

REMARK ON JUSTIFICATION OF ASYMPTOTICS OF SPECTRA OF CYLINDRICAL WAVEGUIDES WITH PERIODIC SINGULAR PERTURBATIONS OF BOUNDARY AND COEFFICIENTS

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To perform an asymptotic analysis of spectra of singularly perturbed periodic waveguides, it is required to estimate remainders of asymptotic expansions of eigenvalues of a model problem on the periodicity cell uniformly with respect to the Floquet parameter. We propose two approaches to this problem. The first is based on the max–min principle and is sufficiently easily realized, but has a restricted application area. The second is more universal, but technically complex since it is required to prove the unique solvability of the problem on the cell for some value of the spectral parameter and the Floquet parameter in a nonempty closed segment, which is verified by constructing an almost inverse operator of the operator of an inhomogeneous model problem in variational setting. We consider boundary value problems on the simplest periodicity cell: a rectangle with a row of fine holes.

1 Introduction

1.1. Motivation. In this paper, we study the model problem on a periodicity cell coming from the Floquet–Bloch theory [1]–[4] in the analysis of the spectrum of a periodically perforated waveguide (cf. Figure 1). Eigenvalues of the model problem depending on the Floquet parameter $\eta \in [-\pi, \pi]$ (the dual variable of the Gelfand transform [5]) determine the location and size of spectral segments and gap opening between them, i.e., generate the band-gap structure of the waveguide spectrum. The main (and essentially new) problem appearing in the justification of asymptotic expansions of eigenvalues of model problems on the periodicity cell is to derive η -uniform estimates for remainders of asymptotic expansions since only such estimates provide a competent information about spectral segments. We note that estimates of such a quality are not necessary in the usual situation of a single spectral problem, but they are required if we deal with a family of problems parametrized by the Floquet parameter. The known justification schemes involve elements that, after adaptation to the class of problems under consideration, do not guarantee the required uniformity. In this paper, we discuss approaches to compensate this lack.

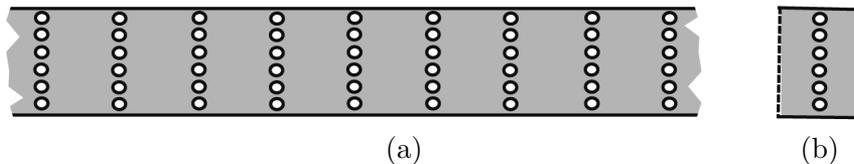


Figure 1. Periodic waveguide (a) and its periodicity cell (b).

We study several boundary value problems on the same cell of the simplest shape as in Figure 1 (b) for which we have different formulations and different proofs of the corresponding results. In Sections 2 and 3, calculations and arguments are rather simple since they are based on variational methods, whereas a new rather complicated and laborious analysis including new ideas and tools is required to study some problems in Section 4. The approaches we propose can be also applied to other methods of periodic singular perturbations of cylindrical and even originally periodic waveguides.

1.2. Statement of the problem. Let $\Omega = \{x = (x_1, x_2) : |x_1| < 1/2, |x_2| < H\}$ be a rectangle in the plane \mathbb{R}^2 . We consider the family of fine holes

$$\omega_j^\varepsilon = \{x : \xi^j := \varepsilon^{-1}(x_1, x_2 - 2H\varepsilon j) \in \omega\} \subset \Omega, \quad j = -N, \dots, N. \quad (1.1)$$

Here, ω is a domain such that its closure $\bar{\omega} = \omega \cup \partial\omega$ is contained in the rectangle $\square = (-h, h) \times (-H, H)$, h, H are fixed positive numbers, and $\varepsilon = (1 + 2N)^{-1}$ is a small parameter, i.e., N is a large natural number. In the perforated cell of a periodic waveguide (cf. Figure 1 (a), (b))

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{j=-N}^N \bar{\omega}_j^\varepsilon, \quad (1.2)$$

we consider the differential equation

$$-\Delta u^\varepsilon(x; \eta) = \lambda^\varepsilon(\eta)u^\varepsilon(x; \eta), \quad x \in \Omega^\varepsilon, \quad (1.3)$$

where Δ is the Laplace operator and λ^ε is the spectral parameter. We consider Equation (1.3)

with the Neumann condition and quasiperiodicity condition on the rectangle sides

$$\frac{\partial u^\varepsilon}{\partial n}(x_1, \pm H; \eta) := \pm \frac{\partial u^\varepsilon}{\partial x_2}(x_1, \pm H; \eta) = 0, \quad |x_1| < \frac{1}{2}, \quad (1.4)$$

$$\frac{\partial^p u^\varepsilon}{\partial x_1^p} \left(+\frac{1}{2}, x_2; \eta \right) = e^{i\eta} \frac{\partial^p u^\varepsilon}{\partial x_1^p} \left(-\frac{1}{2}, x_2; \eta \right), \quad |x_2| < H, \quad p = 0, 1, \quad (1.5)$$

where $\eta \in [-\pi, \pi]$ is the above-mentioned Floquet parameter caused by applying the Gelfand transform [5] to the problem in an infinite waveguide as in Figure 1 (a) and $\partial_n := \partial/\partial n$ is the normal derivative with respect to the inward normal on the boundaries of the holes (1.1) that are assumed to be piecewise smooth, consequently, n is the outward normal to the boundary $\partial\Omega^\varepsilon$ of the perforated rectangle. On the contours $\partial\omega_{-N}^\varepsilon, \dots, \partial\omega_N^\varepsilon$, we impose the Dirichlet conditions

$$u^\varepsilon(x; \eta) = 0, \quad x \in \partial\omega^\varepsilon := \bigcup_{j=-N}^N \partial\omega_j^\varepsilon \quad (1.6)$$

or the Neumann conditions

$$\partial_n u^\varepsilon(x; \eta) = 0, \quad x \in \partial\omega^\varepsilon. \quad (1.7)$$

In what follows, we deal with a composite rectangle. Together with Equation (1.3), we also consider the differential equation

$$-T\Delta u_{T\circ}^\varepsilon(x; \eta) = T\lambda^\varepsilon(\eta)u_{T\circ}^\varepsilon(x; \eta), \quad x \in \omega_j^\varepsilon, \quad j = -N, \dots, N, \quad (1.8)$$

in the holes (1.1) and the transmission conditions on the boundaries of the inclusions

$$u_{T\circ}^\varepsilon(x; \eta) = u_{T\Box}^\varepsilon(x; \eta), \quad T\partial_n u_{T\circ}^\varepsilon(x; \eta) = \partial_n u_{T\Box}^\varepsilon(x; \eta), \quad x \in \partial\omega_j^\varepsilon, \quad j = -N, \dots, N, \quad (1.9)$$

connecting the restrictions $u_{T\circ}^\varepsilon$ on ω^ε and $u_{T\Box}^\varepsilon$ on Ω^ε of the function u_T^ε defined in Ω and the normal derivatives of these restrictions. Moreover, $T \in (0, +\infty)$ is also a parameter of the problem. We emphasize that, in the particular case $T = 1$, the singular perturbation vanishes since the transmission conditions (1.9) are transformed to the continuity conditions, whereas Equations (1.3) and (1.8) remain unchanged (cf. Remark 1.2). For the sake of brevity we sometimes omit the subscripts T , η and \circ , \Box .

The problems (1.3)–(1.6) and (1.3)–(1.5), (1.7) will be denoted by $\mathcal{P}_D^\varepsilon(\eta)$ and $\mathcal{P}_N^\varepsilon(\eta)$ respectively, whereas the problem (1.3)–(1.5), (1.8), (1.9) will be mentioned as Problem $\mathcal{P}_T^\varepsilon(\eta)$. The variational statement of the last problem is associated with the integral identity

$$a_T^\varepsilon(u_T^\varepsilon, \psi; \Omega) = \lambda_T^\varepsilon b_T^\varepsilon(u_T^\varepsilon, \psi; \Omega), \quad \psi \in H_\eta^1(\Omega), \quad (1.10)$$

where $H_\eta^1(\Omega)$ is the Sobolev space of functions satisfying the stable ($p = 0$) quasiperiodicity condition (1.5). Furthermore, (1.10) involves the bilinear forms

$$\begin{aligned} a_T^\varepsilon(u_T^\varepsilon, \psi; \Omega) &= (\nabla_x u_T^\varepsilon, \nabla_x \psi)_{\Omega^\varepsilon} + T(\nabla_x u_T^\varepsilon, \nabla_x \psi)_{\omega^\varepsilon}, \\ b_T^\varepsilon(u_T^\varepsilon, \psi; \Omega) &= (u_T^\varepsilon, \psi)_{\Omega^\varepsilon} + T(u_T^\varepsilon, \psi)_{\omega^\varepsilon}, \end{aligned} \quad (1.11)$$

where $\nabla = \text{grad}$ and $(\cdot, \cdot)_{\Omega^\varepsilon}$ denotes the natural (scalar or vector) inner product in the Lebesgue space $L^2(\Omega^\varepsilon)$. The variational problem (1.10) will be mentioned as Problem $\mathcal{J}_T^\varepsilon(\eta)$. The integral

identities of Problem $\mathcal{J}_M^\varepsilon(\eta)$ corresponding to Problem $\mathcal{P}_M^\varepsilon(\eta)$ with $M = D$ and $M = N$ have the form

$$(\nabla u_M^\varepsilon, \nabla \psi^\varepsilon)_{\Omega^\varepsilon} = \lambda_M^\varepsilon (u_M^\varepsilon, \psi^\varepsilon)_{\Omega^\varepsilon}, \quad \psi^\varepsilon \in H_{\eta, M}^1(\Omega^\varepsilon). \quad (1.12)$$

As above, $H_{\eta, N}^1(\Omega^\varepsilon)$ is the Sobolev space of functions satisfying the first ($p = 0$) quasiperiodicity condition in (1.5), and functions of the subspace $H_{\eta, D}^1(\Omega^\varepsilon) \subset H_{\eta, N}^1(\Omega^\varepsilon)$ additionally satisfy the Dirichlet conditions (1.4) on the boundaries of fine holes.

Remark 1.1. The integral identity of Problem $\mathcal{J}_N^\varepsilon(\eta)$ can be obtained from Problem $\mathcal{J}_T^\varepsilon(\eta)$ as $T \rightarrow +0$, but the limit passage to “absolutely rigid” inclusions as $T \rightarrow +\infty$ does not lead to Problem $\mathcal{J}_D^\varepsilon(\eta)$. This situation is discussed in Subsection 5.1.

The left-hand sides of the integral identities (1.10) and (1.12) are positive symmetric and closed bilinear forms in the spaces $H_\eta^1(\Omega)$ and $H_{\eta, N}^1(\Omega^\varepsilon)$, $H_{\eta, D}^1(\Omega^\varepsilon)$ respectively. Hence Problems $\mathcal{P}_T^\varepsilon(\eta)$ and $\mathcal{P}_N^\varepsilon(\eta)$, $\mathcal{P}_D^\varepsilon(\eta)$ are associated (cf. [6, Chapter 10]) with unbounded selfadjoint positive operators $\mathcal{A}_T^\varepsilon(\eta)$ and $\mathcal{A}_N^\varepsilon(\eta)$, $\mathcal{A}_D^\varepsilon(\eta)$ in Hilbert spaces $L^2(\Omega)$ and $L^2(\Omega^\varepsilon)$. By [6, Theorems 10.1.5 and 10.2.2], the spectrum $\sigma_M^\varepsilon(\eta)$ of the operator $\mathcal{A}_M^\varepsilon(\eta)$ with $M = T, N, D$ forms a monotone unbounded sequence of eigenvalues

$$0 \leq \lambda_{M1}^\varepsilon \leq \lambda_{M2}^\varepsilon \leq \lambda_{M3}^\varepsilon \leq \dots \leq \lambda_m^\varepsilon \leq \dots \rightarrow +\infty, \quad (1.13)$$

where the multiplicity is taken into account.

We can assume that the eigenfunctions $u_{M1}^\varepsilon, u_{M2}^\varepsilon, u_{M3}^\varepsilon, \dots, u_m^\varepsilon, \dots$ of the problems (1.10) with $M = T$ and (1.12) with $M = N, D$ respectively satisfy the orthonormality conditions

$$b_T^\varepsilon(u_{Mm}^\varepsilon, u_{Mn}^\varepsilon)_\Omega = \delta_{m,n} \quad (1.14)$$

and

$$(u_{Mm}^\varepsilon, u_{Mn}^\varepsilon)_{\Omega^\varepsilon} = \delta_{m,n}, \quad M = N, D. \quad (1.15)$$

Here, $\delta_{m,n}$ is the Kronecker symbol and $m, n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Moreover, $\lambda_{D1}^\varepsilon > 0$ and the operator $\mathcal{A}_D^\varepsilon(\eta)$ is positive definite. The problem with the Dirichlet conditions is studied in Section 3, but the most difficult point of the paper appears when we study Problem $\mathcal{P}_T^\varepsilon(\eta)$ which is used to pass continuously from Problem $\mathcal{P}_0^\varepsilon(\eta)$ with the Neumann perforation to Problem $\mathcal{P}_1^\varepsilon(\eta)$ on the whole cell.

Remark 1.2. For $T = 1$ the forms (1.11) become $(\nabla u, \nabla \psi)_\Omega$ and $(u, \psi)_\Omega$, i.e., the foreign inclusions disperse, whereas Problem $\mathcal{P}_T^\varepsilon(\eta)$ with $T = 1$, denoted by $\mathcal{P}(\eta)$, loses its dependence on the small parameter ε . As a result, the eigenpairs $\{\lambda(\eta), u(\cdot; \eta)\}$ satisfying the differential equation in the whole rectangle

$$-\Delta u(x; \eta) = \lambda(\eta)u(x; \eta), \quad x \in \Omega, \quad (1.16)$$

and the conditions (1.4) and (1.5) on its sides (the superscript ε is not necessary now) take the explicit form

$$\begin{aligned} \lambda_{p,q}(\eta) &= \frac{\pi^2 p^2}{4H^2} + (\eta + 2\pi q)^2, \\ u_{p,q}(x; \eta) &= \cos\left(\frac{\pi p}{2H}(x_2 + H)\right) e^{i(\eta + 2\pi q)x_1}, \end{aligned} \quad (1.17)$$

where $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $q \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. It is clear that the eigenvalues (1.17) with two indices should be re-enumerated to obtain a monotone sequence (1.13). The re-enumerated

eigenpairs are denoted by $\{\lambda_j(\eta), u_j(\cdot; \eta)\}$ with one index $j \in \mathbb{N}$, and the eigenfunctions satisfy the orthonormality conditions (cf. (1.14) in the case $T = 1$)

$$(u_j(\cdot; \eta), u_k(\cdot; \eta))_\Omega = \delta_{j,k}, \quad j, k \in \mathbb{N}. \quad (1.18)$$

1.3. Two scenarios of obtaining uniform estimates for remainders. Usually, a scheme for justifying asymptotic expansions of eigenvalues and eigenfunction (eigenvectors) of singularly perturbed spectral problems consists of three steps. First, formal asymptotic expansion of eigenpairs is constructed. Second, the global asymptotic approximation to the eigenfunction is obtained when we glue together the external and internal expansions obtained by the method of matched asymptotic expansions (cf., for example, [7, 8]) or summarize smooth type solutions with boundary layer type solutions within the framework of the method of composite asymptotic expansions (cf., for example, [9, 10]). Then it is possible to calculate the residuals generated by the approximate eigenpair in the singularly perturbed initial problem and, finally, to apply the lemma about almost eigenvalues and eigenvectors [11] provided by the spectral representation of the resolvent (cf., for example, [6, Chapter 6]). Thus, at the second step, in a small neighborhood of the $\varkappa_k(\eta)$ -multiple eigenvalue $\lambda_{Mk}^0(\eta)$ of the limit problem, we can find the terms

$$\lambda_{MP_k^\varepsilon(\eta)}^\varepsilon(\eta), \dots, \lambda_{MP_k^\varepsilon(\eta) + \varkappa_k(\eta) - 1}^\varepsilon(\eta) \quad (1.19)$$

of the subsequence of eigenvalues of the initial problem $\mathcal{P}_M^\varepsilon(\eta)$. It is important that the factor $c(\eta)$ in the final estimates

$$|\lambda_{MP_k^\varepsilon(\eta) + p}^\varepsilon(\eta) - \lambda_{Mk}^0(\eta)| \leq c_k(\eta)\varepsilon^{\alpha_0}, \quad p = 0, \dots, \varkappa_k(\eta) - 1, \quad (1.20)$$

can be made independent of the Floquet parameter $\eta \in [\eta_0 - \delta_0, \eta_0 + \delta_0]$ for some $\alpha_0 \in (0, 1]$ and $\eta_0 \in (-\pi, \pi]$, $\delta_0 > 0$ (we refer to Subsection 5.2 for details).

It remains to verify that $P_k^\varepsilon(\eta) = k$ in the list (1.19), i.e., to realize the third step of the scheme. Traditionally, for this purpose one uses the so-called convergence theorems establishing the limit passage as $\varepsilon \rightarrow +0$ (usually, the spaces in (1.21) should be specified, but it is not necessary for our goals):

$$\begin{aligned} \lambda_{Mj}^\varepsilon(\eta) &\rightarrow \lambda_{MJ(j)}^0(\eta), \\ u_{Mj}^\varepsilon(\cdot; \eta) &\rightarrow u_{MJ(j)}^0(\cdot; \eta) \text{ weakly in } H^1 \text{ and strongly in } L^2. \end{aligned} \quad (1.21)$$

The number $J(j) \in \mathbb{N}$ of the eigenpair $\{\lambda_{MJ(j)}^0(\eta), u_{MJ(j)}^0(\cdot; \eta)\}$ of the limit problem is not specified at this step, but it becomes j after completing proofs of all necessary results. Thus, owing to formula (1.21) for any $j \in \mathbb{N}$ and $\eta \in [-\pi, \pi]$, it is possible to prove by contradiction that, in the case $\varepsilon \in (0, \varepsilon_j(\eta)]$, there are no ‘‘superfluous’’ eigenvalues satisfying (1.20). Therefore, it is easy to prove that $P_k^\varepsilon(\eta) = k$ in the list (1.19) for sufficiently small ε .

We note the brittleness of the third step: we do not know the character of dependence of $\varepsilon_j(\eta)$ on j and η . Consequently, it is not clear whether the estimates (1.20) with $P_k^\varepsilon(\eta) = k$ are uniform with respect to the Floquet parameter.

In this paper, we propose two approaches to overcoming the above-mentioned difficulty arising in the justification of asymptotic expansions of spectral segments

$$v_{Mk}^\varepsilon = \{\lambda = \lambda_{Mk}^\varepsilon(\eta) \mid \eta \in [-\pi, \pi]\}, \quad k \in \mathbb{N}, \quad (1.22)$$

generating the spectrum of an infinite waveguide with the periodicity cell (1.2) (cf. Figure 1 (a)). We emphasize that if estimates for the remainder in the asymptotic expansions of the eigenvalues $\lambda_{Mk}^\varepsilon(\eta)$ are not uniform with respect to η , then we cannot make any conclusion about geometric characteristics of the connected compact set v_{Mk}^ε for $\varepsilon \in (0, \varepsilon_{Mk}]$ even if $\varepsilon_{Mk} > 0$ is small.

The first approach (cf. Sections 2 and 3) is to somehow use the classical max–min principle (cf., for example, [6, Theorem 10.2.2]) to prove the inequalities

$$\lambda_{Mk}^0(\eta) - c(\eta)\varepsilon^\theta \leq \lambda_{Mk}^\varepsilon(\eta) \leq \lambda_{Mk}^0(\eta) + c(\eta)\varepsilon^\theta, \quad (1.23)$$

where $\theta > 0$ and

$$c(\eta) \leq c_\theta, \quad \eta \in [-\pi, \pi]. \quad (1.24)$$

By (1.23) and (1.24), it is easy to show that it is not necessary to use convergence theorems in the justification scheme and we can verify that $P_k^\varepsilon(\eta) = k$ in the list (1.19) for every η (we refer the reader for details to [12], where this approach is realized for a singularly perturbed cell of some other shape in the space \mathbb{R}^d , $d = 3$).

According to the second approach, it is not necessary to exclude the above-mentioned convergence theorems from the scheme, but we can ignore the fact that we do not know whether the convergences (1.21) are uniform by verifying the following assertion: if λ_\bullet is not eigenvalue of the limit problem $\mathcal{P}_T^0(\eta)$ for any $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, then the point λ_\bullet does not belong to the spectrum (1.13) of Problem $\mathcal{P}_T^\varepsilon(\eta)$ for $\varepsilon \in (0, \varepsilon(\eta_\bullet, \delta_\bullet)]$ and $\varepsilon(\eta_\bullet, \delta_\bullet) > 0$. Thus, the arcs $\{\lambda = \lambda_k^\varepsilon(\eta) \mid \eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]\}$ of dispersion curves do not intersect the segment

$$\Lambda_\bullet := \{(\lambda, \eta) \mid \lambda = \lambda_\bullet, \eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]\}. \quad (1.25)$$

Hence for $|\eta - \eta_\bullet| \leq \delta_\bullet$ the multiplicity of the spectrum $\sigma_T^\varepsilon(\eta)$ on the closed interval $[0, \lambda_\bullet]$ is constant for all $\varepsilon \in (0, \varepsilon(\eta_\bullet, \delta_\bullet)]$. Thus, it becomes indifferent for which value of the parameter the multiplicity is calculated.

In Subsection 5.2, we present another way to use this fact. Namely, since the eigenvalues $\lambda_{Tk}^\varepsilon(\eta)$ of Problem $\mathcal{P}_T^\varepsilon(\eta)$ continuously depend on two parameters $T \in [0, 1]$ and $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, the dispersion curves cannot intersect the horizontal segment (1.25), and, consequently, on the segment $[0, \lambda_\bullet]$, the multiplicity $\#\sigma(\eta)$ of the spectrum $\sigma(\eta)$ of Problem $\mathcal{P}(\eta) = \mathcal{P}_T^\varepsilon(\eta)|_{T=0}$ on the whole cell Ω coincides with the multiplicity $\#\sigma_N^\varepsilon(\eta)$ of the spectrum $\sigma_N^\varepsilon(\eta)$ of Problem $\mathcal{P}_N^\varepsilon(\eta) = \mathcal{P}_T^\varepsilon(\eta)|_{T=0}$ on the cell with the Neumann perforation. Thus, the asymptotics of the spectral segments (1.22) is justified without convergence theorems since the lemma about almost eigenvalues and eigenvectors asserts that there are at least $\#\sigma(\eta)$ eigenvalues (1.19) of the problem (1.3)–(1.5), (1.7) on $[0, \lambda_\bullet]$. By the aforesaid, the number of eigenvalues is equal to $\#\sigma(\eta)$.

1.4. Structure of the paper. In Section 2, we use the max–min principle to obtain the right inequality in (1.23) for the eigenvalues of Problem $\mathcal{P}_T^\varepsilon(\eta)$; here, we use rather elementary calculations. This method does not provide the required result completely, but we can slightly modify Equation (1.8) (cf. Subsection 2.2), and then use the same principle to obtain the left inequality in (1.23) for $T > 1$. Unfortunately, the modified problem is useless to study Problem $\mathcal{P}_N^\varepsilon(\eta)$ (cf. Remark 5.1).

In Section 3, we study Problem $\mathcal{P}_D^\varepsilon(\eta)$. Owing to the Dirichlet conditions on the hole boundaries, we can derive weighted estimates (Lemma 3.1 and Proposition 3.1). Based on these estimates, we use the max–min principle to prove (1.23) completely. In a sense, the problem

with the conditions (1.6) is simpler than other problems. In Subsection 5.3, we show that the second approach is also applicable to obtain uniform estimates for remainders of asymptotic expansions.

Section 4 represents a technically difficult result. We show how the second approach is realized by considering Problem $\mathcal{P}_N^\varepsilon(\eta)$. To establish the unique solvability of Problem $\mathcal{P}_T^\varepsilon(\eta)$ with parameters $\lambda = \lambda_\bullet$ and $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, we construct almost inverse operators for the operators $A_T^\varepsilon(\lambda_\bullet; \eta)$ of the family of problems. We first use the trick [13] of smoothing the right-hand side of the singularly perturbed problem and then, in fact, repeat the procedure for constructing asymptotics for the solutions and estimate the appeared small residuals.

The proposed method for verifying whether there are eigenvalues on the segment (1.25) can be also used in other approaches to the study of singular perturbations of periodicity cells. Therefore, we mention some works dealing with differential equations with strongly contrast coefficients [14]–[17], the case where the periodicity cells split in limit [18]–[21], the case of thin domains [22]–[25], and the case of regular and singular perturbations of boundaries [26]–[28].

In Section 5, we describe the second approach by considering Problems $\mathcal{P}_T^\varepsilon(\eta)$, $T \in [0, 1]$, and $\mathcal{P}_D^\varepsilon(\eta)$. This section also contains auxiliary results used in this paper.

2 Max–Min Principle

2.1. Problem $\mathcal{P}_T^\varepsilon(\eta)$. By [6, Theorem 10.2.2], the term numbered by $k \in \mathbb{N}$ in the subsequence (1.14) $_T$ is expressed by

$$\lambda_{Tk}^\varepsilon(\eta) = \max_{\mathcal{E}_T^k(\eta)} \inf_{u^\varepsilon \in \mathcal{E}_T^k(\eta) \setminus \{0\}} \frac{a_T^\varepsilon(u^\varepsilon, u^\varepsilon; \Omega)}{b_T^\varepsilon(u^\varepsilon, u^\varepsilon; \Omega)}, \quad (2.1)$$

where we have the bilinear forms (1.11) and $\mathcal{E}_T^k(\eta)$ is any subspace of the space $H_\eta^1(\Omega)$ of codimension $k - 1$, in particular, $\mathcal{E}_T^1(\eta) = H_\eta^1(\Omega)$.

We denote by $\mathcal{L}^k(\eta)$ the linear span of the eigenfunctions $u_1(\cdot; \eta), \dots, u_k(\cdot; \eta)$ of the problem (1.16), (1.4), (1.5) (cf. Remark 1.2). Since $\dim \mathcal{L}^k(\eta) = k$, the intersection $\mathcal{E}_T^k(\eta) \cap \mathcal{L}^k(\eta)$ is not empty: it contains a nontrivial linear combination

$$\mathcal{U}^{\mathcal{E}_T^k(\eta)}(x; \eta) = \sum_{j=1}^k \alpha_j^{\mathcal{E}_T^k(\eta)} u_j(x; \eta), \quad (2.2)$$

where $\sum_{j=1}^k |\alpha_j^{\mathcal{E}_T^k(\eta)}|^2 = 1$. By (1.18), (2.1), (2.2),

$$\lambda_{Tk}^\varepsilon(\eta) \leq \max_{\mathcal{E}_T^k(\eta)} \frac{a_T^\varepsilon(\mathcal{U}^{\mathcal{E}_T^k(\eta)}, \mathcal{U}^{\mathcal{E}_T^k(\eta)}; \Omega)}{b_T^\varepsilon(\mathcal{U}^{\mathcal{E}_T^k(\eta)}, \mathcal{U}^{\mathcal{E}_T^k(\eta)}; \Omega)}. \quad (2.3)$$

In what follows, we omit the superscript $\mathcal{E}_T^k(\eta)$ in the notation. Since

$$|a_T^\varepsilon(\mathcal{U}, \mathcal{U}; \Omega) - (\nabla \mathcal{U}, \nabla \mathcal{U})_\Omega| = |T - 1|(\nabla \mathcal{U}, \nabla \mathcal{U})_{\omega^\varepsilon} \leq c_k |T - 1| \text{mes } 2\omega^\varepsilon, \quad (2.4)$$

where $\text{mes } 2\omega^\varepsilon = (1 + 2N)\varepsilon^2 \text{mes } 2\omega = O(\varepsilon)$ and $\text{mes } 2\omega$ is the area of ω , we see that the majorant in (2.4) does not exceed $C_k |T - 1| \varepsilon$; moreover, C_k is independent of k and can be taken the same

for all $\eta \in [-\pi, \pi]$. Thus,

$$\begin{aligned} |a_T^\varepsilon(\mathcal{U}, \mathcal{U}; \Omega) - (\nabla \mathcal{U}, \nabla \mathcal{U})_\Omega| &\leq C_k |T - 1| \varepsilon, \\ |b_T^\varepsilon(\mathcal{U}, \mathcal{U}; \Omega) - (\mathcal{U}, \mathcal{U})_\Omega| &\leq C_k |T - 1| \varepsilon, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} (\nabla \mathcal{U}, \nabla \mathcal{U})_\Omega &= \sum_{j=1}^k \lambda_j(\eta) |\alpha_j|^2 \leq \lambda_k(\eta), \\ (\mathcal{U}, \mathcal{U})_\Omega &= \sum_{j=1}^k |\alpha_j|^2 = 1 \end{aligned} \quad (2.6)$$

since $\lambda_j(\eta) \leq \lambda_k(\eta)$, $j = 1, \dots, k$. Thus, from (2.3)–(2.6) we obtain the following assertion.

Proposition 2.1. *The eigenvalues (1.13) of Problem $\mathcal{P}_T^\varepsilon(\eta)$ satisfy the estimate*

$$\lambda_k^\varepsilon(\eta) \leq \lambda_k(\eta) + C_k |T - 1| \varepsilon, \quad (2.7)$$

where $k \in \mathbb{N}$, $\lambda_k(\eta)$ are the eigenvalues of the limit problem $\mathcal{P}(\eta)$ (cf. Remark 1.2) and C_k are independent of $\eta \in [-\pi, \pi]$ and $T \geq 0$.

The case $T = 0$ corresponding to Problem $\mathcal{P}_N^\varepsilon(\eta)$ with the Neumann condition (cf. Remark 1.2 and Subsection 5.1) is covered by Proposition 2.1 because no essential modifications are required to verify formula (2.7) in this case. As shown in Subsection 3.3, the eigenvalues of Problem $\mathcal{P}_D^\varepsilon(\eta)$ with the Dirichlet conditions satisfy an equality similar to (2.7).

2.2. The modified problem $\mathcal{P}_T^\varepsilon(\eta)$. Unfortunately, the max–min principle does not yield immediately the left inequality in (1.23) for the eigenvalues of Problem $\mathcal{P}_T^\varepsilon(\eta)$. We discuss one problem to which this approach is applicable by using elementary calculations. Namely, for $T > 1$ we replace Equation (1.8) with the following:

$$-T \Delta u_T^{\varepsilon \flat}(x; \eta) = \lambda_T^{\varepsilon \flat}(\eta) u_T^{\varepsilon \flat}(x; \eta), \quad x \in \omega_j^\varepsilon, \quad j = -N, \dots, N. \quad (2.8)$$

In other words, we eliminate T from the right-hand sides of the differential equations (1.8), and thereby the “material density” of the cell Ω becomes constant. We assign the symbol \flat denoting this operation to ingredients of the relations (1.3)–(1.5), (1.9) generating Problem $\mathcal{P}_T^{\varepsilon \flat}(\eta)$. The eigenvalues of this problem are denoted by $\lambda_{Tj}^{\varepsilon \flat}(\eta)$. We assume that the corresponding eigenfunctions $u_{Tj}^{\varepsilon \flat}(\cdot; \eta)$ satisfy the orthonormality conditions (1.18).

We apply the max–min principle to the operator $\mathcal{A}(\eta)$ of Problem $\mathcal{P}(\eta) = \mathcal{P}_T^\varepsilon(\eta)|_{T=1}$, consisting of Equations (1.16), (1.4), (1.5) without the parameter ε and make the required changes in formula (2.1). The linear span $\mathcal{L}_k^{\varepsilon \flat}(\eta)$ of the eigenfunctions $u_{T1}^{\varepsilon \flat}(\cdot; \eta), \dots, u_{Tk}^{\varepsilon \flat}(\cdot; \eta)$ of Problem $\mathcal{P}_T^{\varepsilon \flat}(\eta)$ intersects each subspace $\mathcal{E}_{Nk}(\eta)$ in the max–min principle, whereas the intersection of these sets contains the following linear combination similar to (2.2):

$$\mathcal{W}^\flat(x; \eta) = \sum_{j=1}^k \alpha_j^\flat(\eta) u_j^{\varepsilon \flat}(\cdot; \eta).$$

Since $\|\mathcal{W}^\flat(\cdot; \eta); L^2(\Omega)\| = 1$ in view of (1.18) and the second identity in (2.2), we have

$$\lambda_k(\eta) \leq \max_{\mathcal{E}_{Nk}(\eta)} \frac{(\nabla \mathcal{W}^\flat(\cdot; \eta), \nabla \mathcal{W}^\flat(\cdot; \eta))_\Omega}{(\mathcal{W}^\flat(\cdot; \eta), \mathcal{W}^\flat(\cdot; \eta))_\Omega}$$

$$= \max_{\mathcal{E}_{Nk}^\varepsilon(\eta)} (a_T^\varepsilon(\mathcal{W}^b(\cdot; \eta), \mathcal{W}^b(\cdot; \eta); \Omega) - (T-1) \|\nabla \mathcal{W}^b(\cdot; \eta); L^2(\Omega)\|^2) \leq \lambda_{Tk}^{\varepsilon^b}(\eta). \quad (2.9)$$

Proposition 2.2. *If $T > 1$, then the eigenvalues of the problems (1.3)–(1.5), (2.8), (1.9) and (1.16), (1.4), (1.5) are connected by*

$$\lambda_k(\eta) \leq \lambda_{Tk}^{\varepsilon^b}(\eta) \leq \lambda_k(\eta) + C_k^b(T-1)\varepsilon, \quad (2.10)$$

where C_k^b is independent of the Floquet parameter $\eta \in [-\pi, \pi]$ and small parameter $\varepsilon \in (0, \varepsilon_{Tk}^b)$ with some $\varepsilon_{Tk}^b > 0$.

Proof. It suffices to note that the transformations providing Proposition 2.1 can be also used for the Rayleigh fraction in the max–min principle

$$\lambda_{Tk}^{\varepsilon^b}(\eta) = \max_{\mathcal{E}_T^{\varepsilon^b}(\eta)} \inf_{u^\varepsilon \in \mathcal{E}_T^{\varepsilon^b}(\eta) \setminus \{0\}} \frac{a_T^\varepsilon(u^\varepsilon, u^\varepsilon; \Omega)}{(u^\varepsilon, u^\varepsilon)_\Omega} \quad (2.11)$$

for the eigenvalues of the operator $\mathcal{A}_T^{\varepsilon^b}(\eta)$, i.e., in fact, the second inequality in (2.10) is not different from (2.7). The first inequality in (2.10) is contained in (2.9). \square

3 Perforations with Dirichlet Conditions

3.1. The limit problem \mathcal{P}_D^0 . By the conditions (1.6) on the boundaries of the densely located fine holes (1.1), Problem $\mathcal{P}_D^\varepsilon(\eta)$ significantly differs from other problems considered in the paper because of the Poincaré–Friedrichs inequality

$$\varepsilon^{-2} \|u^\varepsilon; L^2(\square_j^{2\varepsilon h} \setminus \omega_j^\varepsilon)\|^2 \leq c_{h,\omega} \|\nabla u^\varepsilon; L^2(\square_j^{2\varepsilon h} \setminus \omega_j^\varepsilon)\|^2, \quad (3.1)$$

valid in view of the Dirichlet condition on $\partial\omega_j^\varepsilon$, where $\square_j^{2\varepsilon h} = \{x : |x_1| < 2h\varepsilon, |x_2 - 2H\varepsilon j| < 2\varepsilon H\}$; moreover, $\overline{\omega_j^\varepsilon} \subset \square_j^{\varepsilon h}$. The inequality (3.1) is verified by stretching variables $x \mapsto \xi^j$ (cf. formula (1.1)). Summarizing the inequality (3.1) with respect to $j = -N, \dots, N$, we get the estimate

$$\varepsilon^{-2} \|u^\varepsilon; L^2(\Omega_{2\varepsilon h}^\varepsilon)\|^2 \leq c_{h,\omega} \|\nabla u^\varepsilon; L^2(\Omega_{2\varepsilon h}^\varepsilon)\|^2, \quad (3.2)$$

where $\Omega_\ell^\varepsilon = \{x \in \Omega^\varepsilon : |x_1| < \ell\}$. Since the factor on the left-hand side of (3.2) is large, we can conclude that the limit problem of $\mathcal{P}_D^\varepsilon(\eta)$ is Problem $\mathcal{P}_D^0(\eta)$ consisting of the differential equation

$$-\Delta u^0(x; \eta) = \lambda^0 u^0(x; \eta), \quad x \in \Omega \setminus \Upsilon, \quad (3.3)$$

the conditions (1.4), (1.5) on the lateral sides of the rectangle Ω , and the Dirichlet condition on its vertical mean line $\Upsilon = \{x : x_1 = 0, |x_2| < H\}$

$$u^0(x; \eta) = 0, \quad x \in \Upsilon. \quad (3.4)$$

A detailed analysis of the limit passage as $\varepsilon \rightarrow +0$ can be found in [29].

The problem (3.3), (3.4), (1.4), (1.5) has the explicit solution

$$\lambda_{p,q}^0 = \pi^2 q^2 + \frac{\pi^2 p^2}{4H^2}, \quad p \in \mathbb{N}, \quad q \in \mathbb{Z}, \quad (3.5)$$

$$u_{p,q}^0(x; \eta) = \begin{cases} \sin(\pi q x_1) \cos(\pi p(2H)^{-1}(x_2 + H)), & x_1 > 0, \\ e^{-i\eta} \sin(\pi q x_1) \cos(\pi p(2H)^{-1}(x_2 + H)), & x_1 < 0. \end{cases} \quad (3.6)$$

It is remarkable that the eigenvalues (3.5) are independent of the Floquet parameter. Owing to this fact, the asymptotic analysis in [29] shows that there are narrow spectral segments (cf. Figure 2) separated by wide gaps in the spectrum of the periodic waveguide with Dirichlet perforation (cf. Figure 1 (b)). We emphasize that the dependence of the eigenfunctions (3.6) on η is largely fictitious; the periodicity cell in the waveguide is taken in an arbitrary way and, by the quasiperiodicity conditions (1.5), the eigenfunctions in the shifted cell $\Omega^{\leftrightarrow} = (0, 1) \times (-H, H)$ take the form $\sin(\pi q x_1) \cos(\pi p(2H)^{-1}(x_2 + H))$ without the Floquet parameter.

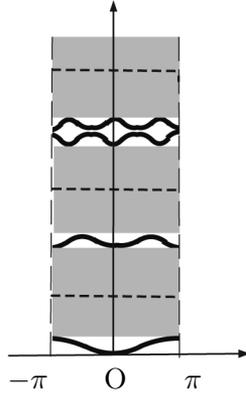


Figure 2. The dispersion curves for Problem $\mathcal{P}_D^\varepsilon$ are shown with bold lines. The wide gaps are the projections of tinted rectangles on the ordinate axis. The dashed lines indicate admissible values λ_\bullet of the spectral parameter for which it is easy to construct the almost inverse operators (cf. Subsection 5.3).

We omit the argument η even in the notation of the functions (3.6) and denote by \mathcal{P}_D the limit problem (3.3), (3.4), (1.4), (1.5). We enumerate the eigenvalues (3.5) in nondescending order and re-numerate terms of the subsequence $\{\lambda_j^0\}_{j \in \mathbb{N}}$ with one index. We assume that the corresponding eigenfunctions u_j^0 satisfy the orthonormality conditions (1.18).

3.2. Weighted estimates. We first apply the one-dimensional Hardy inequality

$$\int_0^R t^{-2} |V(t)|^2 dt \leq 4 \int_0^R \left| \frac{dV}{dt}(t) \right|^2 dt \quad (3.7)$$

which is valid for any $R > 0$ and $V \in C_c^1(0, R]$ vanishing at $t = 0$. Indeed,

$$\begin{aligned} \int_0^R t^{-2} |V(t)|^2 dt &= 2 \int_0^R t^{-2} \int_0^t V(\tau) \frac{dV}{d\tau}(\tau) d\tau dt \leq 2 \int_0^R |V(\tau)| \left| \frac{dV}{d\tau}(\tau) \right| \int_\tau^R t^{-2} dt d\tau \\ &= 2 \int_0^R |V(\tau)| \left| \frac{dV}{d\tau}(\tau) \right| \left(\frac{1}{\tau} - \frac{1}{R} \right) d\tau \leq 2 \left(\int_0^R \tau^{-2} |V(\tau)|^2 d\tau \right)^{1/2} \left(\int_0^R \left| \frac{dV}{d\tau}(\tau) \right|^2 d\tau \right)^{1/2}. \end{aligned}$$

Lemma 3.1. For $u^\varepsilon \in H_{\eta, D}^1(\Omega^\varepsilon)$

$$\|(\varepsilon^2 + x_1^2)^{-1/2} u^\varepsilon; L^2(\Omega^\varepsilon)\| \leq c_\omega \|\nabla u^\varepsilon; L^2(\Omega^\varepsilon)\|, \quad (3.8)$$

where c_ω is independent of ε and u^ε .

Proof. Let $\mathcal{X}_\varepsilon \in C_c^\infty(\mathbb{R})$ be a cut-off function such that

$$\begin{aligned} \mathcal{X}_\varepsilon(x_1) &= \begin{cases} 1, & |x_1| > 2\varepsilon h, \\ 0, & |x_1| < \varepsilon h, \end{cases} \quad 0 \leq \mathcal{X}_\varepsilon \leq 1, \\ \left| \frac{\partial \mathcal{X}_\varepsilon}{\partial x_1}(x_1) \right| &\leq \frac{C\mathcal{X}}{\varepsilon h}, \quad |x_1| \in [\varepsilon h, 2\varepsilon h], \\ \frac{\partial \mathcal{X}_h^\varepsilon}{\partial x_1}(x_1) &= 0, \quad |x_1| \notin [\varepsilon h, 2\varepsilon h]. \end{aligned} \quad (3.9)$$

By (3.3), the product $\mathcal{X}_\varepsilon(x_1)u^\varepsilon(x_1, x_2)$ satisfies the inequalities

$$\|\nabla(\mathcal{X}_\varepsilon u^\varepsilon); L^2(\Omega)\|^2 \leq c(\|\nabla u^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \varepsilon^{-2}\|u^\varepsilon; L^2(\text{supp } |\nabla \mathcal{X}_\varepsilon|)\|^2) \leq c\|\nabla u^\varepsilon; L^2(\Omega^\varepsilon)\|^2.$$

Integrating (3.7) with $V(t, x_2) = \mathcal{X}_h^\varepsilon(t)u^\varepsilon(t, x_2)$ with respect to $x_2 \in (-H, H)$, we get

$$\| |x_1|^{-1}V^\varepsilon; L^2(\Omega^\varepsilon \setminus \Omega_h^\varepsilon) \|^2 \leq c\| |x_1|^{-1}V^\varepsilon; L^2(\Omega) \|^2 \leq c\|\nabla V^\varepsilon; L^2(\Omega) \|^2 \leq c\|\nabla u^\varepsilon; L^2(\Omega^\varepsilon) \|^2. \quad (3.10)$$

From (3.1) and (3.10) we obtain the estimate (3.8). \square

In the following assertion, we use the trick proposed in [30] and used, in particular, in [12].

Proposition 3.1. *Let $u_k^\varepsilon(\cdot; \eta) \in H_{\eta, D}^1(\Omega^\varepsilon)$ be the normalized (by (1.18)) eigenfunction of Problem $\mathcal{P}_D^\varepsilon(\eta)$ corresponding to the eigenvalue $\lambda_k^\varepsilon(\eta)$. Then there are $\theta \in (0, 1]$, $\varepsilon_0 > 0$, $c_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ and $\eta \in [-\pi, \pi]$*

$$\|(\varepsilon^2 + x_1^2)^{-\theta} \nabla u_k^\varepsilon(\cdot; \eta); L^2(\Omega^\varepsilon)\| + \|(\varepsilon^2 + x_1^2)^{-1-\theta} u_k^\varepsilon(\cdot; \eta); L^2(\Omega^\varepsilon)\| \leq c_0 \lambda_k^\varepsilon(\eta). \quad (3.11)$$

Proof. We substitute $\psi^\varepsilon = R_\varepsilon^{-2\theta} u_k^\varepsilon \in H_{\eta, D}^1(\Omega^\varepsilon)$, where $R_\varepsilon(x) = (\varepsilon^2 + x_1^2)^{1/2}$, into the integral identity (1.12)_D. In what follows, we omit η in the notation. We set $U_k^\varepsilon = R_\varepsilon^{-\theta} u_k^\varepsilon$. By (3.8), we have

$$\|U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq \|R_\varepsilon^{-1} u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq c\|\nabla u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 = c\lambda_k^\varepsilon(\eta)$$

since $R_\varepsilon(x)^{1-\theta} \leq (\varepsilon^2 + 1/4)^{1-\theta} \leq 1$ for $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 \leq \sqrt{3}/2$. A simple transformation shows that

$$\begin{aligned} \lambda_k^\varepsilon \|U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 &= \lambda_k^\varepsilon (U_k^\varepsilon, \psi^\varepsilon)_{\Omega^\varepsilon} = (\nabla u_k^\varepsilon, \nabla \psi^\varepsilon)_{\Omega^\varepsilon} \\ &= (R_\varepsilon^{-\theta} \nabla u_k^\varepsilon, \nabla U_k^\varepsilon)_{\Omega^\varepsilon} + \theta (R_\varepsilon^{-\theta} \nabla u_k^\varepsilon, U_k^\varepsilon R_\varepsilon^{-1} \nabla R_\varepsilon)_{\Omega^\varepsilon} \\ &= \|\nabla U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 - \theta (U_k^\varepsilon R_\varepsilon^{-1} \nabla R_\varepsilon, \nabla U_k^\varepsilon)_{\Omega^\varepsilon} \\ &\quad + \theta (\nabla U_k^\varepsilon, U_k^\varepsilon R_\varepsilon^{-1} \nabla R_\varepsilon)_{\Omega^\varepsilon} - \theta^2 \|U_k^\varepsilon R_\varepsilon^{-1} \nabla R_\varepsilon; L^2(\Omega^\varepsilon)\|^2. \end{aligned} \quad (3.12)$$

Two terms on the right-hand side cancel, and the remaining two terms satisfy the relations

$$\|\nabla U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \geq c_\omega^{-2} \|R_\varepsilon^{-1} U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2,$$

$$R_\varepsilon^{-1}(x) |\nabla R_\varepsilon(x)| \leq R_\varepsilon^{-1}(x) \quad \Rightarrow \quad \|U_k^\varepsilon R_\varepsilon^{-1} \nabla R_\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq \|R_\varepsilon^{-1} U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2.$$

Thus, for $\theta \leq (2c_\omega)^{-1}$ the right-hand side of (3.12) is estimated from below by $\frac{3}{4} \|\nabla U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2$ which, in turn, exceeds $c \|R_\varepsilon^{-1-\theta} u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2$ with $c > 0$ in view of Lemma 3.1. We have

$$\|R_\varepsilon^{-\theta} \nabla u_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 \leq 2(\|\nabla U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + \|U_k^\varepsilon R_\varepsilon^\theta \nabla R_\varepsilon^{-\theta}; L^2(\Omega^\varepsilon)\|^2) \leq C \|\nabla U_k^\varepsilon; L^2(\Omega^\varepsilon)\|^2$$

which completes the proof. \square

Remark 3.1. By construction, the exponent θ in Proposition 3.1 cannot be large. Thus, in the limit problem \mathcal{P}_D without small parameter ε , but with the Dirichlet condition (3.4), we have $\theta < 1/2$ in view of smoothness of the eigenfunctions (3.6).

3.3. Application of the max–min principle. In the following representation of the eigenvalues of the operator of Problem $\mathcal{P}_D^\varepsilon(\eta)$, similar to (2.1) and (2.11),

$$\lambda_{Dk}^\varepsilon(\eta) = \max_{\mathcal{E}_D^k(\eta)} \inf_{u^\varepsilon \in \mathcal{E}_D^k(\eta) \setminus \{0\}} \frac{\|\nabla u^\varepsilon; L^2(\Omega^\varepsilon)\|}{\|u^\varepsilon; L^2(\Omega^\varepsilon)\|} \quad (3.13)$$

each subspace $\mathcal{E}_D^k(\eta) \subset H_{\eta,D}^1(\Omega^\varepsilon)$ of codimension $k - 1$ intersects the k -dimensional span $\mathcal{L}_k^\varepsilon$ of the functions $\mathbf{U}_j^\varepsilon = \mathcal{X}_\varepsilon u_{Dj}^0$, $j = 1, \dots, k$. We introduced the cut-off function (3.9) to satisfy the Dirichlet conditions (1.6) and guarantee that $\mathcal{X}_\varepsilon u_{Dj}^0$ belong to the space $H_{\eta,D}^1(\Omega^\varepsilon)$. At the same time, for the eigenfunctions (3.6) (which are enumerated and satisfy (1.18)) we have the elementary estimates

$$|u_{D,j}^0(x)| \leq c_j^0 |x_1|, \quad |\nabla u_{D,j}^0(x)| \leq c_j^1 \quad (3.14)$$

which imply that the smoothing functions $\mathbf{U}_1^\varepsilon, \dots, \mathbf{U}_k^\varepsilon$ with small ε inherit the linear independence property of $u_{D1}^0, \dots, u_{Dk}^0$. Furthermore, for the linear combination

$$\mathcal{U}^{\mathcal{E}_D^k(\eta)}(\cdot; \eta) = \sum_{j=1}^k \alpha_j^{\mathcal{E}_D^k(\eta)} \mathbf{U}_j^\varepsilon \in \mathcal{E}_D^k(\eta) \cap \mathcal{L}_k^\varepsilon, \quad \sum_{j=1}^k |\alpha_j^{\mathcal{E}_D^k(\eta)}|^2 = 1, \quad (3.15)$$

we find

$$\begin{aligned} \|\mathcal{U}; L^2(\Omega^\varepsilon)\|^2 &= \sum_{j,q=1}^k \alpha_j \alpha_q ((u_j^0, u_q^0)_\Omega + ((\mathcal{X}_\varepsilon^2 - 1)u_j^0, u_q^0)_\Omega) \\ &\Rightarrow \left| \|\mathcal{U}; L^2(\Omega^\varepsilon)\|^2 - 1 \right| \leq c_k \varepsilon^3, \\ \|\nabla \mathcal{U}; L^2(\Omega^\varepsilon)\|^2 &= \sum_{j,q=1}^k \alpha_j \alpha_q ((\nabla u_j^0, \nabla u_q^0)_\Omega + ((\mathcal{X}_\varepsilon^2 - 1)\nabla u_j^0, \nabla u_q^0)_\Omega) \\ &\quad + (\mathcal{X}_\varepsilon \nabla u_j^0, u_q^0 \nabla \mathcal{X}_\varepsilon)_\Omega + (u_j^0 \nabla \mathcal{X}_\varepsilon, \mathcal{X}_\varepsilon \nabla u_q^0)_\Omega + (u_j^0 \nabla \mathcal{X}_\varepsilon, u_q^0 \nabla \mathcal{X}_\varepsilon)_\Omega \\ &\Rightarrow \left| \|\nabla \mathcal{U}; L^2(\Omega^\varepsilon)\|^2 - \sum_{j=1}^k |\alpha_j|^2 \lambda_j^\varepsilon \right| \leq C_k \varepsilon. \end{aligned} \quad (3.16)$$

For the sake of brevity we omit the argument η and do not indicate that the linear combination (3.15) belongs to $\mathcal{E}_D^k(\eta)$ in the notation. In (3.16), we used formulas (3.9), (3.14) and their consequence $|u_{D,j}^0(x)| \leq 2c_j^0 h \varepsilon$, $x \in \text{supp } |\nabla \mathcal{X}_\varepsilon|$. From (3.13) and (3.16) it follows that

$$\lambda_{Dk}^\varepsilon(\eta) \leq \max_{\mathcal{E}_D^k(\eta)} \frac{\|\nabla \mathcal{U}; L^2(\Omega^\varepsilon)\|}{\|\mathcal{U}; L^2(\Omega^\varepsilon)\|} \leq \lambda_k^0(\eta) + C_k \varepsilon.$$

A similar calculation for the max–min principle (3.13) at $\varepsilon = 0$, i.e., for the eigenvalues of the problem (3.4), (1.4), (1.5) without parameters η and ε , yields the estimate

$$\lambda_{Dk}^0 \leq \max_{\mathcal{E}_D^k(\eta)} \frac{\|\nabla \mathcal{U}_k^\varepsilon; L^2(\Omega)\|}{\|\mathcal{U}_k^\varepsilon; L^2(\Omega^\varepsilon)\|} \leq \lambda_{Dk}^\varepsilon(\eta) + C_k \varepsilon^{2\theta}. \quad (3.17)$$

We explain necessary replacements. First of all, $\mathcal{E}_D^k(\eta)$ is the subspace of the space $H_{\eta,D}^1(\Omega)$ of functions vanishing on the mean line Υ of the rectangle Ω and some test functions in this subspace have the form

$$\mathcal{U}_k^\varepsilon = \mathcal{X}_\varepsilon \sum_{j=1}^k \alpha_j^\varepsilon u_j^\varepsilon, \quad \sum_{j=1}^k |\alpha_j^\varepsilon|^2 = 1,$$

where u_j^ε are the eigenfunctions of Problem $\mathcal{P}_D^\varepsilon(\eta)$ satisfying the orthonormality conditions (1.15), and, as above, the cut-off function (3.9) satisfies the Dirichlet conditions on Υ .

By the weighted estimate (3.11) in Proposition 3.1,

$$\varepsilon^{-1} \|u_{Dj}^\varepsilon; L^2(\text{supp } |\nabla \mathcal{X}_\varepsilon|)\| + \|\nabla u_{Dj}^\varepsilon; L^2(\text{supp } |\nabla \mathcal{X}_\varepsilon|)\| \leq c_j \varepsilon^\theta.$$

Taking into account this estimate, we obtain the following assertion from (3.17).

Theorem 3.1. *The eigenvalues of the problems (1.3)–(1.6) and (3.3), (1.4), (1.5), (3.4) are connected by the inequalities*

$$\lambda_{Dk}^0 - C_k \varepsilon^{2\theta} \leq \lambda_{Dk}^\varepsilon(\eta) \leq \lambda_{Dk}^0 + C_k \varepsilon, \quad (3.18)$$

where $\theta \in (0, 1)$ is the exponent in Proposition 3.1 and C_k is independent of $\eta \in [-\pi, \pi]$ and $\varepsilon \in (0, \varepsilon_k]$ for some $\varepsilon_k > 0$.

As was shown in Subsection 1.3, Theorem 3.1 is sufficient to realize the first approach for obtaining uniform estimates for remainders of asymptotic expansions. This fact was verified in [29] by other arguments.

We note that the max–min principle was also used in [12] to prove analogues of (1.23). This max–min principle is based on a priori weighted estimates for eigenfunctions of the model problem on a three-dimensional periodicity cell with the Neumann condition on the boundary of a single small cavity. The multidimensional ($d > 3$) case can be treated in the same way, whereas the plane problem has not been studied yet.

4 Solvability of Singularly Perturbed Problem in Perforated Cell

4.1. Almost inverse operator. If the spectral parameter λ_\bullet is fixed, we can associate the variational statement of the inhomogeneous problem $\mathcal{P}_T^\varepsilon(\eta)$

$$a_T^\varepsilon(u_T^\varepsilon, \psi; \Omega) - \lambda_\bullet b_T^\varepsilon(u_T^\varepsilon, \psi; \Omega) = f_T^\varepsilon(\psi), \quad \psi \in H_{\eta,T}^1(\Omega), \quad (4.1)$$

with the mapping

$$H_{\eta,T}^1(\Omega) \ni u_T^\varepsilon \mapsto f_T^\varepsilon = A_T^\varepsilon(\eta; \lambda_\bullet) u_T^\varepsilon \in H_{\eta,T}^1(\Omega)^*, \quad (4.2)$$

where $H_{\eta,T}^1(\Omega)$ is the Sobolev space $H_\eta^1(\Omega)$ equipped with the norm depending on the parameter $T \in (0, 1]$

$$\begin{aligned} \|u_T^\varepsilon; H_{\eta,T}^1(\Omega)\| &= (a_T^\varepsilon(u_T^\varepsilon, u_T^\varepsilon; \Omega) + b_T^\varepsilon(u_T^\varepsilon, u_T^\varepsilon; \Omega))^{1/2} \\ &= (\|\nabla u_T^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + T \|\nabla u_T^\varepsilon; L^2(\omega^\varepsilon)\|^2 + \|u_T^\varepsilon; L^2(\Omega^\varepsilon)\|^2 + T \|u_T^\varepsilon; L^2(\omega^\varepsilon)\|^2)^{1/2}. \end{aligned} \quad (4.3)$$

Here, a_T^ε and b_T^ε denote the bilinear forms (1.11) and $f_T^\varepsilon \in H_{\eta,T}^1(\Omega)^*$ is a linear continuous functional on $H_{\eta,T}^1(\Omega)$. In this subsection, we assume that $T > 0$. The case $T = 0$ is considered in Subsection 4.6.

Let $\eta_\bullet \in [-\pi, \pi]$ and $\lambda_\bullet > 0$ be such that the operator $A(\eta_\bullet; \lambda_\bullet)$ of the problem in Ω is invertible for small $\varepsilon > 0$. Then there is $\delta_\bullet > 0$, such that the invertibility property of $A(\eta; \lambda_\bullet)$ is preserved for all $\varepsilon \in (0, \varepsilon_\bullet]$ and $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, where ε_\bullet is independent of η . We emphasize that, in the case $\eta_\bullet = \pm\pi$, to make sense of the last inclusion, the objects should be periodically continued to a neighborhood of $[-\pi, \pi]$.

The goal of this section is to prove the following assertion.

Theorem 4.1. *Under the above invertibility assumption on $A(\eta_\bullet; \lambda_\bullet)$, there exist positive numbers ε_\bullet and δ_\bullet such that the operator $A_T^\varepsilon(\eta; \lambda_\bullet)$ is an isomorphism for all $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, $\varepsilon \in (0, \varepsilon_\bullet]$ and, $T \in [0, 1]$.*

We emphasize that the operator $A_T^\varepsilon(\eta; \lambda_\bullet)$ is selfadjoint, i.e., to prove Theorem 4.1, it suffices to verify the invertibility of this operator.

To prove Theorem 4.1, we construct the so-called almost inverse operator

$$R_T^\varepsilon(\eta; \lambda_\bullet) : H_{\eta,T}^1(\Omega)^* \rightarrow H_{\eta,T}^1(\Omega) \quad (4.4)$$

such that

$$\|A_T^\varepsilon(\eta; \lambda_\bullet)R_T^\varepsilon(\eta; \lambda_\bullet) - \text{Id}; H_{\eta,T}^1(\Omega)^* \rightarrow H_{\eta,T}^1(\Omega)^*\| \leq c_\bullet \varepsilon^{\alpha_\bullet}, \quad (4.5)$$

where $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$, $\alpha_\bullet > 0$ is the exponent and Id is the identity mapping. Then the operator $A_T^\varepsilon(\eta; \lambda_\bullet)R_T^\varepsilon(\eta; \lambda_\bullet)$ is invertible for small $\varepsilon \in (0, \varepsilon_\bullet]$ and $R_T^\varepsilon(\eta; \lambda_\bullet)(A_T^\varepsilon(\eta; \lambda_\bullet)R_T^\varepsilon(\eta; \lambda_\bullet))^{-1}$ is the usual inverse of the mapping (4.2). It is important that α_\bullet and c_\bullet in (4.5) are independent of the Floquet parameter.

We fix a functional $f_T^\varepsilon \in H_{\eta,T}^1(\Omega)^*$ and construct $R_T^\varepsilon(\eta; \lambda_\bullet)f_T^\varepsilon \in H_{\eta,T}^1(\Omega)$ step by step. We assume that $T \in (0, 1]$.

4.2. Auxiliary problem near perforation. We impose the artificial Dirichlet conditions on $\Upsilon_\ell^\pm = \{x : x_1 = \pm\ell, |x_2| < H\}$ and consider the problem on the narrowed ($\ell < 1/2$) rectangle $\Omega_\ell = \{x : |x_1| < \ell, |x_2| < H\}$

$$a_T^\varepsilon(u_T^{\varepsilon\sharp}, \psi^\sharp; \Omega_\ell) - \lambda_\bullet b_T^\varepsilon(u_T^{\varepsilon\sharp}, \psi^\sharp; \Omega_\ell) = f_T^{\varepsilon\sharp}(\psi^\sharp), \quad \psi^\sharp \in H_{\eta,T}^{1\sharp}(\Omega_L). \quad (4.6)$$

The superscript \sharp means, for example, that a test function ψ^\sharp satisfies the additional boundary condition

$$\psi^\sharp(\pm\ell, x_2) = 0, \quad |x_2| < H.$$

The right-hand side $f_T^{\varepsilon\sharp}(\psi) = f_T^\varepsilon(\chi_0^\sharp \psi)$ of (4.6) is a functional, where $\chi_0^\sharp \in C_c^\infty(\mathbb{R})$ is a cut-off function such that $0 \leq \chi_0^\sharp \leq 1$,

$$\chi_0^\sharp(x_1) = \begin{cases} 1, & |x_1| < \ell/2, \\ 0, & |x_1| > 2\ell/3. \end{cases} \quad (4.7)$$

It is clear that

$$\|f_T^{\varepsilon\sharp}; H_{\eta,T}^{1\sharp}(\Omega_\ell)^*\| \leq c \|f_T^\varepsilon; H_{\eta,T}^1(\Omega)^*\|.$$

Proposition 4.1. For $\psi^\sharp \in H_{\eta,T}^{1,\sharp}(\Omega_\ell)$

$$\|\psi^\sharp; L^2(\Omega_\ell^\varepsilon)\|^2 + T\|\psi^\sharp; L^2(\omega^\varepsilon)\|^2 \leq c_\sharp(\ell + \varepsilon)^2\|\psi^\sharp; H_{\eta,T}^{1,\sharp}(\Omega_\ell)\|^2, \quad (4.8)$$

where $\Omega_\ell^\varepsilon = \{x \in \Omega^\varepsilon : |x_1| < \ell\} = \Omega_\ell \setminus \overline{\omega^\varepsilon}$ is the perforated rectangle Ω_ℓ and c_\sharp is independent of $\ell \in (0, 1/2]$ and $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.

Proof. We begin by constructing a suitable extension Ψ^ε to the holes (1.1) of the restriction $\psi^\sharp|_{\Omega_\ell^\varepsilon}$ such that

$$\|\Psi^\varepsilon; L^2(\Omega_\ell)\| \leq c\|\nabla\psi^\sharp; L^2(\Omega_\ell^\varepsilon)\| \leq c\|\psi^\sharp; H_{\eta,T}^{1,\sharp}(\Omega_\ell)\|. \quad (4.9)$$

Namely, we set $\Psi^\varepsilon = \psi^\sharp$ on Ω_ℓ^ε and

$$\Psi^\varepsilon(x) = \overline{\psi}_j^\sharp + \widehat{\psi}_j^\sharp(\varepsilon^{-1}x_1, \varepsilon^{-1}(x_2 - 2H\varepsilon j)), \quad x \in \omega_j^\varepsilon, \quad j = -N, \dots, N, \quad (4.10)$$

where $\overline{\psi}_j^\sharp$ is the mean value of ψ^\sharp over the contour $\partial\omega$:

$$\overline{\psi}_j^\sharp = \frac{1}{\text{mes}_1 \partial\omega_j^\varepsilon} \int_{\partial\omega_j^\varepsilon} \psi^\sharp(x) ds_x \quad (4.11)$$

and $\widehat{\psi}_j^\sharp$ is an extension of $\square^h \setminus \overline{\omega} \ni \xi \mapsto \widehat{\psi}_j^\sharp(\xi) := \psi^\sharp(\varepsilon\xi_1, 2H\varepsilon j + \varepsilon x_2) - \overline{\psi}_j^\sharp$ on the rectangle $\square^{2h} = (-2h, 2h) \times (-H, H)$ admitting the natural estimate in the $H^1(\omega)$ -norm. We have

$$\begin{aligned} \|\nabla_x \widehat{\psi}_j^\sharp; L^2(\square^h)\|^2 &= \|\nabla_\xi \widehat{\psi}_j^\sharp; L^2(\square^h)\|^2 \leq c_{h\omega}(\|\nabla_\xi \psi^\sharp; L^2(\square^h)\|^2 + \|\psi^\sharp - \overline{\psi}_j^\sharp; L^2(\square^h)\|^2) \\ &\leq C\|\nabla_\xi \psi^\sharp; L^2(\square^h \setminus \omega)\| = C\|\nabla_x \psi^\sharp; L^2(\square^h \setminus \omega_j^\varepsilon)\|. \end{aligned} \quad (4.12)$$

Here, we used the Poincaré inequality for functions $\widehat{\psi}_j^\sharp$ with zero mean over the contour $\partial\omega$ in accordance with the definition (4.11):

$$\|\psi^\sharp - \overline{\psi}_j^\sharp; L^2(\square^h \setminus \overline{\omega})\| \leq c\|\nabla_\xi(\psi^\sharp - \overline{\psi}_j^\sharp); L^2(\square^h \setminus \overline{\omega})\| = c\|\nabla_\xi \psi^\sharp; L^2(\square^h \setminus \overline{\omega})\|.$$

Summarizing the relation (4.12) with respect to $j = -N, \dots, N$, we obtain the estimate (4.9). Integrating the one-dimensional Friedrichs inequality on $(-\ell, \ell) \ni x_1$ with respect to $x_2 \in (-H, H)$, we get

$$\|\psi^\sharp; L^2(\Omega_\ell^\varepsilon)\|^2 \leq \|\Psi^\varepsilon; L^2(\Omega_\ell)\|^2 \leq \frac{\pi^2}{4\ell^2} \left\| \frac{\partial\Psi^\varepsilon}{\partial x_1}; L^2(\Omega_\ell) \right\|^2 \leq \frac{C}{\ell^2} \|\nabla\psi^\sharp; L^2(\Omega_\ell^\varepsilon)\|^2. \quad (4.13)$$

It remains to consider the norm $\sqrt{T}\|\psi^\sharp; L^2(\omega^\varepsilon)\|$ in (4.3) and the left-hand side of (4.8). Since $T \leq 1$ and the difference $\psi^\sharp - \Psi^\varepsilon$ vanishes on $\partial\omega_j^\varepsilon$, we find

$$\begin{aligned} T \sum_{j=-N}^N \|\psi^\sharp - \Psi^\varepsilon; L^2(\omega_j^\varepsilon)\|^2 &\leq c\varepsilon^2 T \sum_{j=-N}^N \|\nabla(\psi^\sharp - \Psi^\varepsilon); L^2(\omega_j^\varepsilon)\|^2 \\ &\leq 2c\varepsilon^2 \sum_{j=-N}^N (T\|\nabla\psi^\sharp; L^2(\omega_j^\varepsilon)\|^2 + \|\nabla\Psi^\varepsilon; L^2(\square_j^{\varepsilon h})\|^2) \leq C\varepsilon^2\|\psi^\sharp; H_{\eta,D}^{1,\sharp}(\Omega_\ell)\|^2, \end{aligned} \quad (4.14)$$

$$\begin{aligned} T\|\psi^\sharp; L^2(\omega^\varepsilon)\|^2 &\leq 2T(\|\Psi^\varepsilon; L^2(\omega^\varepsilon)\|^2 + \|\psi^\sharp - \Psi^\varepsilon; L^2(\omega^\varepsilon)\|^2) \\ &\leq c(L^2\|\nabla\Psi^\varepsilon; L^2(\Omega_\ell)\|^2 + \varepsilon^2\|\psi^\sharp; H_{\eta,D}^{1,\sharp}(\Omega_\ell)\|^2). \end{aligned} \quad (4.15)$$

From (4.14), (4.15), and (4.9) we obtain the required estimate for the norm $\sqrt{T}\|\psi^\sharp; L^2(\omega^\varepsilon)\|$, which completes the proof of (4.8) in view of (4.13). \square

The problem (4.6) is uniquely solvable for small ℓ and ε in view of the Riesz representation theorem and the following inequality obtained from Proposition 4.1:

$$a_T^\varepsilon(\psi^\sharp, \psi^\sharp; \Omega_\ell) - \lambda_\bullet b_T^\varepsilon(\psi^\sharp, \psi^\sharp; \Omega_\ell) \geq \frac{1}{2}a_T^\varepsilon(\psi^\sharp, \psi^\sharp; \Omega_\ell) + \left(\frac{1}{2c^\sharp} \frac{1}{\varepsilon^2 + \ell^2} - \lambda_\bullet\right)b_T^\varepsilon(\psi^\sharp, \psi^\sharp; \Omega_\ell).$$

Moreover, for $u_T^{\varepsilon^\sharp} \in H_{\eta, T}^1(\Omega_\ell)$ we have

$$\|u_T^{\varepsilon^\sharp}; H_{\eta, T}^1(\Omega_\ell)\| \leq c\|f_T^{\varepsilon^\sharp}; H_{\eta, T}^1(\Omega_\ell)^*\| \leq C\|f_T^\varepsilon; H_{\eta, T}^1(\Omega)^*\|.$$

To conclude the first step, we define the first term

$$\mathcal{R}_T^{\varepsilon^\sharp} = \chi_1^\sharp u_T^{\varepsilon^\sharp} \quad (4.16)$$

in the coming representation of the almost inverse operator

$$R_T^\varepsilon(\eta; \lambda_\bullet)f_T^\varepsilon = \mathcal{R}_T^{\varepsilon^\sharp} + \mathcal{R}_T^{\varepsilon\Omega} + \mathcal{R}_T^\omega. \quad (4.17)$$

The cut-off function $\chi_1^\sharp \in C_c^\infty(\mathbb{R})$ satisfying $\chi_1^\sharp \chi_0^\sharp = \chi_0^\sharp$ was introduced to (4.16) since

$$\chi_1^\sharp(x_1) = \begin{cases} 1, & |x_1| \leq 2\ell/3, \\ 0, & |x_1| \geq 5\ell/6. \end{cases} \quad (4.18)$$

4.3. Smooth solution in Ω . Taking $\psi \in H_{\eta, T}^1(\Omega)$ and substituting the test function $\psi^\sharp = \psi\chi_1^\sharp$ into the integral identity (4.6), we get

$$\begin{aligned} f_T^\varepsilon(\psi\chi_0^\sharp) &= f_T^{\varepsilon^\sharp}(\psi\chi_1^\sharp) = a_T^\varepsilon(u_T^{\varepsilon^\sharp}, \psi\chi_1^\sharp; \Omega_L) - \lambda_\bullet b_T^\varepsilon(u_T^{\varepsilon^\sharp}, \psi\chi_1^\sharp; \Omega_L) \\ &= a_T^\varepsilon(\chi_1^\sharp u_T^{\varepsilon^\sharp}, \psi; \Omega) - \lambda_\bullet b_T^\varepsilon(\chi_1^\sharp u_T^{\varepsilon^\sharp}, \psi; \Omega) + (\nabla u_T^{\varepsilon^\sharp}, \psi \nabla \chi_1^\sharp)_\Omega - (u_T^{\varepsilon^\sharp} \nabla \chi_1^\sharp, \nabla \psi)_\Omega. \end{aligned} \quad (4.19)$$

Thus, the residual

$$f_T^{\varepsilon\Omega}(\psi) = f_T^\varepsilon((1 - \chi_0^\sharp)\psi) - (\nabla u_T^{\varepsilon^\sharp}, \psi \nabla \chi_1^\sharp)_\Omega + (u_T^{\varepsilon^\sharp} \nabla \chi_1^\sharp, \nabla \psi)_\Omega \quad (4.20)$$

left by the first term (4.16) in the expression (4.17) satisfies the estimate

$$\|f_T^{\varepsilon\Omega}; H_\eta^1(\Omega)^*\| \leq C\|f_T^\varepsilon; H_{\eta, T}^1(\Omega)^*\|. \quad (4.21)$$

At the same time, an extremely important property of the functional (4.20) obtained precisely at the first step is that, according to the definitions (4.18) and (4.7), the functional $f_T^{\varepsilon\Omega}$ vanishes on test functions that vanish in the rectangle $\Omega_{\ell/3} = (-\ell/3, \ell/3) \times (-H, H)$, i.e., $\text{supp } f_T^{\varepsilon\Omega} \subset \overline{\Omega \setminus \Omega_{\ell/2}}$. Thus, the solution $u_T^{\varepsilon\Omega} \in H_\eta^1(\Omega)$ to the problem

$$(\nabla u_T^{\varepsilon\Omega}, \nabla \psi)_\Omega - \lambda_\bullet (u_T^{\varepsilon\Omega}, \psi)_\Omega = f_T^{\varepsilon\Omega}(\psi), \quad \psi \in H_\eta^1(\Omega), \quad (4.22)$$

existing by the assumption of Theorem 4.1 and satisfying the estimate

$$\|u_T^{\varepsilon\Omega}; H_\eta^1(\Omega)\| \leq c\|f_T^{\varepsilon\Omega}; H_\eta^1(\Omega)^*\|, \quad (4.23)$$

becomes infinitely differentiable, at least, in the rectangle $\overline{\Omega_{\ell/3}}$. In particular, in view of local estimates for solutions to elliptic equations, we have

$$\|u_T^{\varepsilon\Omega}; C^m(\Omega_{\ell/3})\| \leq c_m \|f_T^{\varepsilon\Omega}; H_\eta^1(\Omega)^*\|, \quad m \in \mathbb{N}_0. \quad (4.24)$$

Furthermore, Equation (1.3) holds in $\Omega_{\ell/3}$ and Equation (1.8) holds in ω_j^ε . Therefore, the residuals left by the sum $\mathcal{R}_T^{\varepsilon\#} + \mathcal{R}_T^{\varepsilon\Omega}$ in the representation (4.17) are concentrated on the boundaries of the inclusions (1.1). Indeed,

$$\begin{aligned} & a_T^\varepsilon(u_T^{\varepsilon\Omega}, \psi; \Omega) - \lambda_\bullet b_T^\varepsilon(u_T^{\varepsilon\Omega}, \psi; \Omega) - f_T^{\varepsilon\Omega}(\psi) \\ &= (T-1) \sum_{j=-N}^N ((\nabla u_T^{\varepsilon\Omega}, \nabla \psi)_{\omega_j^\varepsilon} - \lambda_\bullet (u_T^{\varepsilon\Omega}, \psi)_{\omega_j^\varepsilon}) = (T-1) \sum_{j=-N}^N (\partial_n u_T^{\varepsilon\Omega}, \psi)_{\partial\omega_j^\varepsilon}. \end{aligned} \quad (4.25)$$

To compensate such residuals, we construct a boundary layer.

4.4. Boundary layer problem. As usual, near the row of fine holes we have the boundary layer phenomenon described by the solutions to the problem in the strip $\Pi = (-H, H) \times \mathbb{R}$:

$$-\Delta_\xi w_\square(\xi) = 0, \quad \xi \in \Pi_\square = \Pi \setminus \bar{\omega}, \quad (4.26)$$

$$-T\Delta_\xi w_\circ(\xi) = 0, \quad \xi \in \omega, \quad (4.27)$$

$$w_\square(\xi) - w_\circ(\xi) = 0, \quad \partial_n w_\square(\xi) - T\partial_n w_\circ(\xi) = g(\xi), \quad \xi \in \partial\omega, \quad (4.28)$$

$$w_\square(\xi_1, H) = w_\square(\xi_1, -H), \quad \frac{\partial w_\square}{\partial \xi_2}(\xi_1, H) = \frac{\partial w_\square}{\partial \xi_2}(\xi_1, -H), \quad \xi_1 \in \mathbb{R}. \quad (4.29)$$

Here, w_\square and w_\circ are the restrictions of w on Π_\square and ω respectively. The Laplace equations (4.26) and (4.27) are obtained from the Helmholtz equations (1.3) and (1.8) by passing to the stretched variables $\xi = \varepsilon^{-1}x$, where $\Delta_x + \lambda = \varepsilon^{-2}\Delta_\xi + \lambda$, i.e., the Laplacian Δ_ξ is the leading part of the asymptotics of the Helmholtz operator $\Delta_x + \lambda$. The transmission conditions (4.28) are inherited by the similar conditions (1.9), whereas the periodicity conditions (4.29) are artificial, are not related to the quasiperiodicity conditions (1.4), and will be used to construct a global solution of boundary layer type.

The variational statement of the problem (4.26)–(4.29) has the form

$$(\nabla_\xi w, \nabla_\xi \varphi)_{\Pi_\circ} + T(\nabla_\xi w, \nabla_\xi \varphi)_\omega = (g, \varphi)_{\partial\omega}, \quad \varphi \in \mathcal{H}_{T,per}(\Pi), \quad (4.30)$$

on the space $\mathcal{H}_{T,per}(\Pi)$ obtained by completion of the linear set $C_{c,per}^\infty(\overline{\Pi})$ of infinitely differentiable compactly supported functions that are $2H$ -periodic with respect to ξ_2 in the energy norm

$$\|\varphi; \mathcal{H}_{T,per}(\Pi)\| = (\|\nabla_\xi \varphi; L^2(\Pi_\circ)\|^2 + T\|\nabla_\xi \varphi; L^2(\omega)\|^2 + \|\varphi_\square; L^2(\partial\omega)\|^2)^{1/2}. \quad (4.31)$$

By the Poincaré inequality on the sets $\{\xi \in \Pi_\square : |\xi_1| < 2h\}$ and ω and the one-dimensional Hardy inequality (3.7) with $R = +\infty$, the norm (4.31) is equivalent to the weight norm

$$(\|\nabla_\xi \varphi; L^2(\Pi_\circ)\|^2 + T\|\nabla_\xi \varphi; L^2(\omega)\|^2 + \|(1 + \xi_2^2)^{-1/2} \varphi; L^2(\Pi_\circ)\|^2 + T\|\varphi; L^2(\omega)\|^2)^{1/2} \quad (4.32)$$

uniformly with respect to the parameter $T \in (0, 1]$. As known, the constants belong to the space $\mathcal{H}_{T,per}(\Pi)$, in particular, the norms (4.32) are finite. Thus, the following assertion holds (cf., for example, [31, Section 3]).

Proposition 4.2. *The problem (4.30) with $g \in L^2(\partial\omega)$ has a solution $w \in \mathcal{H}_{T,per}(\Pi)$ if and only if*

$$\int_{\partial\omega} g(\xi) ds_\xi = 0. \quad (4.33)$$

This solution is defined up to an additive constant and is unique provided that

$$\int_{\partial\omega} w_\square(\xi) ds_\xi = 0; \quad (4.34)$$

moreover,

$$\|w; \mathcal{H}_{T,per}(\Pi)\| \leq c \|g; L^2(\partial\omega)\|,$$

where c is independent of g and $T \in (0, 1]$.

If the contour $\partial\omega$ and the right-hand side g of the second transmission condition in (4.28) are smooth, then the components w_\square and w_\circ of the solution $w \in \mathcal{H}_{T,per}(\Pi)$ are also smooth on the sets $\overline{\Pi} \setminus \omega$ and $\overline{\omega}$ respectively. In the general case, the component w_\square is smooth outside any neighborhood of the compact set $\overline{\omega}$, whereas the component w_\circ is smooth inside the domain ω . These obvious facts are true because of local estimates for solutions to elliptic equations: based on the periodicity condition, it is possible to reduce the problem (4.26)–(4.29) to the case of the cylindrical surface $\mathbb{R} \times \mathbb{S}_{2H}$, where \mathbb{S}_L is the circle of length L .

By the Fourier method, we have the representation

$$w(\xi) = \tilde{w}(\xi) + \sum_{\pm} \pm \chi_{\pm}(\xi_1) c_w \quad (4.35)$$

and the following estimate for the remainder:

$$|\nabla_{\xi}^m \tilde{w}(\xi)| \leq c_m e^{-(2H)^{-1}\pi|\xi_1|}, \quad |\xi_1| \geq h, \quad m \in \mathbb{N}_0. \quad (4.36)$$

Moreover, c_w is a constant and $\chi_{\pm} \in C^\infty(\mathbb{R})$ are cut-off functions vanishing on $\overline{\omega}$,

$$\chi_{\pm}(\xi_1) = \begin{cases} 1, & \pm\xi_1 \geq 2h, \\ 0, & \pm\xi_1 \leq h. \end{cases}$$

We emphasize that the function (4.35) is stabilized, generally speaking, to different constants c_w^\pm as $\xi_1 \rightarrow \pm\infty$, i.e., the representation (4.35) providing the zero sum of these constants $c_w^\pm = \pm c_w$ distinguishes the solution $w \in \mathcal{H}_{T,per}(\Pi)$, i.e., it is a counterpart of the orthogonality conditions (4.34) in Proposition 4.2.

In what follows, we need two ($q = 1, 2$) special solutions \mathcal{W}_q to the problem (4.26)–(4.29) (or (4.30)) with $g^q(\xi) = -\partial_n \xi_q$. In both cases, the solvability conditions (4.33) are satisfied, $\mathcal{W}_q \in \mathcal{H}_{T,per}(\Pi)$, and (4.35), (4.36) hold.

4.5. Component of boundary layer type. Thus, in the construction (4.17) of an almost inverse operator, we set

$$\begin{aligned} \mathcal{R}_T^{\varepsilon\Omega}(x) &= u_T^{\varepsilon\Omega}(x), \\ \mathcal{R}_T^\omega(x) &= \varepsilon(T-1)\chi_0^\sharp(x_1) \sum_{q=1,2} \mathcal{W}_q(\varepsilon^{-1}x_1, \varepsilon^{-1}x_2) \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(0, x_2) \end{aligned} \quad (4.37)$$

$$=: \varepsilon \chi_0^\sharp(x_1) \mathcal{S}^\varepsilon(x) = \varepsilon \chi_0^\sharp(x_1) \sum_{q=1,2} \mathcal{S}_q^\varepsilon(x). \quad (4.38)$$

We recall that one of the factors in (4.38), namely, the trace on Υ of the derivative of solution $u_T^{\varepsilon\Omega}$ to the problem (4.22) is a smooth function on the segment $[-H, H] \ni x_2$. Taking into account formulas (4.24) and (4.36), we obtain the estimate

$$\|\mathcal{R}_T^\omega; H_{\eta,T}^1(\Omega)\| \leq c \|f_T^\varepsilon; H_{\eta,T}^1(\Omega)^*\|. \quad (4.39)$$

Furthermore,

$$(\nabla \mathcal{R}_T^\omega, \nabla \psi^\varepsilon)_\Omega = \varepsilon (\mathcal{S}^\varepsilon \nabla \chi_0^\sharp, \nabla \psi^\varepsilon)_\Omega - \varepsilon (\nabla \mathcal{S}^\varepsilon, \psi^\varepsilon \nabla \chi_0^\sharp)_\Omega + \varepsilon (\nabla \mathcal{S}^\varepsilon, \nabla (\chi_0^\sharp \psi^\varepsilon))_\Omega. \quad (4.40)$$

The moduli of the first two inner products on the right-hand side of (4.40) do not exceed

$$c \varepsilon \|u_T^{\varepsilon\Omega}; C^1(\Omega_{L/3})\| \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\|$$

since the estimate (4.36) applied to the solutions \mathcal{W}^q implies

$$|\mathcal{W}_q(\varepsilon^{-1}x)| \leq c_{\mathcal{W}}, \quad |\nabla_x \mathcal{W}_q(\varepsilon^{-1}x)| \leq C_{\mathcal{W}} \varepsilon^{-1} e^{-\tau/\varepsilon}, \quad \tau > 0, \quad x \in \text{supp} |\nabla_x \chi_0^\sharp|. \quad (4.41)$$

Furthermore,

$$\begin{aligned} \varepsilon (\nabla_x \mathcal{S}_q^\varepsilon, \nabla (\chi_0^\sharp \psi^\varepsilon))_{\Omega_L} &= -\varepsilon (\Delta_x \mathcal{S}_q^\varepsilon, \chi_0^\sharp \psi^\varepsilon)_{\Omega_L} + \varepsilon (\partial_{n(x)} \mathcal{S}_q^\varepsilon, \chi_0^\sharp \psi^\varepsilon)_{\omega^\varepsilon} \\ &- \sum_{\pm} \int_{-1/2}^{1/2} \frac{\partial \mathcal{S}_q^\varepsilon}{\partial x_2}(x_1, \pm H) \chi_0^\sharp(x_1) \psi^\varepsilon(x_1, \pm H) dx_1 =: \mathcal{S}_q^1(\psi^\varepsilon) + \mathcal{S}_q^2(\psi^\varepsilon) + \mathcal{S}_q^3(\psi^\varepsilon). \end{aligned} \quad (4.42)$$

We consider each term $\mathcal{S}_m^\varepsilon(\psi^\varepsilon)$. Since the smooth function

$$\frac{\partial \mathcal{W}_q}{\partial \xi_2}(\xi, \pm H) = \varepsilon^{-1} \frac{\partial \mathcal{W}_q}{\partial x_2}\left(\frac{x_1}{\varepsilon}, \pm H\right)$$

decays at infinity at a rate $O(e^{-(2H)^{-1}\pi|x_1|/\varepsilon})$ (cf. (4.35), (4.36) and (4.41)), we find

$$\begin{aligned} |\mathcal{S}_q^2(\psi^\varepsilon)| &\leq c \|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\| \left(\int_{-1/2}^{1/2} e^{-H^{-1}\pi|x_1|/\varepsilon} dx_1 \right)^{1/2} \sum_{\pm} \|\psi^\varepsilon(\cdot, \pm H); L^2(-1/2, 1/2)\| \\ &\leq c \varepsilon^{1/2} \|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\| \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\|. \end{aligned} \quad (4.43)$$

Note that the estimates for the $L^2(-1/2, 1/2)$ -norm of the test function ψ^ε are obtained from the usual trace inequality for the extension (4.10) of the restriction $\psi^\varepsilon|_{\Omega^\varepsilon}$ on Ω in the Sobolev class H^1 satisfying the relation

$$\|\Psi^\varepsilon; H^1(\Omega)\| \leq c \|\psi^\varepsilon; H^1(\Omega^\varepsilon)\| \leq c \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\|.$$

Furthermore, since \mathcal{W}_q is harmonic in $\Pi \setminus \bar{\omega}$ and ω , we get

$$\begin{aligned} \mathcal{S}_q^1(\psi^\varepsilon) &= -(T-1)\varepsilon \left(\left(\mathcal{W}_q \Delta_x \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q} \Big|_{\Upsilon}, \chi_0^\sharp \psi^\varepsilon \right)_{\Omega^\varepsilon} + T \left(\mathcal{W}_q \Delta_x \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q} \Big|_{\Upsilon}, \chi_0^\sharp \psi^\varepsilon \right)_{\omega^\varepsilon} \right) \\ &- 2(T-1)\varepsilon \left(\left((\nabla_x \mathcal{W}_q)^\top \nabla_x \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q} \Big|_{\Upsilon}, \chi_0^\sharp \psi^\varepsilon \right)_{\Omega^\varepsilon} + T \left((\nabla_x \mathcal{W}_q)^\top \nabla_x \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q} \Big|_{\Upsilon}, \chi_0^\sharp \psi^\varepsilon \right)_{\omega^\varepsilon} \right). \end{aligned}$$

Taking into account formulas (4.35) and (4.36), we derive the estimate

$$\begin{aligned}
|\mathcal{I}_q^1(\psi^\varepsilon)| &\leq c\|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \sup_{x \in \Upsilon} (|\nabla_x u_T^{\varepsilon\Omega}(x)| + |\nabla_x^2 u_T^{\varepsilon\Omega}(x)|) \\
&\times (\|\mathcal{W}_q; L^2(\Omega_{L/3}^\varepsilon)\|^2 + T\|\mathcal{W}_q; L^2(\omega^\varepsilon)\|^2 + \|\nabla_x \mathcal{W}_q; L^2(\Omega_{L/3}^\varepsilon)\|^2 + T\|\nabla_x \mathcal{W}_q; L^2(\omega^\varepsilon)\|^2)^{1/2} \\
&\leq c\varepsilon\|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\| \left(1 + \frac{1}{\varepsilon} \int_{-L/3}^{L/3} e^{-H^{-1}\pi|x_1|/\varepsilon} dx_1\right)^{1/2} \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \\
&\leq c\varepsilon^{1/2}\|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\| \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\|. \tag{4.44}
\end{aligned}$$

We represent the remaining terms $\mathcal{I}_q^2(\psi^\varepsilon)$ in (4.42) as

$$\begin{aligned}
\mathcal{I}_q^2(\psi^\varepsilon) &= \varepsilon(T-1) \sum_{j=-N}^N \int_{\partial\omega_j^\varepsilon} \psi^\varepsilon(x) \frac{\partial}{\partial n} \left(\frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x_1, 0) \mathcal{W} \left(\frac{x}{\varepsilon} \right) \right) ds_x \\
&= \varepsilon(T-1) \sum_{j=-N}^N \int_{\partial\omega_j^\varepsilon} \psi^\varepsilon(x) \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x) \frac{\partial \mathcal{W}}{\partial n(x)} \left(\frac{x}{\varepsilon} \right) ds_x \\
&+ \varepsilon(T-1) \sum_{j=-N}^N \int_{\partial\omega_j^\varepsilon} \psi^\varepsilon(x) \mathcal{W} \left(\frac{x}{\varepsilon} \right) \frac{\partial^2 u_T^{\varepsilon\Omega}}{\partial n \partial x_q}(x_1, 0) ds_x \\
&+ \varepsilon \sum_{j=-N}^N \int_{\partial\omega_j^\varepsilon} \psi^\varepsilon(x) \frac{\partial \mathcal{W}}{\partial n(x)} \left(\frac{x}{\varepsilon} \right) \left(\frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x_1, 0) - \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x) \right) ds_x \\
&=: \mathcal{I}_q^{20}(\psi^\varepsilon) + \mathcal{I}_q^{21}(\psi^\varepsilon) + \mathcal{I}_q^{22}(\psi^\varepsilon). \tag{4.45}
\end{aligned}$$

By the definition of the special solutions \mathcal{W}_q (cf. Subsection 4.4), the sum

$$\begin{aligned}
&\mathcal{I}_1^{20}(\psi^\varepsilon) + \mathcal{I}_1^{20}(\psi^\varepsilon) \\
&= -\varepsilon(T-1) \sum_{j=-N}^N \int_{\partial\omega_j^\varepsilon} \psi^\varepsilon(x) \left(\frac{\partial \xi}{\partial n(\xi)} \right)^\top \nabla_x u_T^{\varepsilon\Omega}(x) ds_x = -(T-1) \sum_{j=-N}^N (\partial_n u_T^{\varepsilon\Omega}, \psi)_{\partial\omega_j^\varepsilon} \tag{4.46}
\end{aligned}$$

coincides with the expression (4.25) taken with the opposite sign, i.e., the residual generated by the smooth solution $u_T^{\varepsilon\Omega}$ are eliminated.

We deal with another pair of terms on the right-hand side of (4.45) as follows:

$$\begin{aligned}
\mathcal{I}_q^{21}(\psi^\varepsilon) &\leq c\varepsilon\|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \varepsilon^{-1/2} \|\mathcal{W}_q; L^2(\partial\omega)\| \sup_{x \in \Upsilon} |\nabla_x^2 u_T^{\varepsilon\Omega}(x)| \\
&\leq c\varepsilon^{3/2} \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\|, \\
\mathcal{I}_q^{22}(\psi^\varepsilon) &\leq c\varepsilon\|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \varepsilon^{-1/2} \|\partial_n \mathcal{W}_q; L^2(\partial\omega)\| \sup_{x \in \Upsilon} \left| \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x) - \frac{\partial u_T^{\varepsilon\Omega}}{\partial x_q}(x_1, 0) \right| \\
&\leq c\varepsilon^{-1/2} \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)\| \varepsilon \|u_T^{\varepsilon\Omega}; C^2(\Omega_{L/3})\|. \tag{4.47}
\end{aligned}$$

We summarize the above calculation. First, the estimates (4.18), (4.23), and (4.39) show that the norm of the mapping (4.4) is uniformly bounded. Second, the sum (4.17) of the three terms (4.16), (4.37), and (4.38) generates the residual $\tilde{f}_T^\varepsilon(\psi^\varepsilon)$ in the problem (4.1) satisfying the inequality

$$|\tilde{f}_T^\varepsilon(\psi^\varepsilon)| \leq c\varepsilon^{1/2} \|f_T^\varepsilon; H_{\eta,T}^1(\Omega)\| \|\psi^\varepsilon; H_{\eta,T}^1(\Omega)^*\|$$

in view of (4.19), (4.20), (4.25), (4.46), and also (4.21), (4.24), and (4.43)–(4.44), (4.47). It is this inequality (4.5) with exponent $\alpha_\bullet = 1/2$ that was required to construct the almost inverse operator (4.4) and, consequently, the true inverse operator (4.2) of the problem (4.1). Thus, Theorem 4.1 is proved for $T \in (0, 1]$.

4.6. Case $T = 0$. All calculations and arguments of the previous subsections can be easily adapted to Problem $\mathcal{P}_N^\varepsilon(\eta)$. In fact, it suffices to set $T = 0$ in all formulas and remove the first identities from the transmission conditions (1.9) and (4.28), i.e., transform them to the Neumann conditions (1.7) and $\partial_n w(\xi) = g(\xi)$ for $\xi \in \partial\omega$ respectively. It is obvious that the obtained estimates are uniform with respect to the Floquet parameter $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$. The fact that $\varepsilon_\bullet > 0$ can be different in the situations $T \in (0, 1]$ and $T = 0$ does not affect the final formulation of Theorem 4.1 which thereby is valid for all $T \in [0, 1]$.

5 Comments

5.1. Limit as $T \rightarrow +0$. We fix $\varepsilon > 0$, i.e., the inclusions (1.1) are not assumed to be small, and construct the formal asymptotic expansions of the eigenvalues $(1.13)_T$ of Problem $\mathcal{P}_T^\varepsilon(\eta)$ as $T \rightarrow +0$. We refer, for example, to [32] for asymptotic procedures including the proof of estimates for remainders of asymptotic expansions (the uniformity in the Floquet parameter is not necessary).

The transmission conditions (1.9) split into two boundary conditions as $T \rightarrow +0$, so that the Neumann conditions are imposed at the more rigid inclusion, whereas the Dirichlet conditions are related to the softer one. Thus, the asymptotic ansätze for eigenpairs $\{\lambda_{Tk}^\varepsilon(\eta), u_{Tk}^\varepsilon(\cdot; \eta)\}$ of the problem (1.3)–(1.5), (1.8), (1.9)

$$\lambda_{Tk}^\varepsilon(\eta) = \lambda_{0k}^\varepsilon(\eta) + T\lambda_k^{\varepsilon'}(\eta) + \dots, \quad (5.1)$$

$$u_{Tk\Box}^\varepsilon(x; \eta) = u_{0k\Box}^\varepsilon(x; \eta) + Tu_{k\Box}^{\varepsilon'}(x; \eta) + \dots, \quad (5.2)$$

$$u_{Tk\circ}^\varepsilon(x; \eta) = u_{0k\circ}^\varepsilon(x; \eta) + Tu_{k\circ}^{\varepsilon'}(x; \eta) + \dots \quad (5.3)$$

contain eigenpairs of the limit problem consisting of the equation

$$-\Delta u_{0k\Box}^\varepsilon(x; \eta) = \lambda_{0k}^\varepsilon(\eta) u_{0k\Box}^\varepsilon(x; \eta), \quad x \in \Omega^\varepsilon, \quad (5.4)$$

the Neumann boundary conditions (1.4), (1.7), the quasiperiodicity conditions (1.9), and the equations

$$-\Delta u_{0k\circ}^\varepsilon(x; \eta) = \lambda_{0k}^\varepsilon(\eta) u_{0k\circ}^\varepsilon(x; \eta), \quad x \in \omega_j^\varepsilon, \quad j = -N, \dots, N, \quad (5.5)$$

with the Dirichlet boundary conditions (1.6). The dots in (5.1)–(5.3) mean lower-order terms that are not essential in our formal analysis.

The eigenvalues $\lambda_{j\circ k}^\varepsilon$ of the Dirichlet problems (5.5), (1.6) are independent of η and satisfy the estimate

$$\lambda_{j\circ k}^\varepsilon \geq \lambda_{\circ 1}^\varepsilon = \varepsilon^{-2} \Lambda_\omega, \quad j = -N, \dots, N, \quad (5.6)$$

where $\Lambda_\omega > 0$ is the first (least) eigenvalue of the Dirichlet problem in the domain ω . Consequently, for $\varepsilon \in (0, \varepsilon_\omega)$ and some $\varepsilon_\omega > 0$ (which is acceptable) the required segment $[0, \lambda_\bullet]$ is free from the spectrum of the problem (5.5), (1.6). This segment can contain the eigenvalues of the problem (5.4), (1.4), (1.7), (1.9) which depend on $\eta \in [\eta_\bullet - \delta_\bullet, \eta_\bullet + \delta_\bullet]$ in a complex way (cf. Figure 3 (a) and Subsection 1.3). In particular, they can be multiple for some values of the Floquet parameter η . Let $\lambda_{Nk}^\varepsilon(\eta)$ be a $\varkappa_k(\eta)$ -multiple eigenvalue, i.e.,

$$\lambda_{Nk-1}^\varepsilon(\eta) < \lambda_{Nk}^\varepsilon(\eta) = \cdots = \lambda_{Nk+\varkappa_k(\eta)-1}^\varepsilon(\eta) < \lambda_{Nk+\varkappa_k(\eta)}^\varepsilon(\eta)$$

in the ordered subsequence of eigenvalues. We consider the asymptotic ansätze (5.1) for $\lambda_{Tp}^\varepsilon(\eta)$ with correction terms $\lambda_p^{\varepsilon'}(\eta)$. In what follows, $p = k, \dots, k + \varkappa_k(\eta) - 1$. The leading terms $\mathbf{u}_{0p\Box}^\varepsilon$ of the ansatz (5.2) for the restrictions $u_{Tp\Box}^\varepsilon$ of the eigenfunctions u_{Tp}^ε onto the perforated rectangle Ω^ε are looked for in the form

$$\mathbf{u}_{0p\Box}^\varepsilon(x; \eta) = c_k^{(p)} u_{0k\Box}^\varepsilon(x; \eta) + \cdots + c_{k+\varkappa_k(\eta)-1}^{(p)} u_{0k+\varkappa_k(\eta)-1\Box}^\varepsilon(x; \eta), \quad (5.7)$$

and corrections in these ansätze are denoted by $\mathbf{u}_{p\Box}^{\varepsilon'}$. Here, $u_{0k\Box}^\varepsilon, \dots, u_{0k+\varkappa_k(\eta)-1\Box}^\varepsilon$ are eigenfunctions of Problem $\mathcal{P}_N^\varepsilon(\eta)$ satisfying (1.15), and $c^{(p)} = (c_k^{(p)}, \dots, c_{k+\varkappa_k(\eta)-1}^{(p)})^\top \in \mathbb{R}^{\varkappa_k(\eta)}$ are such that $(c^{(q)})^\top c^{(p)} = \delta_{p,q}$, $p, q = k, \dots, k + \varkappa_k(\eta) - 1$. The leading terms $\mathbf{u}_{0kj_o}^\varepsilon$ of the ansätze (5.3) for the functions $u_{Tk_o}^\varepsilon = u_{Tk}^\varepsilon|_{\omega_j^\varepsilon}$ are solutions to the problems in the fine domains (1.1)

$$\begin{aligned} -\Delta \mathbf{u}_{0p_o}^\varepsilon(x; \eta) &= \lambda_{0k}^\varepsilon(\eta) \mathbf{u}_{0p_o}^\varepsilon(x; \eta), \quad x \in \omega_j^\varepsilon, \\ \mathbf{u}_{0p_o}^\varepsilon(x; \eta) &= \mathbf{u}_{0p\Box}^\varepsilon(x; \eta) = \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} u_{0m\Box}^\varepsilon(x; \eta), \quad x \in \partial\omega_j^\varepsilon, \end{aligned} \quad (5.8)$$

where $j = -N, \dots, N$. These problems are uniquely solvable since $\lambda_{0k}^\varepsilon < \lambda_{o1}^\varepsilon$ in view of (5.6). It is obvious that

$$\mathbf{u}_{0p_o}^\varepsilon(x; \eta) = \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} u_{0m_o}^\varepsilon(x; \eta). \quad (5.9)$$

Finally, the correction terms $\mathbf{u}_{p\Box}^{\varepsilon'}$ are found from the equations

$$\begin{aligned} -\Delta \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) - \lambda_{0k}^\varepsilon(\eta) \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) &= \lambda_p^{\varepsilon'}(\eta) \mathbf{u}_{0p\Box}^\varepsilon(x; \eta) \\ &:= \lambda_p^{\varepsilon'}(\eta) \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} u_{0m\Box}^\varepsilon(x; \eta), \quad x \in \Omega^\varepsilon, \end{aligned} \quad (5.10)$$

with the conditions (1.4), (1.5) on the unilateral sides of the rectangle Ω and the inhomogeneous Neumann conditions on the boundaries of the holes (1.1)

$$\partial_n \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) = \partial_n \mathbf{u}_{0p_o}^\varepsilon(x; \eta) := \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} \partial_n u_{0m_o}^\varepsilon(x; \eta), \quad x \in \partial\omega_j^\varepsilon, \quad j = -N, \dots, N. \quad (5.11)$$

The equalities (5.10) and (5.11) are obtained by substituting the ansätze for eigenpairs into Equation (1.3) and the second transmission condition in (1.9) and collecting after that coefficients of the small parameter T .

The solvability conditions for the problem (5.10), (5.11), (1.4), (1.5) are obtained with the help of the Green formula in the domain Ω^ε for functions $\mathbf{u}_{p\Box}^{\varepsilon'}$ and u_{Nq}^ε :

$$\begin{aligned}
\lambda_p^{\varepsilon'} c_q^{(p)} &= \lambda_p^{\varepsilon'}(\eta) \int_{\Omega^\varepsilon} u_{Nq}^\varepsilon(x; \eta) \mathbf{u}_{0p\Box}^\varepsilon(x; \eta) dx \\
&= - \int_{\Omega^\varepsilon} u_{Nq}^\varepsilon(x; \eta) (\Delta \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) + \lambda_{0k}^\varepsilon(\eta) \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta)) dx \\
&= \int_{\partial\omega^\varepsilon} (\mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) \partial_n u_{Nq}^\varepsilon(x; \eta) - u_{Nq}^\varepsilon(x; \eta) \partial_n \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta)) ds_x \\
&= - \int_{\partial\omega^\varepsilon} u_{Nq}^\varepsilon(x; \eta) \partial_n \mathbf{u}_{p\Box}^{\varepsilon'}(x; \eta) ds_x = - \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} \int_{\partial\omega^\varepsilon} u_{Nq}^\varepsilon(x; \eta) \partial_n u_{Nm\circ}^\varepsilon(x; \eta) ds_x \\
&= - \sum_{m=k}^{k+\varkappa_k(\eta)-1} c_m^{(p)} \int_{\partial\omega^\varepsilon} u_{Nq\circ}^\varepsilon(x; \eta) \partial_n u_{Nm\circ}^\varepsilon(x; \eta) ds_x =: \sum_{m=k}^{k+\varkappa_k(\eta)-1} M_{qm}^\varepsilon(\eta) c_m^{(p)}.
\end{aligned}$$

Using (5.8), (5.9) and integrating by parts over the domains ω_j^ε (recall that n is the inward normal on $\partial\omega_j^\varepsilon$), we get

$$M_{qm}^\varepsilon(\eta) = \lambda_{0k}^\varepsilon(\eta) (u_{Nq}^\varepsilon(\cdot; \eta), u_{Nm\circ}^\varepsilon(\cdot; \eta))_{\omega^\varepsilon} - (\nabla u_{Nq}^\varepsilon(\cdot; \eta), \nabla u_{Nm\circ}^\varepsilon(\cdot; \eta))_{\omega^\varepsilon}, \quad (5.12)$$

where $q, m = k, \dots, k + \varkappa_k(\eta) - 1$. Thus, the asymptotic corrections $\lambda_k^{\varepsilon'}(\eta), \dots, \lambda_{k+\varkappa_k(\eta)-1}^{\varepsilon'}(\eta)$ are eigenvalues of the symmetric $(\varkappa_k(\eta) \times \varkappa_k(\eta))$ -matrices $M^\varepsilon(\eta)$ with entries (5.12).

Thus, we have constructed the leading terms of the asymptotic expansions (5.1) of the eigenvalues of Problem $\mathcal{P}_T^\varepsilon(\eta)$. The estimates

$$|\lambda_{Tk}^\varepsilon(\eta) - \lambda_{Nk}^\varepsilon(\eta) - T \lambda_p^{\varepsilon'}(\eta)| \leq C_k(\eta) T^{3/2}, \quad k \in \mathbb{N},$$

obtained in [32] provide the required continuity of $T \mapsto \lambda_{Tk}^\varepsilon(\eta)$ at $T = 0$.

By [33, Theorem 7.1.8], if $\lambda_{Nk}^\varepsilon(\eta)$ is a simple eigenvalue ($\varkappa_k(\eta) = 1$ in the list (5.7)), then this function is an analytic real-valued function on $[0, T_0(\varepsilon, \eta)]$. This property is important, but plays no role in this paper.

Remark 5.1. 1. At the first glance, in view Proposition 2.2, it is much easier to deal with the modified problem $\mathcal{P}_T^{\varepsilon\flat}(\eta)$ in Subsection 2.2 which can be transformed to Problems $\mathcal{P}_N^\varepsilon(\eta)$ and $\mathcal{P}(\eta)$ by passing to the limit as $T \rightarrow +0$ and $T \rightarrow 1 - 0$ respectively. However, first, the right inequality in (2.10) holds only if $T > 1$, i.e., it is useless in the limit passage and, second, additional difficulties arise while constructing the almost inverse operator for the operator $A_T^{\varepsilon\flat}(\lambda_\bullet; \eta)$ of the problem (1.3)–(1.5), (2.8), (1.9).

2. As known, passing to the limit as $T \rightarrow +\infty$, we obtain the problem (1.3)–(1.5) with the following integro-differential boundary conditions on the boundaries of the holes (1.1):

$$u_\infty^\varepsilon(x; \eta) = c_j^\varepsilon, \quad x \in \partial\omega_j^\varepsilon, \quad \int_{\partial\omega_j^\varepsilon} \partial_n u_\infty^\varepsilon(x; \eta) ds_x = 0, \quad j = -N, \dots, N.$$

Moreover, the constants c_j^ε are not a priori fixed, but are found when we solve the problem. The variational statement (1.12) of the problem is considered in the subspace

$$H_{\eta,\infty}^1(\Omega^\varepsilon) = \{u_\infty^\varepsilon \in H_\eta^1(\Omega^\varepsilon) : u_\infty^\varepsilon(x; \eta) = c_j^\varepsilon, x \in \partial\omega_j^\varepsilon, j = -N, \dots, N\}.$$

5.2. Justification of the asymptotics the spectral segments with the Neumann perforations. A simple result presented in [32] and rewritten in Subsection 5.1 leads to the following important conclusion: Theorem 4.1 shows that the multiplicities of the spectra on $[0, \lambda_\bullet]$ coincide for Problem $\mathcal{P}_N^\varepsilon(\eta)$ with the Neumann conditions on the boundaries of holes (1.1) and Problem $\mathcal{P}(\eta)$ on the whole cell. We consider an example to show how to apply this result.

Formula (1.17) for the eigenvalues of the limit problem $\mathcal{P}(\eta)$ involving Equation (1.16) in the rectangle $\Omega = (-1/2, 1/2) \times (-H, H)$ and the conditions (1.4), (1.5) on the sides of Ω shows that the corresponding dispersion curves form a truss of a rather complex structure. The lower part of the truss is shown in Figure 3 (b), (c) in the case $1/6 < H < 1/4$, where two gaps γ_{N1}^ε and γ_{N2}^ε (the projections of tinted rectangles on the ordinate axis in Figure 3 (a)) can appear in the spectrum of an infinite periodic waveguide with the Neumann perforation (cf. Figure 1 (a)). These gaps are located in $c\varepsilon$ -neighborhoods of the points $\lambda_{N1}^\varepsilon = \pi^2$ and $\lambda_{N2}^\varepsilon = (2H)^{-2}\pi^2$. To identify the gaps γ_{N1}^ε , we need asymptotic formulas for the upper and lower bounds of the spectral segments v_{N1}^ε and v_{N2}^ε respectively. Due to the intertwining of dispersion curves, the justification of asymptotics for $\lambda_{N1}^\varepsilon(\eta)$ and $\lambda_{N2}^\varepsilon(\eta)$ is performed in two steps. First, Theorem 4.1 applied to the segment $\Lambda_{\bullet 1} = \{(\lambda_{\bullet 1}, \eta) \mid |\eta| \leq \delta_1\}$ (cf. the definition (1.25)) shows that only one dispersion curve can pass through the rectangle $[0, \lambda_{\bullet 1}] \times [-\delta_1, \delta_1]$ in Figure 3 (a). Second, Theorem 4.1 applied to the segment $\Lambda_{\bullet 2+} = \{(\lambda_{\bullet 2}, \eta) \mid \eta \in [\pi - 2\delta_2, \pi]\}$ yields the opposite observation: the rectangle $[0, \lambda_{\bullet 2}] \times [\pi - 2\delta_2, \pi]$ in Figure 3 (a) contains arcs of two dispersion curves. We emphasize that the rectangle is not necessarily symmetrically located because the function $\eta \mapsto \lambda_{Mk}^\varepsilon(\eta)$ is even. Since we can choose $\lambda_{\bullet 1} < \lambda_{\bullet 2}$ and $\delta_1 > \pi - 2\delta_2$ (cf. Figure 3 (b), where the segments $\Lambda_{\bullet 1}$ and $\Lambda_{\bullet 2\pm}$ are marked with dash-dotted lines ending with the symbol \bullet), we can derive the required uniform estimates with respect to the Floquet parameter for the remainders in the asymptotic representations of the eigenvalues and obtain exhaustive information about the spectral segments $(1.22)_N$ with $k = 1$ and $k = 2$.

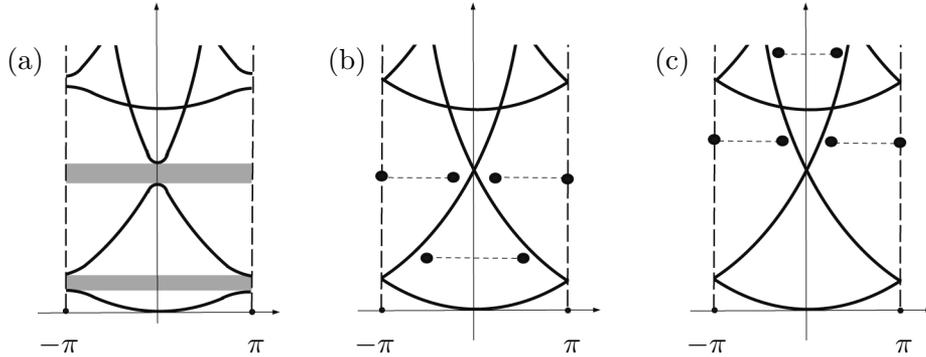


Figure 3. The dispersion curves in the model problem with the Neumann perforation (a) and on the whole cell (b) and (c). The dash-dotted line ended with the symbol \bullet represents the segment (1.25) which is free from the spectrum of Problem $\mathcal{P}_T^\varepsilon(\eta)$.

To study the opening of the gap γ_{N1}^ε , we use Figure 3 (c). The upper and lower bounds of

the segments v_{N2}^ε and v_{N3}^ε are determined by the eigenvalues $\lambda_{N2}^\varepsilon(\eta)$ and $\lambda_{N3}^\varepsilon(\eta)$, but, in view of the structure of the truss of dispersion curves, we need to justify the asymptotic expansions of four eigenvalues $\lambda_{N1}^\varepsilon(\eta), \dots, \lambda_{N4}^\varepsilon(\eta)$ on $\eta \in [-\delta_3, \delta_3]$. The segments $\Lambda_{\bullet 3}$ and $\Lambda_{\bullet 4\pm}$ are presented in Figure 3 (c) as above. It is natural to expect that similar coverings of the range interval of the Floquet parameter can be also constructed for more complicated trusses.

5.3. Another approach to justifying asymptotics for spectral segments with the Dirichlet perforation. The asymptotics of the eigenvalues of the problem (1.3)–(1.6) is constructed in [29]. The asymptotics of narrowed spectral segments justified in (1.22) (cf. Figure 2) can be done by different methods, in particular, by using the inequality (3.18) in Theorem 3.1.

The second above-discussed approach is also applicable. Namely, the construction of an almost inverse operator for mapping

$$A_D^\varepsilon(\eta; \lambda_\bullet) : H_{\eta,D}^1(\Omega^\varepsilon) \rightarrow H_{\eta,D}^1(\Omega^\varepsilon)^* \quad (5.13)$$

of the inhomogeneous problem $\mathcal{P}_D^\varepsilon(\eta)$ in the variational setting

$$(\nabla u_D^\varepsilon, \nabla \psi_D^\varepsilon)_{\Omega^\varepsilon} - \lambda_\bullet (u_D^\varepsilon, \psi_D^\varepsilon)_{\Omega^\varepsilon} = f_D^\varepsilon(\psi_D^\varepsilon), \quad \psi_D^\varepsilon \in H_{\eta,D}^1(\Omega^\varepsilon)$$

repeats (with some simplifications) the arguments of Section 4. Moreover, because of the simplicity of dispersion curves, in Problem \mathcal{P}_D^0 we deal with horizontal segments of level $\lambda = \lambda_{Dk}$. Therefore, we choose points $\lambda_\bullet \notin \{\lambda_{Dk}\}_{k \in \mathbb{N}}$ for which the operator (5.13) realizes an isomorphism for all $\eta \in [-\pi, \pi]$, $\varepsilon \in (0, \varepsilon_{k_\bullet})$ and some $\varepsilon_{k_\bullet} > 0$. As a result, for such values of ε the multiplicities of the discrete spectra of Problems $\mathcal{P}_D^\varepsilon(\eta)$ and \mathcal{P}_D^0 on $[0, \lambda_\bullet]$ coincide. Since the asymptotic expansions of the eigenvalues $\lambda_{Dk}^\varepsilon(\eta)$ are obtained in [29], it is obvious that the estimates for the remainders are uniform in $\eta \in [-\pi, \pi]$.

Acknowledgments

The work has partially been supported by MICINN PGC2018-098178-B-I00, MTM2017-89976-P, and MINECO “Severo Ochoa Programme for Centres of Excellence in RaD” (SEV-2015-0554).

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