^{3} \left\| e_{ij,y}(W^{l,\hat{x}}) \right\|_{L^2(B(0,R_0)\cap\mathbb{R}^{3+})} \leq C,$$

which concludes with the proof of the proposition.

Proposition 6 *For each fixed $\hat{x} \in \Sigma$, $\mathcal{C}(\hat{x})$ defined by* (25) *is a symmetric and positive definite matrix. In addition, its coefficients depend continuously on $\hat{x} \in \overline{\Sigma}$.*

Proof Considering (77) the symmetry and positivity of $\mathcal{C}$ are due to (5): see the reasoning in Proposition 2. In addition, from (77) and Proposition 5, $\mathcal{C}_{ij}$, for $i,j = 1,2,3$, are continuous functions on $\overline{\Sigma}$, and the proposition is proved.

Theorem 5 *For each $\hat{x} \in \overline{\Sigma}$ and $l = 1,2,3$, the solution $W^{l,\hat{x}} \in \Psi^l + \mathfrak{V}_1$ of problem* (75) *can be represented in terms of the Green matrix-function $G^{\hat{x}}_{ij}(y)$ as follows*

$$W^{l,\hat{x}}_i(y_1,y_2,y_3) = \left\langle \sigma^{l,\hat{x}}_j, \quad G^{\hat{x}}_{ij}(y_1 - \cdot, y_2 - \cdot, y_3)\right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \qquad (80)$$

where $\sigma^{l,\hat{x}}$ is defined by

$$\sigma^{l,\hat{x}} = (\sigma^{l,\hat{x}}_1, \sigma^{l,\hat{x}}_2, \sigma^{l,\hat{x}}_3) := (\sigma^{\hat{x}}_{13,y}(W^{l,\hat{x}}), \sigma^{\hat{x}}_{23,y}(W^{l,\hat{x}}), \sigma^{\hat{x}}_{33,y}(W^{l,\hat{x}}),$$

and $G^{\hat{x}}$ by (37), *with $\Phi^{\hat{x}}(\omega)$ satisfying* (38). *In addition, there is a positive constant C independent of $\hat{x}$, such that for $y \in \mathbb{R}^{3+}$, with $|y|$ large enough, we have*

$$|W^{l,\hat{x}}_i(y)| \leq C\frac{1}{|y|} \qquad \text{and} \qquad \left| \frac{\partial W^{l,\hat{x}}_i}{\partial y_p}(y) \right| \leq C\frac{1}{|y|^2}\,, \quad i,p = 1,2,3. \qquad (81)$$

Proof On account of (22) and (37), we follow the proof of Theorem 4.1 of [21], using a density argument, with minor modifications, to obtain the representation (80) for the solution of (75).

Estimates (81) are a consequence of (80) and the chains of inequalities

$$|W_i^{l,\hat{x}}(y^*)| \le C\Big\|\sigma^{\hat{x}}_{j3,y}(W^{l,\hat{x}})\Big\|_{H^{-1/2}(T)}\Big\|G^{\hat{x},y^*}_{ij}\Big\|_{H^1(T)} \le C\left(\frac{1}{d(y^*,\overline{T})}+\frac{1}{d(y^*,\overline{T})^2}\right)$$

and

$$\left|\frac{\partial W_i^{l,\hat{x}}}{\partial y_p}(y^*)\right| \le C\Big\|\sigma^{\hat{x}}_{j3,y}(W^{l,\hat{x}})\Big\|_{H^{-1/2}(T)}\Big\|\frac{\partial G^{\hat{x},y^*}_{ij}}{\partial y_p}\Big\|_{H^1(T)}$$
$$\le C\left(\frac{1}{d(y^*,\overline{T})^2}+\frac{1}{d(y^*,\overline{T})^3}\right),$$

where y^* is any point with $y_3^*>0$, C is a constant independent of both y^* and $\hat{x}$ and $G^{\hat{x},y^*}_{ij}$ is defined by

$$G^{\hat{x},y^*}_{ij}(\xi_1,\xi_2) = G^{\hat{x}}_{ij}(y_1^*-\xi_1,\, y_2^*-\xi_1,\, y_3^*) \qquad \forall(\xi_1,\xi_2)\in T$$

To obtain the above estimates, we follow the technique in Proposition 4.1 of [21] when the media is isotropic, with minor modifications, using the continuous embedding of $H^{1/2}(T)\subset H^1(T)$, formula (37) and the uniform bounds (38) and (78) (cf. also the proof of Corollary 1 for a more smooth vector-function $\sigma^{l,\hat{x}}$). Thus, the theorem is proved.

7.2 The convergence of solutions of stationary problems

Throughout this section, we employ $\widetilde{W}^{l,\varepsilon}$ constructed as in (44) replacing $W^{l,M,\widetilde{x}_{\mathrm{k}}}$ by $W^{l,\widetilde{x}_{\mathrm{k}}}$ in (43), namely,

$$W^{l,\mathrm{k},\varepsilon}(x) = W^{l,\widetilde{x}_{\mathrm{k}}}\left(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\right)\varphi^\varepsilon(x) \quad \text{for} \quad x\in B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon+\frac{\varepsilon}{4}\right). \tag{82}$$

For simplicity, we use the same notation for the global function $\widetilde{W}^{l,\varepsilon}$. Properties (45)-(49) in Proposition 3 also hold using (82) instead of (43). The same occurs with the convergence (52) in Proposition 4.

Hence, for $\phi\in(C^1(\overline{\Omega}))^3$, $\phi=0$ on Γ_Ω, we take the *test function* $v(x)=\phi_l(x)\widetilde{W}^{l,\varepsilon}(x)$ in (16). Since v vanishes on $\bigcup T^\varepsilon$, we have

$$\int_\Omega \sigma_{ij,x}(u^\varepsilon)e_{ij,x}(\phi_l\widetilde{W}^{l,\varepsilon})\,dx = \int_\Omega f_i\phi_l\widetilde{W}^{l,\varepsilon}_i\,dx. \tag{83}$$

Applying the same arguments as in (55)-(57), the passage to the limit in (83), gives

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij,x}(\phi)\,dx - \int_\Omega f_i\phi_i\,dx = -\lim_{\varepsilon\to0}\int_\Omega \sigma_{ij,x}(u^\varepsilon)e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx. \tag{84}$$

Accepting that the limit in (84) is given by $r_0 \int_\Sigma \mathcal{C}_{ij} u_i^0 \phi_j d\hat{x}$ (cf. Theorem 7 below), u^0 satisfies

$$\int_\Omega \sigma_{ij,x}(u^0) e_{ij,x}(\phi)\, dx + r_0 \int_\Sigma \mathcal{C}_{ij} u_i^0 \phi_j\, d\hat{x} = \int_\Omega f_i \phi_i\, dx \quad \forall \phi \in (C^1(\overline{\Omega}))^3,\ \phi\big|_{\Gamma_\Omega} = 0.$$

By a density argument, we conclude that u^0 is the unique solution of (24). Therefore, we have proved the following result.

Theorem 6 *The solution u^ε of* (16) *converges weakly in $(H^1(\Omega))^3$, as $\varepsilon \to 0$, towards the solution u^0 of* (24).

Finally, we obtain the limit in the right hand side of (84) as a consequence of the following theorem.

Theorem 7 *For any $u^0 \in \mathbf{V}$ which is the weak limit in $(H^1(\Omega))^3$ of a subsequence of u^ε, still denoted by ε, cf.* (18)*, we construct a sequence $\widetilde{u}^\varepsilon \in \mathbf{V}$, such that*

$$\widetilde{u}^\varepsilon = 0 \ \text{ on } \bigcup T^\varepsilon, \qquad \widetilde{u}^\varepsilon \xrightarrow{\varepsilon \to 0} u^0 \ \text{ weakly in } (H^1(\Omega))^3, \tag{85}$$

and, for any $\phi \in (C^1(\overline{\Omega}))^3$ with $\phi = 0$ on Γ_Ω the following convergences occur:

$$\lim_{\varepsilon \to 0} \int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx = r_0 \int_\Sigma \mathcal{C}_{ij} u_i^0 \phi_j d\hat{x}, \tag{86}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon - u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx = 0. \tag{87}$$

Proof We follow the steps in Section 6.2 with suitable modifications which we outline below. As a matter of fact, some integrals on T transform into dual products in $H^{-1/2}(T) \times H^{1/2}(T)$ and the corresponding proof must be changed.

First, note that the construction of $\widetilde{u}^\varepsilon$ satisfying (85) and (86) repeats the proof in Section 6.2.2: indeed, it suffices to take into account that all the integrals over T^ε (T respect.) containing $\widetilde{u}^\varepsilon$ ($\widetilde{W}^{l,\varepsilon}$ respect.) vanish, as well as the definition (77) of $\mathcal{C}$.

Now, we show (87) as follows. We repeat the proof in Section 6.2.1 to obtain

$$\begin{aligned} \mathbf{I}_\varepsilon &= - \int_\Omega \sigma_{ij,x}(u^\varepsilon - \widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx \\ &= r_\varepsilon \sum_{\widetilde{x}_k} \int_{B^+(0,1+\frac{\varepsilon}{4r_\varepsilon})} \sigma_{ij,y}^{\widetilde{x}_k}(W^{l,\widetilde{x}_k}) e_{ij,y}(d^\varepsilon \phi_l \varphi^\varepsilon) dy + o(1). \end{aligned}$$

Then, since $\varphi^\varepsilon = 0$ on $\partial B^+(\widetilde{x}_{\rm k}, r_\varepsilon + \frac{\varepsilon}{4}) \cap \{x_3 > 0\}$, $\varphi^\varepsilon = 1$ on $T^\varepsilon_{\widetilde{x}_{\rm k}}$, and $\widetilde{u}^\varepsilon = 0$ on $T^\varepsilon_{\widetilde{x}_{\rm k}}$, by applying the Green formula and relation (78) we get

$$\begin{aligned}
|\mathbf{I}_\varepsilon| &= r_\varepsilon \Big| \sum_{\widetilde{x}_{\rm k}} \big\langle \sigma^{\widetilde{x}_{\rm k}}_{i3,y}(W^{l,\widetilde{x}_{\rm k}}),\, \tau_y(d^\varepsilon_i \phi_l) \big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} + o(1) \Big| \\
&\leq r_\varepsilon \sum_{\widetilde{x}_{\rm k}} \big\| \sigma^{\widetilde{x}_{\rm k}}_{i3,y}(W^{l,\widetilde{x}_{\rm k}}) \big\|_{H^{-1/2}(T)} \big\| \tau_y(u^\varepsilon_i \phi_l) \big\|_{H^{1/2}(T)} + o(1) \\
&\leq C r_\varepsilon \varepsilon^{-1} \sum_{i=1}^{3} \Big(\sum_{\widetilde{x}_{\rm k}} \Big\| \tau_y u^\varepsilon_i \Big\|^2_{H^{1/2}(T)} \Big)^{1/2} + o(1).
\end{aligned}$$

Hence, performing the change $y \mapsto x$ in the integrals

$$\big\| \tau_y u^\varepsilon_i \big\|^2_{H^{1/2}(T)} = \int_T |\tau_y u^\varepsilon_i|^2 d\hat{y} + \int_T \int_T \frac{|u^\varepsilon_i(\hat{y}) - u^\varepsilon_i(\hat{y}')|^2}{|\hat{y} - \hat{y}'|^3} d\hat{y} d\hat{y}',$$

cf. (23), we obtain

$$\begin{aligned}
|\mathbf{I}_\varepsilon| &\leq C\varepsilon^{-1} \sum_{i=1}^{3} \Big(\sum_{\widetilde{x}_{\rm k}} \Big\| u^\varepsilon_i \Big\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_{\rm k}})} \Big)^{1/2} + o(1) \\
&\leq C \frac{1}{\varepsilon\beta(\varepsilon)} \sum_{i=1}^{3} \Big\| \beta(\varepsilon) \chi_{\bigcup T^\varepsilon} u^\varepsilon_i \Big\|_{H^{1/2}(\Sigma)} + o(1) \\
&= C \frac{1}{\varepsilon\beta(\varepsilon)} \sum_{i=1}^{3} \Big\| \sigma^\varepsilon_{i3} \Big\|_{H^{-1/2}(\Sigma)} + o(1) \leq C \frac{1}{\varepsilon\beta(\varepsilon)} + o(1).
\end{aligned}$$

Here, $\chi_{\bigcup T^\varepsilon}$ denotes the characteristic function of the set $\bigcup_{{\rm k}\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_{\rm k}}$, and we have used the equation on Σ in (16), cf. (9), the continuity of M, the trace embedding theorem and (17). Now, since $\beta^0 = +\infty$ in (2) and $r_0 > 0$ in (1), we have that $\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = 0$, and (87) holds. Thus, the theorem is proved.

7.3 The spectral convergence

In this section, we show the convergence of the eigenpairs of (11), when $r_0 > 0$ and $\beta^0 = +\infty$.

Theorem 8 *For each $k = 1, 2, 3 \ldots$, λ^ε_k in (12) and λ^0_k in (33) satisfy*

$$\lambda^\varepsilon_k \to \lambda^0_k, \quad \text{as } \varepsilon \to 0,$$

where $\{\lambda^0_k\}_{k=1}^\infty$ are the eigenvalues of (32) with $\mathcal{B}(\hat{x}) = \mathcal{C}(\hat{x})$. In addition, for each sequence, we can extract a subsequence, still denoted by ε, such that the corresponding eigenfunctions $u^{\varepsilon,k}$ converge towards $u^{0,k}$ in $(L^2(\Omega))^3$, where $u^{0,k}$ is an eigenfunction of (32) corresponding to λ^0_k, and the set $\{u^{0,k}\}_{k=1}^\infty$ forms an orthogonal basis in $(L^2(\Omega))^3$.

Proof We follow the scheme of the proof of Theorem 4 with minor modifications. We mainly apply Lemma 1 using the result and proof of Theorem 6 instead of Theorem 3.

Remark 1 In connection with the convergence of solutions in the rest of the cases stated in Section 3, we observe that when $r_0 = 0$ the convergence (48) takes place in $(H^1(\Omega))^3$, and the proof of convergence simplifies providing that u^0 in (18) is the solution of (28). When $r_0 = +\infty$, a very different technique should be applied to show convergence towards the solution of (27): cf. e.g., [15] in the case of scalar problem in porous media.

Remark 2 It should be emphasized that our technique allows us to apply and extend the results in [21] and [6], the regions T^ε being stuck to the plane, to the case where the media is heterogeneous and anisotropic. The technique can also be applied to other boundary homogenization problems, both scalar and vector, in heterogeneous media.

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