Localization effects for Dirichlet problems in domains surrounded by thin stiff and heavy bands

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Abstract

We consider a Dirichlet spectral problem for a second order differential operator, with piecewise constant coefficients, in a domain Ω_ε in the plane $\mathbb{R}^2$. Here Ω_ε is $\Omega\cup\omega_\varepsilon\cup\Gamma$, where Ω is a fixed bounded domain with boundary Γ, ω_ε is a curvilinear band of width $O(\varepsilon)$, and $\Gamma=\overline{\Omega}\cap\overline{\omega}_\varepsilon$. The *density* and *stiffness* constants are of order ε^{-m-t} and ε^{-t} respectively in this band, while they are of order 1 in Ω; $t\geq 1$, $m>2$, and ε is a small positive parameter. We address the asymptotic behavior, as $\varepsilon\to 0$, for the eigenvalues and the corresponding eigenfunctions. In particular, we show certain localization effects for eigenfunctions associated with low frequencies. This is deeply involved with the extrema of the curvature of Γ.

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1. Introduction and statement of the problem

Let Ω be a bounded domain of the plane $\mathbb{R}^2$ with a smooth boundary Γ and let (ν,τ) be the natural orthogonal curvilinear coordinates in a neighborhood of Γ: τ is the arc length and ν the distance along the normal vector

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to Γ; $\nu < 0$ inside Ω. Let ℓ denote the length of the curve Γ and $\varkappa(\tau)$ its curvature at the point τ. We assume that the domain Ω is surrounded by the thin band $\omega_\varepsilon = \{x : 0 < \nu < \varepsilon h\}$ where $\varepsilon > 0$ is a small parameter and h is a positive constant, cf. (1.4). Let Ω_ε be the domain $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon \cup \Gamma$ and $\Gamma_\varepsilon = \{x : \nu = \varepsilon h\}$ the boundary of Ω_ε (see Figure 1).

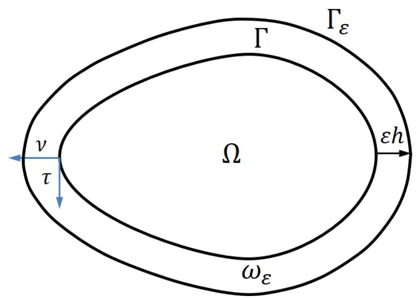

Figure 1: Possible geometry for Ω_ε

We consider the spectral Dirichlet problem in Ω_ε for a second order differential operator with piecewise constants coefficients:

$$-A\Delta_x U^\varepsilon = \lambda^\varepsilon U^\varepsilon \quad \text{in } \Omega, \tag{1.1a}$$
$$-a\varepsilon^{-t}\Delta_x u^\varepsilon = \lambda^\varepsilon \varepsilon^{-t-m} u^\varepsilon \quad \text{in } \omega_\varepsilon, \tag{1.1b}$$
$$U^\varepsilon = u^\varepsilon \quad \text{on } \Gamma, \tag{1.1c}$$
$$\varepsilon^t A\partial_\nu U^\varepsilon = a\partial_\nu u^\varepsilon \quad \text{on } \Gamma, \tag{1.1d}$$
$$u^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon. \tag{1.1e}$$

Here, A and a are two positive constants while ∂_ν denotes the derivative along the outward normal vector ν to the curve Γ; t and m are two positive parameters. We study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalues λ^ε of (1.1) and the corresponding eigenfunctions which we identify with pairs of functions $\{U^\varepsilon, u^\varepsilon\}$. In (1.1), U^ε stands for the restriction of the eigenfunction to Ω and u^ε for the restriction of the eigenfunction to ω_ε.

Problem (1.1) is new in the literature. It is of interest, for instance, in the study of reinforcement problems for solid media and in vibrations for a two-phase system in fluid mechanics. Here, the band ω_ε is both stiffer and heavier. Parameters t and m deal with the physical characteristic of the medium and it seems natural to have a different asymptotic behavior as $\varepsilon \to 0$ for the eigenpairs $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$ of (1.1) depending on their value.

In particular, for $0 < m \leq 2$ the low frequencies are of order 1 while for $m > 2$ they are of order ε^{m-2} (cf. Proposition 1.1 and Remark 5.4). For $m > 2$ and $t \geq 1$, among other things, the paper provides a mathematical proof on how the low frequencies vibrations in reinforcement problems can concentrate around certain points of the boundary.

Usually, the localization phenomena occur near the extrema of the curvature, see e.g. [19, 23, 9, 17] for maxima and [11, 3] for minima. In problems with banded domains, they occur for both, maxima and minima (see [16] and the present paper).

In this respect, let us recall the results in [14, 15, 16] for a very different problem: the Neumann problem (1.1a)–(1.1d) along with

$$\partial_\nu u^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon. \tag{1.2}$$

They are the closest works in the literature for a domain $\omega_\varepsilon = \{x : 0 < \nu < \varepsilon h(\tau)\}$, where h is a strictly positive function of the τ variable ℓ-periodic, $h \in C^\infty(\mathbb{S}_\ell)$, $\mathbb{S}_\ell$ stands for the circumference of length ℓ and ω_ε may vary with the arc length. A characterization of the limiting problems for the eigenpairs of problem (1.1a)–(1.1d), (1.2) for the different values of t and m has been obtained in [14] by means of asymptotic expansions. Sharp bounds for convergence rates of the eigenpairs $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$ in the case where $t = 1$ and $m = 0$ have been given by using the so-called *inverse-direct reduction method* (cf. [24, 25, 20]). A different approach for the eigenpairs is provided in [15] for the case where $t > 1$ and $m = 0$ where, in addition to the convergence, a complete asymptotic expansion for the eigenpairs has been obtained, and a connection of this problem with Wentzell problems with small parameters has been shown. Also, both papers [14, 15] describe precise bounds for convergence rates for the low frequencies and the corresponding eigenfunctions in the cases mentioned above $m = 0$ and $t \geq 1$. We refer to [14, 15] for further references.

Paper [16] deals with the Neumann problem (1.1a)–(1.1d), (1.2) in the case where $t = 1$ and $m > 0$, and considers the low and high frequencies which are of order ε^m and 1 respectively. The limiting problems associated with both kinds of frequencies are obtained and information on the structure of the corresponding eigenfunctions is also provided. These problems appear independently of the geometry of the band ω_ε, but for $m > 2$ there are other limiting problems associated with the so-called middle frequencies, namely eigenvalues of order ε^{m-2}, which strongly depend on this geometry: more precisely whether the function h is constant or not. Moreover, only in the case where h is not a constant, the eigenfunctions corresponding to the middle frequencies are localized asymptotically in small neighborhoods of points τ_0 of the boundary where the function h presents a local maximum.

Here we deal with the Dirichlet problem (1.1) which provides a very different behavior of the spectrum as $\varepsilon \to 0$. We consider the low and high frequencies in the case where $t \geq 1$ and $m > 2$ which are now of order ε^{m-2} and 1 respectively (see Remark 5.4 for other values of m). In contrast with the Neumann problem, when the function h is constant, we show new localization effects for the eigenfunctions of (1.1) at points τ_0 of the boundary where the curvature of Γ presents a local minimum (cf. Theorem 3.3 and Remark 2.2). Besides, these eigenfunctions correspond to low frequencies of (1.1). When the curvature of Γ has a unique global minimum (cf. Figure 2), we also study the convergence of the low frequencies with conservation of the multiplicity, once we have rescaled the eigenvalues and the corresponding eigenfunctions in a suitable way (cf. Theorem 4.1). We note that, for the sake of brevity, we avoid writing proofs in the case where Γ has several curved components (cf., e.g., Figure 2 (d)).

1.1. A priori estimate for the eigenvalues

The weak formulation of problem (1.1) reads: to find λ^ε and $\{U^\varepsilon, u^\varepsilon\} \in H_0^1(\Omega_\varepsilon) \setminus \{0\}$, satisfying

$$\begin{aligned} A \int_\Omega \nabla_x U^\varepsilon \cdot \nabla_x G \, dx \; + \; \frac{a}{\varepsilon^t} \int_{\omega_\varepsilon} \nabla_x u^\varepsilon \cdot \nabla_x g \, dx \\ = \lambda^\varepsilon \left(\int_\Omega U^\varepsilon \, G \, dx + \frac{1}{\varepsilon^{t+m}} \int_{\omega_\varepsilon} u^\varepsilon \, g \, dx \right) \quad \forall \{G, g\} \in H_0^1(\Omega_\varepsilon). \end{aligned} \tag{1.3}$$

Here, and in what follows, we identify a function in $L^2(\Omega_\varepsilon)$ ($H^1(\Omega_\varepsilon)$, respectively) with the pair of functions $\{G, g\}$, where G stands for the restriction of the function to Ω and g for the restriction of the function to ω_ε. In particular, the eigenpairs formed by the eigenvalues λ^ε and the corresponding eigenfunctions read $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$.

For each $\varepsilon > 0$, problem (1.3) is a standard spectral problem in the couple of spaces $H_0^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon)$, with a positive and discrete spectrum. Let us consider

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_k^\varepsilon \leq \cdots \xrightarrow{k \to \infty} \infty$$

the sequence of eigenvalues repeated according to their multiplicity.

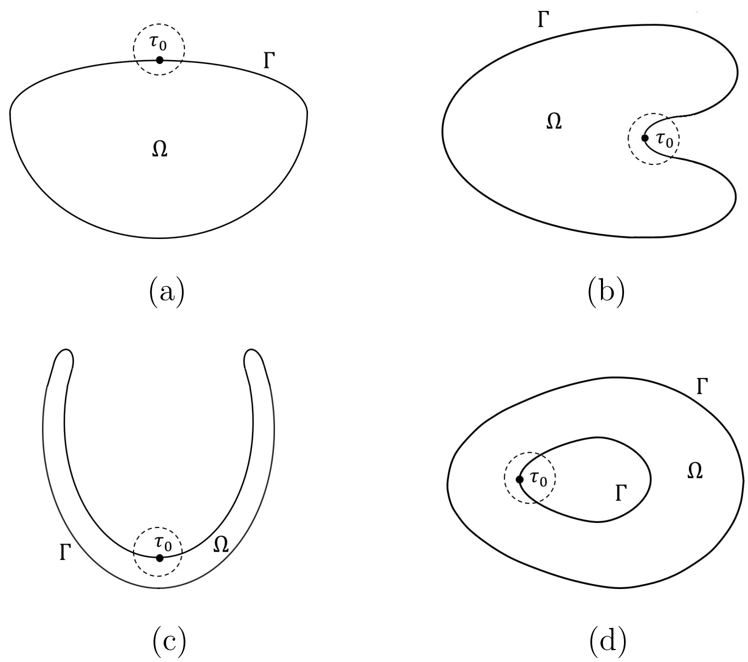

Figure 2: Examples of different domains Ω and points τ_0 where the localization phenomena occur with the global minimum.

Now, we introduce some notations and obtain estimates for the eigenvalues of (1.3) (see, for instance, [12, 14] for the technique).

Let $\Gamma(\tau) = (\Gamma_1(\tau), \Gamma_2(\tau))$ be a parametrization of the boundary $\Gamma = \partial\Omega$ by its arc length $\tau \in [0, \ell)$, namely $(\Gamma_1'(\tau))^2 + (\Gamma_2'(\tau))^2 = 1$; we choose the counterclockwise orientation of the boundary. Let $\varkappa$ be the curvature of Γ, $\varkappa(\tau) = \Gamma_1'(\tau)\Gamma_2''(\tau) - \Gamma_1''(\tau)\Gamma_2'(\tau)$ for $\tau \in [0, \ell)$; note that the curvature is nonnegative if the domain Ω is convex.

For ε small enough, let us consider the change

$$x_1 = \Gamma_1(\tau) + \nu\Gamma_2'(\tau) \qquad \text{and} \qquad x_2 = \Gamma_2(\tau) - \nu\Gamma_1'(\tau), \tag{1.4}$$

where (ν, τ) are the orthogonal curvilinear coordinates, $\nu \in [0, \varepsilon h)$ and $\tau \in [0, \ell)$. The Jacobian of the above transformation is $K(\nu, \tau) = 1 + \nu\varkappa(\tau)$.

In a neighborhood of Γ, we introduce the so-called *local coordinates*

$$(\zeta, \tau), \quad \zeta = \varepsilon^{-1}\nu, \tag{1.5}$$

which transforms the thin domain ω_ε into a band ω_1 of length ℓ and width $O(1)$; namely, $\omega_\varepsilon = \{(\nu, \tau) : \nu \in [0, \varepsilon h), \tau \in \mathbb{S}_\ell\}$ into $\omega_1 = \{(\zeta, \tau) : \zeta \in$

$[0, h), \tau \in \mathbb{S}_\ell\}$. Note that the boundary condition along with the change of variable (1.5) in ω_ε yield

$$\|g\|^2_{L^2(\omega_\varepsilon)} \leq C\varepsilon^2 \|\nabla_x g\|^2_{L^2(\omega_\varepsilon)} \quad \forall \{G, g\} \in H^1_0(\Omega_\varepsilon); \tag{1.6}$$

here and in what follows C denotes a strictly positive constant independent of ε.

Proposition 1.1. *Let $\{\lambda^\varepsilon_k\}_{k=1}^\infty$ be eigenvalues of* (1.3)*. For each fixed $k \in \mathbb{N}$ and a small ε, we have*

$$\begin{aligned} C \leq \lambda^\varepsilon_k \leq C_k \qquad & \textit{when } m \leq 2, \\ C\varepsilon^{m-2} \leq \lambda^\varepsilon_k \leq C_k \varepsilon^{m-2} \quad & \textit{when } m > 2, \end{aligned} \tag{1.7}$$

where the positive constants C and C_k do not depend on ε, but $C_k \to \infty$ as $k \to \infty$.

Proof. The lower bounds hold as a direct consequence of (1.3), the Poincaré inequality, (1.6), and the fact that $m \leq 2$ or $m > 2$, respectively (cf. Proposition 4.2).

As regards the upper bounds, the minimax principle gives the equalities

$$\lambda^\varepsilon_k = \min_{\substack{E_k \subset H^1_0(\Omega_\varepsilon) \\ \dim E_k = k}} \max_{\substack{\{V, v\} \in E_k \\ \{V, v\} \not\equiv 0}} \frac{A \displaystyle\int_\Omega |\nabla_x V|^2 \, dx + \frac{a}{\varepsilon^t} \int_{\omega_\varepsilon} |\nabla_x v|^2 \, dx}{\displaystyle\int_\Omega |V|^2 \, dx + \frac{1}{\varepsilon^{t+m}} \int_{\omega_\varepsilon} |v|^2 \, dx}, \tag{1.8}$$

where the minimum is taken over all the subspaces $E_k \subset H^1_0(\Omega_\varepsilon)$ with $\dim E_k = k$.

Let $\{\mu_k\}_{k=1}^\infty$ be the eigenvalues of the Dirichlet problem in Ω and $\{V_k\}_{k=1}^\infty$ the corresponding eigenfunctions which are assumed to form an orthonormal basis in $L^2(\Omega)$. For each fixed k, $E^\#_k$ is the linear space

$$E^\#_k = [\{V_1, 0\}, \ldots, \{V_k, 0\}] \subset H^1_0(\Omega_\varepsilon),$$

where $\{V_r, 0\}$ denotes the extension of V_r to Ω_ε by 0 in ω_ε, for $r = 1, 2 \ldots, k$. Then, from (1.8), for any $m \in \mathbb{R}$, we derive

$$\lambda^\varepsilon_k \leq \max_{\substack{\{V, v\} \in E^\#_k \\ \{V, v\} \not\equiv 0}} \frac{A \displaystyle\int_\Omega |\nabla_x V|^2 \, dx}{\displaystyle\int_\Omega |V|^2 \, dx} = \mu_k.$$

This inequality provides the upper bound in (1.7) when $m \leq 2$. In order to prove it when $m > 2$, we consider $\{\lambda_{0,k}\}_{k=1}^{\infty}$ the eigenvalues of the spectral problem

$$\left\{ \begin{array}{l} -a y_0'' = \lambda_0 y_0 \quad \zeta \in (0,h), \\ y_0'(0) = y_0(h) = 0 \end{array} \right.$$

and $\{y_{0,k}\}_{k=1}^{\infty}$ the corresponding eigenfunctions (cf. Section 2 for details) and we define the functions $V_k^\varepsilon \in H_0^1(\Omega_\varepsilon)$ as

$$V_k^\varepsilon(x) = \left\{ \begin{array}{ll} y_{0,k}(0) & \text{if } x \in \Omega, \\ y_{0,k}(\nu/\varepsilon) & \text{if } x \in \omega_\varepsilon. \end{array} \right.$$

Then, taking in (1.8) the particular subspace of $H_0^1(\Omega_\varepsilon)$, $E_k^\varepsilon = [V_1^\varepsilon, \ldots, V_k^\varepsilon]$, and making the change of variable (1.5) in ω_ε, we obtain

$$\lambda_k^\varepsilon \leq \max_{\substack{\{V,v\} \in E_k^\varepsilon \\ \{V,v\} \not\equiv 0}} \frac{\dfrac{a}{\varepsilon^2} \displaystyle\int_{\omega_1} \left| y_{0,k}'(\zeta) \right|^2 K_\varepsilon \, d\zeta d\tau}{\dfrac{1}{\varepsilon^m} \displaystyle\int_{\omega_1} |y_{0,k}(\zeta)|^2 K_\varepsilon \, d\zeta d\tau}, \tag{1.9}$$

where $K_\varepsilon(\zeta,\tau) = 1 + \varepsilon\zeta\varkappa(\tau)$. On account of the continuity of $\varkappa(\tau)$, for sufficiently small ε, (1.9) gives

$$\lambda_k^\varepsilon \leq C\varepsilon^{m-2}\lambda_{0,k},$$

C being a constant independent of ε. Therefore, the proposition is proved. □

Relations in (1.7) indicate the order of magnitude of the eigenvalues of problem (1.3) for fixed k, the so-called *low frequencies*. The aim of this paper is to study, for $t \geq 1$ and $m > 2$, its asymptotic behavior as $\varepsilon \to 0$ and that of the corresponding eigenfunctions $\{U_k^\varepsilon, u_k^\varepsilon\}$. We assume that they are subject to the orthonormalization condition

$$\varepsilon^{t+1} A \int_\Omega \nabla_x U_k^\varepsilon \cdot \nabla_x U_l^\varepsilon \, dx + \varepsilon a \int_{\omega_\varepsilon} \nabla_x u_k^\varepsilon \cdot \nabla_x u_l^\varepsilon \, dx = \delta_{k,l},$$

where $\delta_{k.l}$ denotes the Kronecker symbol. It should be noted that, by introducing the change of variable (1.5), the integral identity (1.3) reads

$$\begin{aligned} \varepsilon^{t+1} A \int_\Omega \nabla_x U^\varepsilon \cdot \nabla_x G \, dx + a \int_{\omega_1} \partial_\zeta \mathsf{u}^\varepsilon \partial_\zeta \mathsf{g} K_\varepsilon \, d\zeta d\tau + \varepsilon^2 a \int_{\omega_1} \partial_\tau \mathsf{u}^\varepsilon \partial_\tau \mathsf{g} K_\varepsilon^{-1} \, d\zeta d\tau \\ = \frac{\lambda^\varepsilon}{\varepsilon^{m-2}} \left(\varepsilon^{t+m-1} \int_\Omega U^\varepsilon \, G \, dx + \int_{\omega_1} \mathsf{u}^\varepsilon \, \mathsf{g} \, K_\varepsilon \, d\zeta d\tau \right), \end{aligned} \tag{1.10}$$

where now u^ε and g denote the functions u^ε and g written in the new variables (ζ, τ), and $K_\varepsilon(\zeta, \tau) = 1 + \varepsilon\zeta\varkappa(\tau)$ denotes the Jacobian of the transformation from (x_1, x_2) to (ν, τ) in the (ζ, τ) variables.

In particular, we construct three-term asymptotic expansions of eigenvalues of (1.1) of order ε^{m-2},

$$\lambda^\varepsilon = \varepsilon^{m-2}(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^{3/2}\lambda_{3/2} + o(\varepsilon^{3/2})), \tag{1.11}$$

and show localization effects for the corresponding eigenfunctions in the case where the curvature of Γ is not constant. As a matter of fact, we construct approaches to eigenfunctions corresponding to certain eigenvalues of order ε^{m-2} which concentrate asymptotically their support in $\varepsilon^{1/4}$-neighborhoods of points which are local minima of $\varkappa$ (cf. [19, 23, 8, 9] for very different problems with localization effects in neighborhoods of local extrema of the curvature). Note that, in contrast with the Neumann problem, these localization effects can arise when the thickness of the band is constant (of order ε), and they are associated with low frequencies (cf. [16]). We refer to [19, 23, 10, 11, 6, 27, 3, 4] for different problems in thin domains where localization effects for the eigenfunctions arise: [19, 23, 27] deal with thin plate-like domains while [10, 11, 6] consider a thin rod structure in two and three dimensions respectively. See [21] for references on other quite different localization effects at points for vibrating systems with concentrated masses.

In this respect, it is worthy emphasizing that localization effects for eigenfunctions in the literature are related either to geometrical characteristics of the domains along with the operator under consideration, or to physical characteristics of the material. Here, as happens in [16], they are related to both characteristics. The localization near a point τ_0 can be enlightened by introducing suitable local variables (which somehow isolate the point), rescaled spectral parameters and renormalized eigenfunctions. All this is glimpsed by means of asymptotic expansions. Although the frequency range giving rise to localized eigenfunctions differs from [16], the choice of appropriate scales leads to a certain connection of the operator (cf. (1.1)) with the harmonic oscillator operator (2.37) which also involves the geometrical characteristics of the domain.

We also describe the asymptotic behavior as $\varepsilon \to 0$ of the eigenvalues λ_k^ε of (1.1) for $k \in \mathbb{N}$ fixed. As occurs in other Dirichlet problems posed in thin domains (cf. [5, 10, 11, 3, 4, 2, 27, 18]), for all k fixed, the values $\lambda_k^\varepsilon/\varepsilon^{m-2}$ have a common limit λ_0, the dominant eigenvalue of (2.22), and we have to turn to the second correction term $\lambda_{3/2}$ to show the effect of asymptotic splitting in the eigenvalue sequence, namely,

$$\lambda_k^\varepsilon = \varepsilon^{m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\lambda_{3/2,k} + o(\varepsilon^{3/2})\Big) \quad \text{for } k \in \mathbb{N}, \tag{1.12}$$

τ_0 being the only point where the curvature of Γ has the global minimum. In this case, $\lambda_{3/2,k}$ are the eigenvalues of the harmonic oscillator operator (2.37). The proof is based on a factorization principle which somehow allows us to isolate oscillations. This technique has been used in the literature of homogenization problems (cf. [26] and references therein), but to our knowledge this is the first time that it is used for reinforcement problems. The method involves a rescaling for eigenfunctions which along with a suitable shift and rescaling for eigenvalues lead us to a reformulation of the original problem in terms of new eigenvalues and eigenfunctions.

The structure of the paper is the following: in Section 2 we construct the formal asymptotic expansions (1.11) and identify the values λ_0 as eigenvalues of (2.22), $\lambda_1 = a\varkappa(\tau_0)/h$, and λ_2 as eigenvalues of the harmonic oscillator operator (2.37), τ_0 being a point where $\varkappa$ presents a local minimum and $\varkappa''(\tau_0) > 0$ (cf. (2.38) and Remark 5.3). These asymptotic expansions are justified up to a certain degree in Section 3 where we obtain estimates for convergence rates for the low frequencies and the corresponding eigenfunctions as stated in Theorem 3.3. However, this still does not imply the convergence of the kth eigenvalue of (1.1). The aim of Section 4 is precisely to justify (1.12) when τ_0 is the only point where the curvature of Γ has the global minimum (see Theorem 4.1). We divide the proof into several steps. In (4.1), we verify that the dominant eigenvalue of (2.22) is the common limit of the rescaled eigenvalues of (1.3), $\lambda_k^\varepsilon/\varepsilon^{m-2}$ for $k \in \mathbb{N}$ fixed (cf. Theorem 4.3). Later on, in (4.2), using the principal eigenpair of (2.22) we reformulate the original problem (1.3) in terms of a new spectral parameter and eigenfunctions, problem (4.64), and we show that its eigenvalues converge towards the eigenvalues of (2.37) (see (4.3)). In (4.4), we state the equivalence of the spectral problems (1.3) and (4.64), and show convergence (1.12). Dealing with the local effects for the low frequencies, the technique differs very much from [16]. Finally, the eigenvalues of (1.1) of order 1, that is, the high frequencies, are considered in Section 5.

It should be noted that in (1.11) for λ_0 the dominant eigenvalue of (2.22) and for $\lambda_1 = a\varkappa(\tau_0)/h$ with τ_0 the point where $\varkappa$ presents the global minimum (namely, (1.12)), we are dealing with very low frequencies, while for other values τ_0 where $\varkappa$ presents local minima or for other values of λ_0, eigenvalues of (2.22), we are dealing with larger frequencies but with the same order of magnitude.

2. Asymptotic expansions

In this section, we provide asymptotic expansions for the eigenvalues of (1.1) of order ε^{m-2} and their corresponding eigenfunctions. We determine the terms arising in these expansions from the eigenpairs of two one-dimensional problems (cf. (2.22) and (2.37)). The justification for these expansions is given in Section 3.

Let τ_0 be a point where the function $\varkappa$ has a local minimum. In order to isolate a neighborhood of this point τ_0, it proves useful to introduce some *local variables* defined by

$$\zeta = \varepsilon^{-1}\nu \quad \text{and} \quad \eta = \varepsilon^{-\gamma}(\tau - \tau_0) \tag{2.13}$$

with γ a constant, $\gamma > 0$. For any $d > 0$, the change (2.13) transforms the narrow band $\{(\nu, \tau) : \nu \in [0, \varepsilon h), |\tau - \tau_0| < d\}$ into the band $\{(\zeta, \eta) : \zeta \in [0, h), \eta \in (-d\varepsilon^{-\gamma}, d\varepsilon^{-\gamma})\}$ of width $O(1)$ and length $O(\varepsilon^{-\gamma})$, and it leads us to consider a limiting problem in $[0, h) \times \mathbb{R}$ independent of the geometry.

Taking into account the Taylor expansions of $\varkappa(\tau)$ in a neighborhood of τ_0, we introduce the new variables in the Laplacian in the curvilinear coordinates, namely in

$$\Delta_{\nu,\tau} = K(\nu, \tau)^{-1}\partial_\nu(K(\nu, \tau)\partial_\nu) + K(\nu, \tau)^{-1}\partial_\tau(K(\nu, \tau)^{-1}\partial_\tau), \tag{2.14}$$

being $K(\nu, \tau) = 1 + \nu\varkappa(\tau)$, and gather the different powers of ε. Since $\varkappa'(\tau_0) = 0$, we have

$$\Delta_{\zeta,\eta} = \varepsilon^{-2}\,\partial_\zeta^2 + \varepsilon^{-1}\varkappa(\tau_0)\,\partial_\zeta + \varepsilon^{2\gamma-1}2^{-1}\varkappa''(\tau_0)\,\eta^2\,\partial_\zeta^2 + \varepsilon^{-2\gamma}\partial_\eta^2 + \cdots \tag{2.15}$$

where here and in the sequel the dots denote further asymptotic terms of different powers of ε which in general are not used to derive our results.

Following the idea in [19, 23, 16] for localized eigenfunctions, among the possible choices of γ we consider one that leads us to an eigenvalue problem in $L^2(\mathbb{R})$ for the Hermite differential operator in the "tangencial" variable η (cf. (2.37)). Under the assumption $\varkappa''(\tau_0) > 0$ (cf. Remark 2.1 for other cases), equalizing the exponents of ε in the third and fourth terms on the right hand side of (2.15) yields $\gamma = 1/4$ in (2.13), namely,

$$\zeta = \varepsilon^{-1}\nu \quad \text{and} \quad \eta = \varepsilon^{-1/4}(\tau - \tau_0). \tag{2.16}$$

Now, we consider an asymptotic expansion for the eigenvalues λ^ε and for the corresponding eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ in Ω and ω_ε of the form:

$$\lambda^\varepsilon = \varepsilon^{m-2}(\lambda_0 + \varepsilon^{1/2}\lambda_{1/2} + \varepsilon\lambda_1 + \varepsilon^{3/2}\lambda_{3/2} + \cdots), \tag{2.17}$$

$$U^\varepsilon(x) = \mathtt{V}^\varepsilon(x) + \varepsilon^{1/2}\mathtt{V}^\varepsilon_{1/2}(x) + \varepsilon\mathtt{V}^\varepsilon_1(x) + \varepsilon^{3/2}\mathtt{V}^\varepsilon_{3/2}(x) + \cdots \quad x \in \Omega, \tag{2.18}$$

$$u^\varepsilon(\zeta,\eta) = v_0(\zeta,\eta) + \varepsilon^{1/2}v_{1/2}(\zeta,\eta) + \varepsilon v_1(\zeta,\eta) + \varepsilon^{3/2}v_{3/2}(\zeta,\eta) + \cdots \quad \zeta \in [0,h), \eta \in \mathbb{R}, \tag{2.19}$$

respectively. Besides, we suppose that $\mathtt{V}^\varepsilon$ in (2.18) or v_0 in (2.19) are different from zero. We note that we have assumed that the outer expansion (2.18) can be a non-regular expansion since we allow the terms arising in the expansion to be dependent on ε and x simultaneously and (2.19) is the expansion in the fast variables.

After considering equations (2.15), we replace expansions (2.17)–(2.19) in problem (1.1) and collect coefficients of the same powers of ε. In a first step, we have that the leading terms in (2.17) and (2.19) satisfy the following problem with the parameter $\eta \in \mathbb{R}$:

$$-a\,\partial^2_\zeta v_0 = \lambda_0 v_0 \quad \zeta \in (0,h), \tag{2.20}$$

$$\partial_\zeta v_0(0,\eta) = 0, \quad v_0(h,\eta) = 0. \tag{2.21}$$

From (2.20)–(2.21), we deduce that λ_0 is an eigenvalue of

$$\begin{cases} -ay_0'' = \lambda_0 y_0 \quad \zeta \in (0,h), \\ y_0'(0) = y_0(h) = 0 \end{cases} \tag{2.22}$$

and

$$v_0(\zeta,\eta) = y_0(\zeta)v(\eta) \quad \zeta \in (0,h),\ \eta \in \mathbb{R}, \tag{2.23}$$

where y_0 is an eigenfunction of (2.22) corresponding to λ_0 and v is an arbitrary function of η to be determined. It is clear that the eigenvalues of (2.22) are given by

$$\lambda_{0,k} = \frac{a(2k-1)^2\pi^2}{4h^2} \qquad \text{for } k = 1, 2, \ldots \tag{2.24}$$

and the corresponding eigenfunctions can be chosen to be

$$y_{0,k}(\zeta) = \sin\left(\frac{(2k-1)\pi}{2h}(\zeta - h)\right) \qquad \text{for } k = 1, 2, \ldots. \tag{2.25}$$

In a second step, we obtain the following problem with the parameter $\eta \in \mathbb{R}$:

$$-a\,\partial_\zeta^2 v_{1/2} = \lambda_0 v_{1/2} + \lambda_{1/2} v_0 \quad \zeta \in (0, h), \tag{2.26}$$

$$\partial_\zeta v_{1/2}(0, \eta) = 0, \quad v_{1/2}(h, \eta) = 0. \tag{2.27}$$

Since $v_0(\zeta, \tau) = y_0(\zeta)v(\eta)$ verifies (2.20)–(2.21), the compatibility condition for the non–homogeneous problem (2.26)–(2.27) in the ζ–variable reads

$$0 = \lambda_{1/2} v(\eta) \int_0^h y_0(\zeta)^2 \, d\zeta, \quad \eta \in \mathbb{R},$$

and so $\lambda_{1/2} = 0$. Now, since the eigenvalues of (2.22) are simple, we choose the solution $v_{1/2} \equiv 0$ to be the unique solution which satisfies $\int_0^h v_{1/2}(\zeta, \cdot) y_0(\zeta) d\zeta = 0$.

In the third step, we obtain the following problem with the parameter $\eta \in \mathbb{R}$:

$$-a\partial_\zeta^2 v_1 - a\varkappa(\tau_0)\,\partial_\zeta v_0 = \lambda_0 v_1 + \lambda_{1/2} v_{1/2} + \lambda_1 v_0, \quad \zeta \in (0, h), \tag{2.28}$$

$$\partial_\zeta v_1(0, \eta) = 0, \quad v_1(h, \eta) = 0. \tag{2.29}$$

Since $\lambda_{1/2} = 0$, the compatibility condition in (2.28)–(2.29) reads

$$-a\varkappa(\tau_0)v(\eta) \int_0^h y_0'(\zeta) y_0(\zeta) \, d\zeta = \lambda_1 v(\eta) \int_0^h y_0(\zeta)^2 \, d\zeta, \quad \eta \in \mathbb{R}.$$

The explicit form (2.25) of the solutions of (2.22) gives

$$\int_0^h y_0' y_0 \, d\zeta = \frac{1}{2}(y_0(h)^2 - y_0(0)^2) = -\frac{1}{2} \quad \text{and} \quad \int_0^h y_0^2 \, d\zeta = \frac{h}{2}, \tag{2.30}$$

and we have that

$$\lambda_1 = a h^{-1} \varkappa(\tau_0). \tag{2.31}$$

In addition, any function v_1 satisfying (2.28)–(2.29) can be written in the form

$$v_1(\zeta, \eta) = \varkappa(\tau_0) v(\eta) y_1(\zeta), \quad \zeta \in (0, h),\, \eta \in \mathbb{R},$$

where y_1 is a solution of

$$\begin{cases} -ay_1'' - \lambda_0 y_1 = ay_0' + \dfrac{a}{h} y_0 \quad \zeta \in (0,h), \\ y_1'(0) = y_1(h) = 0. \end{cases} \tag{2.32}$$

In fact, for each fixed eigenpair (λ_0, y_0) of (2.22), we can choose y_1 above to be the unique solution which satisfies $\int_0^h y_1(\zeta) y_0(\zeta)\, d\zeta = 0$, and then, for $(\lambda_0, y_0) = (\lambda_{0,k}, y_{0,k})$ verifying (2.24) and (2.25), we have

$$v_1(\zeta,\eta) = v_{1,k}(\zeta,\eta) = \varkappa(\tau_0) v(\eta) y_{1,k}(\zeta), \quad \zeta \in (0,h),\, \eta \in \mathbb{R}, \tag{2.33}$$

where

$$\begin{aligned} y_{1,k}(\zeta) = &- \left(\frac{\zeta}{2} - \frac{(8k + (2k-1)^2\pi^2)h}{8k(2k-1)\pi^2} \right) \sin\left(\frac{(2k-1)\pi}{2h} (\zeta - h) \right) \\ &+ \frac{1}{(2k-1)\pi} (\zeta - h) \cos\left(\frac{(2k-1)\pi}{2h} (\zeta - h) \right), \quad \text{for } k = 1, 2, \ldots. \end{aligned} \tag{2.34}$$

Following the process, in the next step, we derive the problem for $v_{3/2}$ with the parameter $\eta \in \mathbb{R}$:

$$\begin{aligned} -a\, \partial_\zeta^2 v_{3/2} - a\varkappa(\tau_0) \partial_\zeta v_{1/2} - \frac{a\varkappa''(\tau_0)}{2}\, \eta^2\, \partial_\zeta v_0 - a\partial_\eta^2 v_0 & \\ = \lambda_0 v_{3/2} + \lambda_{1/2} v_1 + \lambda_1 v_{1/2} + \lambda_{3/2} v_0, \quad \zeta \in (0,h), & \qquad (2.35) \\ \partial_\zeta v_{3/2}(0,\eta) = 0, \quad v_{3/2}(h,\eta) = 0. & \qquad (2.36) \end{aligned}$$

Now, the compatibility condition for the non–homogeneous problem (2.35)–(2.36) provides:

$$\begin{aligned} &- \frac{a\varkappa''(\tau_0)}{2} \eta^2 v(\eta) \int_0^h y_0'(\zeta) y_0(\zeta)\, d\zeta - a v''(\eta) \int_0^h y_0(\zeta)^2\, d\zeta \\ &= \lambda_{3/2} v(\eta) \int_0^h y_0(\zeta)^2\, d\zeta, \quad \eta \in \mathbb{R}. \end{aligned}$$

From (2.30) we get the equation for the eigenpair $(\lambda_{3/2}, v)$:

$$\frac{a\varkappa''(\tau_0)}{2h} \eta^2 v(\eta) - a v''(\eta) = \lambda_{3/2} v(\eta), \quad \eta \in \mathbb{R}. \tag{2.37}$$

Under the assumption $\varkappa''(\tau_0) > 0$ (cf. Remark 2.1) and prescribing the condition $v \in L^2(\mathbb{R})$, (2.37) has a discrete spectrum (see, for example, Chapter IX of [7]) and we can compute the eigenvalues $\lambda_{3/2}$ as follows

$$\lambda_{3/2,p} = \left(\frac{\varkappa''(\tau_0)}{2h}\right)^{1/2} a(2p-1) \quad \text{for } p = 1, 2 \ldots,$$

while the corresponding eigenfunctions $v(\eta)$ are

$$v^p(\eta) = C_p \exp\left(-\left(\frac{\varkappa''(\tau_0)}{2h}\right)^{1/2} \frac{\eta^2}{2}\right) H_{p-1}\left(\left(\frac{\varkappa''(\tau_0)}{2h}\right)^{1/4} \eta\right) \quad \text{for } p = 1, 2 \ldots,$$

where C_p are arbitrary constants and H_{p-1} are the Hermite polynomials of degree $p-1$.

In addition, by virtue of (2.23), (2.22), (2.31), (2.33), (2.37), and the fact that $\lambda_{1/2} = 0$ and $v_{1/2} \equiv 0$, equation (2.35) becomes

$$-a\partial_\zeta^2 v_{3/2} - \lambda_0 v_{3/2} = a\frac{\varkappa''(\tau_0)}{2}\eta^2 v(\eta)\Big(y_0'(\zeta) + \frac{1}{h}y_0(\zeta)\Big), \quad \zeta \in (0,h), \eta \in \mathbb{R},$$

and $v_{3/2}(\zeta, \eta)$ can be obtained by separation of variables as

$$v_{3/2}(\zeta, \eta) = \frac{\varkappa''(\tau_0)}{2}\eta^2 v^p(\eta) y_{1,k}(\zeta), \quad \zeta \in (0,h),\ \eta \in \mathbb{R},$$

where $y_{1,k}(\zeta)$ is given by (2.34).

Hence, we have identified the first terms in the expansion (2.17) which shows a splitting of the low frequencies into a double series

$$\lambda^\varepsilon \sim \varepsilon^{m-2}\frac{a(2k-1)^2\pi^2}{4h^2} + \varepsilon^{m-1}\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{m-1/2}\left(\frac{\varkappa''(\tau_0)}{2h}\right)^{1/2} a(2p-1) \qquad k, p = 1, 2, \ldots, \tag{2.38}$$

for which the first terms in (2.19) are also determined by

$$u^\varepsilon(\zeta, \tau) \sim y_{0,k}(\zeta)v^p(\eta) + \varepsilon\varkappa(\tau_0)y_{1,k}(\zeta)v^p(\eta) + \varepsilon^{3/2}\frac{\varkappa''(\tau_0)}{2}\eta^2 y_{1,k}(\zeta)v^p(\eta), \qquad \zeta \in (0,h),\ \eta \in \mathbb{R}, \tag{2.39}$$

while the terms in the outer expansion (2.18) are yet to be computed in order that expansions (2.18) and (2.19) match up to a certain order.

To this end, we note that the *fast* variable in (2.19) and (2.39) is $\eta = (\tau-\tau_0)\varepsilon^{-1/4}$ and the function $v_0(\zeta,\eta)+\varepsilon v_1(\zeta,\eta)+\varepsilon^{3/2}v_{3/2}(\zeta,\eta)$ is somehow localized in a neighborhood of $\eta=0$, namely in $\{x\in\omega_\varepsilon : |\tau-\tau_0| < K\varepsilon^{1/4}, \nu\in(0,\varepsilon h)\}$ with K a positive constant, and it is exponentially small outside. Specifying further, $v(\eta)$ is exponentially small for τ satisfying $|\tau-\tau_0|=\varepsilon^p$ with any $p<1/4$.

Hence, in order to get an approximation of $\{U^\varepsilon,u^\varepsilon\}$ in the whole domain Ω_ε, for a fixed d, $0<d<\ell/2$, we introduce a cut–off function

$$\chi\in C^\infty(\mathbb{R}) \text{ such that } 0\le\chi\le 1,\ \chi(s)=1 \text{ as } |s|<d/2 \text{ and } \chi(s)=0 \text{ as } |s|>d. \tag{2.40}$$

Then, we set

$$u^\varepsilon(\nu,\tau)\sim\chi(\tau-\tau_0)\,v\Big(\frac{\tau-\tau_0}{\varepsilon^{1/4}}\Big)\Big[y_0\Big(\frac{\nu}{\varepsilon}\Big)+\varepsilon\Big(\varkappa(\tau_0)+\frac{\varkappa''(\tau_0)}{2}(\tau-\tau_0)^2\Big)\,y_1\Big(\frac{\nu}{\varepsilon}\Big)\Big] \tag{2.41}$$

for $(\nu,\tau)\in\omega_\varepsilon$, where (λ_0,y_0) is an eigenpair of (2.22), $(\lambda_{3/2},v)$ is an eigenpair of (2.37) and y_1 is the solution of (2.32).

Now, equations (1.1a) and (1.1c), along with (2.17), (2.18) and (2.19), provide the first term in the outer expansion (2.18) to be the solution of the non-homogeneous Dirichlet problem

$$\begin{cases} -A\Delta_x\mathtt{V}^\varepsilon=0 & \text{in } \Omega,\\ \mathtt{V}^\varepsilon(x)=\chi(\tau-\tau_0)\,v\Big(\dfrac{\tau-\tau_0}{\varepsilon^{1/4}}\Big)\Big[y_0(0)+\varepsilon\Big(\varkappa(\tau_0)+\dfrac{\varkappa''(\tau_0)}{2}(\tau-\tau_0)^2\Big)\,y_1(0)\Big] & \text{on } \Gamma. \end{cases} \tag{2.42}$$

On account of the smoothness of the non-homogeneous data on Γ, for each fixed $\varepsilon>0$, problem (2.42) has a unique solution $\mathtt{V}^\varepsilon\in H^2(\Omega)$ and we can set

$$U^\varepsilon(x)\sim\mathtt{V}^\varepsilon(x)\quad\text{for } x\in\Omega. \tag{2.43}$$

In addition, since the data is located at $supp(\chi)$ and the function v decays exponentially with the distance to τ_0, one may expect that $\mathtt{V}^\varepsilon$, as well as its derivatives up to the order k, will be $o(1)$ at a distance $O(1)$ of τ_0 (also, at a distance $O(\varepsilon^{p_k})$ for a certain $p_k<1/4$ depending on k).

Hence, formally, from (2.41) and (2.43), we have localized eigenfunctions corresponding to eigenvalues in (2.38). The support of these eigenfunctions concentrates asymptotically in $C\varepsilon^{1/4}$–neighborhoods of τ_0. In the next section, we justify approximations (2.41) and (2.43) and show the estimates above for $\mathtt{V}^\varepsilon$.

Remark 2.1. Similar results can be obtained when $\varkappa$ presents a local minimum in τ_0 but $\varkappa''(\tau_0) = 0$. If so, it is self-evident that we must introduce different variables and asymptotic expansions. As a matter of fact, if $\varkappa'(\tau_0) = \varkappa''(\tau_0) = \cdots = \varkappa^{(2n-1)}(\tau_0) = 0$ and $\varkappa^{(2n)}(\tau_0) > 0$ for certain $n > 1$, the suitable variables to show the local effects for the eigenfunctions are likely to be $\zeta = \nu/\varepsilon$ and $\eta = (\tau - \tau_0)\varepsilon^{-1/2(n+1)}$.

Remark 2.2. When the domain Ω is a disk, the curvature $\varkappa$ is constant and explicit computations for the eigenpairs of (1.1) can be done by means the Bessel functions. In this case, the corresponding eigenfunctions are significant over the whole domain Ω_ε, and no localization effects arise.

On account of the above remarks, in what follows we make the following assumption:

Assumption 1: *the curvature $\varkappa$ of Γ has a local minimum at τ_0 such that $\varkappa''(\tau_0) > 0$.*

3. Estimates of the asymptotic remainders

In this section, we justify up to a certain degree the asymptotic expansions in Section 2. We obtain estimates which establish the closeness of the eigenvalues $\lambda^\varepsilon = O(\varepsilon^{m-2})$ of (1.1) and the values $\lambda_0 + \varepsilon\lambda_1 + \varepsilon^{3/2}\lambda_{3/2}$ where λ_0 and $\lambda_{3/2}$ are eigenvalues of (2.22) and (2.37) respectively, and λ_1 is given by (2.31) (cf. Theorem 3.3). We also provide information on the structure of the eigenfunctions corresponding to λ^ε. However, this still does not imply the convergence of the kth eigenvalue of (1.1).

We first introduce some notation and results of further use. For each $\varepsilon > 0$, $\mathcal{H}^\varepsilon$ is the space $H_0^1(\Omega_\varepsilon)$ with the scalar product

$$(\{U,u\},\{G,g\})_{\mathcal{H}^\varepsilon} = \varepsilon^{t+1} A \int_\Omega \nabla_x U \cdot \nabla_x G \, dx + \varepsilon a \int_{\omega_\varepsilon} \nabla_x u \cdot \nabla_x g \, dx \qquad \forall \{U,u\},\{G,g\} \in H_0^1(\Omega_\varepsilon). \tag{3.44}$$

Let $\mathcal{A}^\varepsilon$ be a positive, compact and symmetric operator on $\mathcal{H}^\varepsilon$ defined by

$$(\mathcal{A}^\varepsilon\{U,u\},\{G,g\})_{\mathcal{H}^\varepsilon} = \varepsilon^{t+m-1} \int_\Omega UG \, dx + \frac{1}{\varepsilon} \int_{\omega_\varepsilon} ug \, dx \quad \forall \{U,u\},\{G,g\} \in H_0^1(\Omega_\varepsilon).$$

It is clear that the eigenvalues of $\mathcal{A}^\varepsilon$ are $\{\varepsilon^{m-2}/\lambda_k^\varepsilon\}_{k=1}^\infty$ where $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ are the eigenvalues of (1.1).

In order to derive estimates, we use a classical result on "almost eigenvalues and eigenvectors" from the spectral perturbation theory, namely, Lemma 3.1, and a result which describes the behavior of the solution of problem (2.42), namely, Lemma 3.2 (see [28] and Chapter 6 in [1] for the proof of Lemma 3.1, and [16] for the proof of Lemma 3.2).

Lemma 3.1. *Let $A : H \longrightarrow H$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and $\lambda,\ r > 0$ such that $\|Au - \lambda u\|_H \leq r$. Then, there exists an eigenvalue λ_i of the operator A satisfying the inequality $|\lambda - \lambda_i| \leq r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of the operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$ and such that*

$$\|u - u^*\|_H \leq \frac{2r}{r^*}.$$

Lemma 3.2. *Let $g \in C^\infty(\mathbb{R})$ be a function verifying*

$$|\nabla_s^k g(s)| \leq C_k(1 + s^2)^{-1-k/2} \quad \text{for } s \in \mathbb{R} \text{ and } k = 0, 1, 2 \ldots$$

For $\varepsilon > 0$, let $\mathtt{V}^\varepsilon$ be the solution of the problem

$$\begin{cases} -\Delta_x \mathtt{V}^\varepsilon = 0 & \text{in } \Omega \\ \mathtt{V}^\varepsilon = \chi(\tau - \tau_0) g((\tau - \tau_0)/\varepsilon^\gamma) & \text{on } \Gamma \end{cases} \tag{3.45}$$

where $\gamma > 0$, $\tau_0 \in \Gamma$, $\chi \in C^\infty(\mathbb{R})$ is a cut–off function such that $\chi(s) = 1$ as $|s| < d/2$ and $\chi(s) = 0$ as $|s| > d$ for sufficiently small $d > 0$. Then, the function $\mathtt{V}^\varepsilon$ satisfies

$$|\nabla_x^k \mathtt{V}^\varepsilon(x)| \leq c_{k,\delta}\, \varepsilon^{\gamma(1-\delta)}(\varepsilon^{2\gamma} + r^2)^{(\delta-1-k)/2} \tag{3.46}$$

for any $0 < \delta < 1$, $x \in \Omega$ and $k = 0, 1, 2 \ldots$, r being dist(x, τ_0) and $c_{k,\delta}$ a constant independent of ε.

Now, we can state the following result which provides bounds for the convergence rates for the eigenvalues and eigenfunctions of (1.3):

Theorem 3.3. *Let (λ_0, y_0) and $(\lambda_{3/2}, v)$ be eigenelements of (2.22) and (2.37), respectively, such that $\|y_0\|^2_{L^2(0,h)} = \|v\|^{-2}_{L^2(\mathbb{R})} = 1/2$. Let y_1 be the solution of (2.32) orthogonal to y_0 in $L^2(0, h)$, and let $\mathtt{V}^\varepsilon$ be the solution of (2.42), where*

$\chi \in C^\infty(\mathbb{R})$ is defined by (2.40). Let $t \geq 1$ and $m > 2$. Then, under the assumption 1, there are eigenvalues λ^ε of problem (1.3) such that

$$\left| \frac{\lambda^\varepsilon}{\varepsilon^{m-2}} - \lambda_0 - \varepsilon \frac{a\varkappa(\tau_0)}{h} - \varepsilon^{3/2}\lambda_{3/2} \right| \leq C\varepsilon^q, \qquad \text{where } q = \min\{(4t+4m-3-2\delta)/8, 13/8\}, \tag{3.47}$$

C is a constant independent of ε, and $\delta \in (0, 1/8)$. Moreover, there is a linear combination of eigenfunctions $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\} \in H_0^1(\Omega_\varepsilon)$, $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}$ corresponding to the eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1.1) which satisfy $\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m} \in [\lambda_0 - K\varepsilon^\theta, \lambda_0 + K\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < q$, $\|\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = \varepsilon^{1/8}$, such that

$$\varepsilon^{(t+1)/2}\|\widetilde{U}^\varepsilon - \beta^\varepsilon \mathtt{V}^\varepsilon\|_{H^1(\Omega)} + \varepsilon^{1/2}\|\widetilde{u}^\varepsilon - \beta^\varepsilon w^\varepsilon\|_{H^1(\omega_\varepsilon)} \leq C\varepsilon^{q-\theta+1/8}, \tag{3.48}$$

where w^ε is defined by

$$w^\varepsilon(\nu,\tau) = \chi(\tau-\tau_0)\, v\Big(\frac{\tau-\tau_0}{\varepsilon^{1/4}}\Big)\Big[y_0\Big(\frac{\nu}{\varepsilon}\Big) + \varepsilon\Big(\varkappa(\tau_0) + \frac{\varkappa''(\tau_0)}{2}(\tau-\tau_0)^2\Big)y_1\Big(\frac{\nu}{\varepsilon}\Big)\Big] \tag{3.49}$$

if $(\nu,\tau) \in \omega_\varepsilon$, $\beta^\varepsilon = \varepsilon^{1/8}\|\{\mathtt{V}^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}$, and $\beta^\varepsilon \to \lambda_0^{-1/2}$ as $\varepsilon \to 0$.

Proof. For sufficiently small ε, the function $\{W^\varepsilon, w^\varepsilon\}$ is defined by

$$W^\varepsilon(x) = \mathtt{V}^\varepsilon(x) \quad \text{ if } x \in \Omega, \tag{3.50}$$

and (3.49). It is clear that $\{W^\varepsilon, w^\varepsilon\} \in H_0^1(\Omega_\varepsilon)$. In addition, considering Lemma 3.2 for $\delta \in (0,1)$, we take integrals over Ω in (3.46) with $k = 0, 1, 2$, and use polar coordinates around τ_0; then, we obtain the estimate

$$\varepsilon^{\delta/4}\|\mathtt{V}^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/4}\|\nabla_x \mathtt{V}^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla_x^2 \mathtt{V}^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{1/4}. \tag{3.51}$$

In order to apply Lemma 3.1, we prove the estimate

$$\left| \Big(\mathcal{A}^\varepsilon\{\widetilde{W}^\varepsilon, \widetilde{w}^\varepsilon\} - \frac{1}{\lambda_0 + \varepsilon\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\lambda_{3/2}}\{\widetilde{W}^\varepsilon, \widetilde{w}^\varepsilon\}, \{G, g\}\Big)_{\mathcal{H}^\varepsilon} \right| \leq C\varepsilon^q \|\{G, g\}\|_{\mathcal{H}^\varepsilon} \tag{3.52}$$

for all $\{G, g\} \in \mathcal{H}^\varepsilon$, where $\{\widetilde{W}^\varepsilon, \widetilde{w}^\varepsilon\} = \{W^\varepsilon, w^\varepsilon\}\|\{W^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}$, and q is defined in the statement of the theorem.

Taking into account the definition of the operator $\mathcal{A}^\varepsilon$, the scalar product $(\cdot,\cdot)_{\mathcal{H}^\varepsilon}$ and the function $\{W^\varepsilon, w^\varepsilon\}$ and introducing the change of variables (2.16) in the integrals in ω_ε, we can write

$$\Big(\lambda_0+\varepsilon\frac{a\varkappa(\tau_0)}{h}+\varepsilon^{3/2}\lambda_{3/2}\Big)\Big(\mathcal{A}^\varepsilon\{W^\varepsilon,w^\varepsilon\}-\frac{1}{\lambda_0+\varepsilon\frac{a\varkappa(\tau_0)}{h}+\varepsilon^{3/2}\lambda_{3/2}}\{W^\varepsilon,w^\varepsilon\},\{G,g\}\Big)_{\mathcal{H}^\varepsilon}$$
$$= J_1+J_2-J_3,$$

where

$$J_1=\Big(\lambda_0+\varepsilon\frac{a\varkappa(\tau_0)}{h}+\varepsilon^{3/2}\lambda_{3/2}\Big)\varepsilon^{t+m-1}\int_\Omega \mathtt{V}^\varepsilon G\,dx-\varepsilon^{t+1}A\int_\Omega \nabla_x\mathtt{V}^\varepsilon\cdot\nabla_x G\,dx,$$

$$\begin{aligned}J_2=&\varepsilon^{1/4}\Big(\lambda_0+\varepsilon\frac{a\varkappa(\tau_0)}{h}+\varepsilon^{3/2}\lambda_{3/2}\Big)\\ &*\int_{\mathrm{R}}\chi_\varepsilon\Big(y_0+\varepsilon\varkappa(\tau_0)y_1+\varepsilon^{3/2}\frac{\varkappa''(\tau_0)}{2}\eta^2y_1\Big)v\mathrm{g}\widetilde{\mathrm{K}}_\varepsilon\,d\zeta d\eta\end{aligned}$$

and

$$\begin{aligned}J_3=&a\varepsilon^{1/4}\int_{\mathrm{R}}\partial_\zeta\Big(\chi_\varepsilon\Big(y_0+\varepsilon\varkappa(\tau_0)y_1+\varepsilon^{3/2}\frac{\varkappa''(\tau_0)}{2}\eta^2y_1\Big)v\Big)\partial_\zeta\mathrm{g}\widetilde{\mathrm{K}}_\varepsilon\,d\zeta d\eta\\ &+a\varepsilon^{7/4}\int_{\mathrm{R}}\partial_\eta\Big(\chi_\varepsilon\Big(y_0+\varepsilon\varkappa(\tau_0)y_1+\varepsilon^{3/2}\frac{\varkappa''(\tau_0)}{2}\eta^2y_1\Big)v\Big)\partial_\eta\mathrm{g}\widetilde{\mathrm{K}}_\varepsilon^{-1}\,d\zeta d\eta\,;\end{aligned}$$

with g denoting the function $g\in H^1(\omega_\varepsilon)$ in the local variables (ζ,η), $\mathrm{R}=(0,h)\times\mathbb{R}$, $\widetilde{K}_\varepsilon(\zeta,\eta)\equiv 1+\varepsilon\zeta\varkappa(\tau_0+\varepsilon^{1/4}\eta)$ and $\chi_\varepsilon(\eta)=\chi(\varepsilon^{1/4}\eta)$.

To estimate J_1, we take into account the definition of $\mathtt{V}^\varepsilon$, the fact that $G=g$ on Γ and $g=0$ on Γ_ε, the trace inequalities

$$\|G\|^2_{L^2(\Gamma)}=\int_\Gamma\Big(\int_0^{\varepsilon h}\partial_\nu g\,d\nu\Big)^2d\tau\le C\varepsilon\|\nabla_x g\|^2_{L^2(\omega_\varepsilon)}\le C\|\{G,g\}\|^2_{\mathcal{H}^\varepsilon}\qquad \forall\{G,g\}\in H^1_0(\Omega_\varepsilon), \tag{3.53}$$

and

$$\|\partial_\nu U\|_{L^2(\Gamma)}\le C\|U\|_{H^2(\Omega)}\quad \forall U\in H^2(\Omega),$$

estimates (3.51) and equation (3.44). Then,

$$\begin{aligned}|J_1|\le&C\varepsilon^{t+m-1}\|\mathtt{V}^\varepsilon\|_{L^2(\Omega)}\|G\|_{L^2(\Omega)}+\varepsilon^{t+1}C_1\|\partial_\nu\mathtt{V}^\varepsilon\|_{L^2(\Gamma)}\|G\|_{L^2(\Gamma)}\\ \le&C(\varepsilon^{\frac{2t+2m-1-\delta}{4}}+\varepsilon^{t+1-1/4})\|\{G,g\}\|_{\mathcal{H}^\varepsilon}.\end{aligned}$$

To estimate $|J_2-J_3|$, since (λ_0, y_0) is an eigenpair of (2.22), y_1 is a solution of (2.32) and $(\lambda_{3/2}, v)$ is an eigenpair of (2.37), we write

$$
\begin{aligned}
J_2 - J_3 = \varepsilon^{1/4} \Big\{ & \lambda_0 \int_{\mathrm{R}} \chi_\varepsilon v y_0 \mathrm{g} \left(\widetilde{\mathrm{K}}_\varepsilon - 1 - \varepsilon\zeta\varkappa(\tau_0) - \varepsilon^{3/2}\zeta \frac{\varkappa''(\tau_0)}{2}\eta^2 \right) d\zeta d\eta \\
& - a \int_{\mathrm{R}} \partial_\zeta(\chi_\varepsilon v y_0) \partial_\zeta \mathrm{g} \left(\widetilde{\mathrm{K}}_\varepsilon - 1 - \varepsilon\zeta\varkappa(\tau_0) - \varepsilon^{3/2}\zeta \frac{\varkappa''(\tau_0)}{2}\eta^2 \right) d\zeta d\eta \Big\} \\
& + \varepsilon^{5/4} \Big\{ \lambda_0 \int_{\mathrm{R}} \chi_\varepsilon \Big(\varkappa(\tau_0) + \varepsilon^{1/2} \frac{\varkappa''(\tau_0)}{2}\eta^2 \Big) v y_1 \mathrm{g} (\widetilde{\mathrm{K}}_\varepsilon - 1)\, d\zeta d\eta \\
& + \Big(\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{1/2}\lambda_{3/2} \Big) \int_{\mathrm{R}} \chi_\varepsilon v y_0 \mathrm{g} (\widetilde{\mathrm{K}}_\varepsilon - 1)\, d\zeta d\eta \\
& - a \int_{\mathrm{R}} \partial_\zeta \Big(\chi_\varepsilon \Big(\varkappa(\tau_0) + \varepsilon^{1/2} \frac{\varkappa''(\tau_0)}{2}\eta^2 \Big) v y_1 \Big) \partial_\zeta \mathrm{g} (\widetilde{\mathrm{K}}_\varepsilon - 1)\, d\zeta d\eta \Big\} \\
& + \varepsilon^{7/4} \Big\{ -a \int_{\mathrm{R}} \partial_\eta(\chi_\varepsilon v y_0) \partial_\eta \mathrm{g} (\widetilde{\mathrm{K}}_\varepsilon^{-1} - 1)\, d\zeta d\eta \\
& + a \int_{\mathrm{R}} (\varepsilon^{1/2}\chi_\varepsilon'' v - 2\varepsilon^{1/4}\chi_\varepsilon' v') y_0 \mathrm{g}\, d\zeta d\eta \Big\} \\
& + \varepsilon^{9/4} \Big(\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{1/2}\lambda_{3/2} \Big) \int_{\mathrm{R}} \chi_\varepsilon \Big(\varkappa(\tau_0) + \varepsilon^{1/2} \frac{\varkappa''(\tau_0)}{2}\eta^2 \Big) v y_1 \mathrm{g} \widetilde{\mathrm{K}}_\varepsilon\, d\zeta d\eta \\
& - \varepsilon^{11/4} a \int_{\mathrm{R}} \partial_\eta \Big(\chi_\varepsilon \Big(\varkappa(\tau_0) + \varepsilon^{1/2} \frac{\varkappa''(\tau_0)}{2}\eta^2 \Big) v y_1 \Big) \partial_\eta \mathrm{g} \widetilde{\mathrm{K}}_\varepsilon^{-1}\, d\zeta d\eta \,.
\end{aligned}
$$

Now, for fixed ζ and ε, we consider the Taylor series at the point τ_0 of the functions $\widetilde{\mathrm{K}}_\varepsilon(\zeta,\tau) = 1 + \varepsilon\zeta\varkappa(\tau)$, and $\widetilde{\mathrm{K}}_\varepsilon^{-1}(\zeta,\tau) = (1+\varepsilon\zeta\varkappa(\tau))^{-1}$ for $\tau = \tau_0 + \varepsilon^{1/4}\eta$. Then, taking into account the smoothness of $\varkappa$ in $\mathbb{S}_\ell$, and that $\varkappa'(\tau_0) = 0$ and $\|\eta^k v\|_{L^2(\mathbb{R})}$ with $k = 2, 4, 6$ is bounded, we obtain

$$
|J_2 - J_3| \leq C_1 \varepsilon^2 (\|\mathrm{g}\|_{L^2(\mathrm{R})} + \|\partial_\zeta \mathrm{g}\|_{L^2(\mathrm{R})} + \varepsilon^{3/4} \|\partial_\eta \mathrm{g}\|_{L^2(\mathrm{R})})
$$

Moreover, introducing (2.16) in ω_ε and taking into account (1.6) yields

$$
\|\{G,g\}\|^2_{\mathcal{H}^\varepsilon} = \varepsilon^{t+1} A \int_\Omega |\nabla_x G|^2\, dx + \varepsilon^{1/4} a \int_{\mathrm{R}_\varepsilon} |\partial_\zeta \mathrm{g}|^2 \widetilde{\mathrm{K}}_\varepsilon d\zeta d\eta + \varepsilon^{7/4} a \int_{\mathrm{R}_\varepsilon} |\partial_\eta \mathrm{g}|^2 \widetilde{\mathrm{K}}_\varepsilon^{-1} d\zeta d\eta, \tag{3.54}
$$

and

$$
\|\{G,g\}\|^2_{\mathcal{H}^\varepsilon} \geq \varepsilon a \int_{\omega_\varepsilon} |\nabla_x g|^2\, dx \geq \frac{C}{\varepsilon} \int_{\omega_\varepsilon} |g|^2\, dx = C\varepsilon^{1/4} \int_{\mathrm{R}_\varepsilon} |\mathrm{g}|^2 \widetilde{\mathrm{K}}_\varepsilon\, d\zeta d\eta;
$$

here R_ε denotes the domain transformed of ω_ε with the change of variable (2.16). Thus,

$$|J_2 - J_3| \leq C_2 \varepsilon^{15/8} \|\{G, g\}\|_{\mathcal{H}^\varepsilon}.$$

As a result of the above estimates for J_1, $J_2 - J_3$, and the fact that $t \geq 1$, we have

$$\left| \left(\mathcal{A}^\varepsilon \{W^\varepsilon, w^\varepsilon\} - \frac{1}{\lambda_0 + \varepsilon \frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\lambda_{3/2}} \{W^\varepsilon, w^\varepsilon\}, \{G, g\} \right)_{\mathcal{H}^\varepsilon} \right|$$
$$\leq C(\varepsilon^{\frac{2t+2m-1-\delta}{4}} + \varepsilon^{7/4}) \|\{G, g\}\|_{\mathcal{H}^\varepsilon} \quad \{G, g\} \in \mathcal{H}^\varepsilon,$$

where $\delta \in (0, 1/4)$. As regards the normalization of $\{\widetilde{W^\varepsilon}, \widetilde{w^\varepsilon}\}$ in $\mathcal{H}^\varepsilon$, we show

$$\varepsilon^{-1/4} \|\{W^\varepsilon, w^\varepsilon\}\|^2_{\mathcal{H}^\varepsilon} \xrightarrow{\varepsilon \to 0} a \int_{\mathrm{R}} |\partial_\zeta (y_0 v)|^2 \, d\zeta d\eta = \lambda_0; \tag{3.55}$$

which is obtained taking limits in (3.54), on account of (3.51), the normalization in the statement of the theorem for v and y_0 and the variational formulation of problem (2.22). Consequently, (3.52) holds due to the definition of $\{\widetilde{W^\varepsilon}, \widetilde{w^\varepsilon}\}$ and (3.55).

We apply Lemma 3.1 for $H = \mathcal{H}^\varepsilon$, $A = \mathcal{A}^\varepsilon$, $\lambda = (\lambda_0 + \varepsilon \frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\lambda_{3/2})^{-1}$ and $u = \{\widetilde{W^\varepsilon}, \widetilde{w^\varepsilon}\}$ and $r = C\varepsilon^q$ which provides, for sufficiently small ε, at least one eigenvalue $\lambda^\varepsilon_{k(\varepsilon)}$ of (1.1) verifying $|(\lambda^\varepsilon_{k(\varepsilon)} \varepsilon^{2-m})^{-1} - (\lambda_0 + \varepsilon \frac{a\varkappa(\tau_0)}{2} + \varepsilon^{3/2}\lambda_{3/2})^{-1}| \leq C\varepsilon^q$, and consequently, we deduce (3.47). Moreover, if we take, for instance, $r^* = \varepsilon^\theta$ with $0 < \theta < q$, Lemma 3.1 also provides a function $\{\widehat{U^\varepsilon}, \widehat{u^\varepsilon}\} \in \mathcal{H}^\varepsilon$, with $\|\{\widehat{U^\varepsilon}, \widehat{u^\varepsilon}\}\|_{\mathcal{H}^\varepsilon} = 1$, $\{\widehat{U^\varepsilon}, \widehat{u^\varepsilon}\}$ belonging to the eigenspace associated with all the eigenvalues $(\lambda^\varepsilon_{k(\varepsilon)} \varepsilon^{2-m})^{-1}$ of operator $\mathcal{A}^\varepsilon$ contained in the closed interval

$$\left[\left(\lambda_0 + \varepsilon \frac{a\varkappa(\tau_0)}{2} + \varepsilon^{3/2}\lambda_{3/2} \right)^{-1} - \varepsilon^\theta, \left(\lambda_0 + \varepsilon \frac{a\varkappa(\tau_0)}{2} + \varepsilon^{3/2}\lambda_{3/2} \right)^{-1} + \varepsilon^\theta \right],$$

such that

$$\|\{\widehat{U^\varepsilon}, \widehat{u^\varepsilon}\} - \alpha^\varepsilon \{W^\varepsilon, w^\varepsilon\}\|_{\mathcal{H}^\varepsilon} \leq C\varepsilon^{q-\theta}$$

is satisfied where $\alpha^\varepsilon = \|\{W^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}$. Now, we set $\{\widetilde{U^\varepsilon}, \widetilde{u^\varepsilon}\} = \varepsilon^{1/8}\{\widehat{U^\varepsilon}, \widehat{u^\varepsilon}\}$ and $\beta^\varepsilon = \varepsilon^{1/8}\alpha^\varepsilon$, namely,

$$\beta^\varepsilon = \varepsilon^{1/8} \|\{W^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon},$$

which converge towards $\lambda_0^{-1/2}$ as $\varepsilon \to 0$ (see (3.55)).

Then, from (3.44), (3.49), (3.50), and (3.55) it follows

$$\varepsilon^{(t+1)/2}\|\nabla_x(\widetilde{U}^\varepsilon - \beta^\varepsilon \mathsf{V}^\varepsilon)\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla_x(\widetilde{u}^\varepsilon - \beta^\varepsilon w^\varepsilon)\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{q-\theta+1/8},$$

and, since $(\widetilde{U}^\varepsilon - \beta^\varepsilon \mathsf{V}^\varepsilon)|_\Gamma = (\widetilde{u}^\varepsilon - \beta^\varepsilon w^\varepsilon)_\Gamma$, using again Friedrichs' inequality and the trace inequality (3.53) with $g = \widetilde{u}^\varepsilon - \beta^\varepsilon w^\varepsilon$, yields (3.48), and the theorem is proved. □

Remark 3.4. Let us analyze (3.47). For $t \geq 1$ and $m > 2$ such that $t + m > 15/4$, we can choose an appropriate $\delta \in (0, 1/8)$ to get $q > 3/2$ in (3.47) and the asymptotic expansion (2.17) is justified up to order $\varepsilon^{3/2}$. Nevertheless, estimate (3.47) still does not imply the approach of the kth eigenvalue of (1.3) through the kth eigenvalue of (2.37) where λ_0 is the dominant eigenvalue of (2.22).

In the case where $t \geq 1$ and $m > 2$, but $3 < t + m \leq 15/4$, (3.47) also provides a justification for the first two terms arising in (2.17) while it is necessary to construct explicitly the further terms in (2.18) to improve the estimate of $|J_1|$ and, consequently, the estimate in (3.47).

4. The convergence theorems

The aim of this section is to prove the convergence of the low frequencies, that is, the convergence, as $\varepsilon \to 0$, of the rescaled eigenvalues of (1.3), in the way stated by Theorem 4.1. Here, and in the sequel, we make the following assumption as well as assumption 1:

Assumption 2: *there is an only point $\tau_0 \in \mathbb{S}_\ell$ where the curvature $\varkappa$ of Γ has the global minimum.*

Moreover, for technical reasons, in certain proofs we use the restriction $t + m > 15/4$ (cf. Remark 3.4).

Theorem 4.1. *Let $t \geq 1$ and $m > 2$ such that $t+m > 15/4$, and let $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ be eigenvalues of* (1.3)*. Then, under the assumptions* 1 *and* 2*, for each fixed $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon \to 0} \left(\frac{\lambda_k^\varepsilon}{\varepsilon^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon \frac{a\varkappa(\tau_0)}{h} \right) \frac{1}{\varepsilon^{3/2}} = a \left(\frac{\varkappa''(\tau_0)}{2h} \right)^{1/2} (2k-1). \tag{4.56}$$

We divide the proof of Theorem 4.1 into several steps. First, we prove that the rescaled eigenvalues of (1.3), $\lambda_k^\varepsilon/\varepsilon^{m-2}$, have a common limit as is stated in Theorem 4.3. In a second step, see (4.2), we reformulate the original problem (1.3) in terms of a new spectral parameter and eigenfunctions, problem (4.64), and we show that its eigenvalues $\{\beta_k^\varepsilon\}_{k=1}^\infty$ converge towards the eigenvalues $\{\lambda_{3/2,k}\}_{k=1}^\infty$ of (2.37) with conservation of the multiplicity (see (4.3) and (4.4)). Finally, we state in (4.4) that problem (4.64) is equivalent to the original one (1.3), and convergence (4.56) holds.

4.1. The common limit

In this section we characterize the limit of $\lambda_k^\varepsilon/\varepsilon^{m-2}$ for $k \in \mathbb{N}$ fixed and $m > 2$. To do so, we observe that, as a consequence of Proposition 1.1, the sequence $\lambda_k^\varepsilon/\varepsilon^{m-2}$ is bounded and there are converging subsequences, still denoted by ε, $\lambda_k^\varepsilon/\varepsilon^{m-2} \to \lambda_k^*$ as $\varepsilon \to 0$, for a certain λ_k^*. Moreover, by Proposition 4.2 below, the first eigenvalue of (2.22), namely $\frac{a\pi^2}{4h^2}$, is a lower bound of λ_k^*. Finally, on account of Theorem 3.3, we identify this limit as stated in Theorem 4.3.

Proposition 4.2. *Let $t \geq 1$ and $m > 2$. For a small ε, there exists $C > 0$ such that any eigenvalue λ^ε of* (1.3) *satisfies*

$$\frac{\lambda^\varepsilon}{\varepsilon^{m-2}} \geq \frac{a\pi^2}{4h^2}(1 - C\varepsilon). \tag{4.57}$$

Proof. Let $\{U^\varepsilon, u^\varepsilon\}$ be an eigenfunction of (1.3) corresponding to λ^ε satisfying the normalization condition $\|\{U^\varepsilon, u^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = 1$ where $\|\cdot\|_{\mathcal{H}^\varepsilon}$ is defined by (3.44).

On account of the Friedrichs' inequality, (1.1c), (1.1e), and the trace inequality (3.53), we have

$$\|U^\varepsilon\|^2_{L^2(\Omega)} \leq C\Big(\|\nabla U^\varepsilon\|^2_{L^2(\Omega)} + \|U^\varepsilon\|^2_{L^2(\Gamma)}\Big) \leq C\Big(\|\nabla U^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\omega_\varepsilon)}\Big).$$

Moreover, using the continuity of $\varkappa$, the Poincaré Friedrichs' inequality

$$\int_0^\delta |\varphi'|\, dt \geq \frac{\pi^2}{4\delta^2}\int_0^\delta |\varphi|^2\, dt \qquad \forall \varphi \in H^1(0,\delta), \varphi(\delta) = 0$$

for $\delta = \varepsilon h$, and (1.1e), we obtain

$$\|\nabla u^\varepsilon\|^2_{L^2(\omega_\varepsilon)} \geq (1 - \varepsilon C_1)\int_\Gamma \int_0^{\varepsilon h} |\partial_\nu u^\varepsilon|^2\, d\nu d\tau \geq \frac{1 - \varepsilon C_1}{1 + \varepsilon C_2}\frac{\pi^2}{4h^2\varepsilon^2}\|u^\varepsilon\|^2_{L^2(\omega_\varepsilon)},$$

for certain constants $C_1, C_2 > 0$. Thus, gathering the above estimates and the normalization condition for the eigenfunctions yields

$$\begin{aligned}\varepsilon^{2-m}\lambda^\varepsilon &= (\varepsilon^{t+m-1}\|U^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-1}\|u^\varepsilon\|^2_{L^2(\omega_\varepsilon)})^{-1} \\ &\geq \left(C_3(\varepsilon^{m-2} + \varepsilon^{m+t-1}) + \frac{1+\varepsilon C_2}{1-\varepsilon C_1}\frac{4h^2}{a\pi^2}\right)^{-1}.\end{aligned} \tag{4.58}$$

Finally, since $t \geq 1$ and $m > 2$, by a proper choice of the constant $C > 0$ and for a small ε, (4.57) holds from (4.58), which concludes the proof. □

Theorem 4.3. *Let $t \geq 1$ and $m > 2$ such that $t+m > 15/4$, and let $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ be eigenvalues of* (1.3)*. Then, under the assumption* 1*, for each fixed $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon\to 0}\frac{\lambda_k^\varepsilon}{\varepsilon^{m-2}} = \frac{a\pi^2}{4h^2}. \tag{4.59}$$

Proof. First, let $k = 1$. From Propositions 1.1 and 4.2, we can extract a subsequence $\varepsilon_n \to 0$ such that $\varepsilon_n^{2-m}\lambda_1^{\varepsilon_n} \to \lambda_1^*$ for some $\lambda_1^* \geq \frac{a\pi^2}{4h^2}$. Besides, due to Theorem 3.3 with $\lambda_0 = \frac{a\pi^2}{4h^2}$ and $\lambda_{3/2} = a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}$, there is at least one eigenvalue of (1.3) $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ satisfying (3.47). Now, taking limits as $\varepsilon_n \to 0$ in $\varepsilon_n^{2-m}\lambda_1^{\varepsilon_n} \leq \varepsilon_n^{2-m}\lambda_{k(\varepsilon_n)}^{\varepsilon_n}$ yields $\lambda_1^* \leq \frac{a\pi^2}{4h^2}$, and convergence (4.59) holds for $k = 1$.

Without loss of generality, we prove the convergence for $k = 2$, and the result for any k holds by induction. By Proposition 1.1 we can extract a subsequence $\varepsilon_n \to 0$ such that $\varepsilon_n^{2-m}\lambda_2^{\varepsilon_n} \to \lambda_2^*$ for some λ_2^*. Also, on account of Theorem 3.3 with $\lambda_0 = \frac{a\pi^2}{4h^2}$ and $\lambda_{3/2} = 3a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}$, for sufficiently small ε, there is at least one eigenvalue of (1.3) $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ satisfying (3.47). Note that, for ε small enough and $t + m > 15/4$, $k(\varepsilon) \geq 2$. Thus, taking limits as $\varepsilon_n \to 0$ in $\varepsilon_n^{2-m}\lambda_1^{\varepsilon_n} \leq \varepsilon_n^{2-m}\lambda_2^{\varepsilon_n} \leq \varepsilon_n^{2-m}\lambda^{\varepsilon_n}_{k(\varepsilon_n)}$ gives $\lambda_2^* = \frac{a\pi^2}{4h^2}$, and convergence (4.59) holds for $k = 2$. □

4.2. Reformulation of problem (1.3)

In this section, we reformulate the original problem (1.3) in terms of a new spectral problem (4.64). This involves a rescaling of the eigenfunctions and a shift and rescaling of the eigenvalues (cf. (4.65)). To do this, we introduce the changes

$$\{U^\varepsilon, \mathsf{u}^\varepsilon\} = \{V^\varepsilon, S\mathsf{v}^\varepsilon\} \quad \text{and} \quad \{G, \mathsf{g}\} = \{W, S\mathsf{w}\},$$

in problem (1.3) written in the local variable (1.5), namely, equation (1.10), where

$$S(\zeta) = \sin\left(\frac{\pi}{2h}(\zeta - h)\right) \tag{4.60}$$

is an eigenfunction corresponding to the first eigenvalue $\lambda_{0,1} = \frac{a\pi^2}{4h^2}$ of problem (2.22) and $\mathsf{v}^\varepsilon, \mathsf{w}$ are the fractional functions $\mathsf{v}^\varepsilon = \mathsf{u}^\varepsilon/S, \mathsf{w} = \mathsf{g}/S$, respectively. By Proposition 4.4 below, $\mathsf{v}^\varepsilon, \mathsf{w} \in H^1_{S,per}(\omega_1)$.

Proposition 4.4. *Let $\{U, u\} \in H^1_0(\Omega_\varepsilon)$. Then, the fractional function $\mathsf{v} = \mathsf{u}/S \in H^1_{S,per}(\omega_1)$ where u denotes the function u written in the local coordinates (1.5), S is the function defined by (4.60), and $H^1_{S,per}(\omega_1)$ denotes the weighted space $\{\mathsf{v} : S\mathsf{v}, S\partial_\zeta \mathsf{v}, S\partial_\tau \mathsf{v} \in L^2(\omega_1),\ \mathsf{v}(\zeta, 0) = \mathsf{v}(\zeta, \ell)$ for $\zeta \in (0, h)\}$.*

Proof. Note that the only non trivial assertion is the fact that $S\partial_\zeta \mathsf{v} \in L^2(\omega_1)$. Besides, since $S \in C^\infty([0, h])$ and S only vanishes at $\zeta = h$, it suffices to show that $\mathsf{u}/S \in L^2(\omega_1)$, and more precisely, $\mathsf{u}/S \in L^2(\omega_1 \cap \{\zeta > h/2\})$. To prove this, we consider a cut-off function

$$\psi \in C^\infty(\mathbb{R}) \text{ such that } 0 \le \psi \le 1,\ \psi(\zeta) = 0 \text{ as } |\zeta| < 0 \text{ and } \psi(\zeta) = 1 \text{ as } |\zeta| > h/2, \tag{4.61}$$

and use that

$$\exists C > 0 \text{ such that } Cy^2 \le \sin^2\left(\frac{\pi}{2h} y\right) \quad \forall y \in (0, h), \tag{4.62}$$

and the Hardy inequality

$$\int_0^h \frac{1}{y^2} |\varphi(y)|^2 \, dy \le 4 \int_0^h |\varphi'(y)|^2 \, dy \qquad \forall \varphi \in H^1(0, h),\ \varphi(0) = 0.$$

Thus, making the change of variable $y = h - \zeta$, we obtain

$$\begin{aligned} \int_{h/2}^h \left|\frac{\mathsf{u}}{S}\right|^2 d\zeta &\le C \int_0^h \left|\frac{\mathsf{u}\psi}{\zeta - h}\right|^2 d\zeta = C \int_0^h \left|\frac{\mathsf{u}\psi}{y}\right|^2 dy \\ &\le 4C \int_0^h |\partial_y(\mathsf{u}\psi)|^2 \, dy \le C_1 \|\mathsf{u}\|_{H^1(0,h)}, \end{aligned}$$

which implies that $\mathsf{u}/S \in L^2(\omega_1 \cap \{\zeta > h/2\})$. Therefore, the proposition is proved. $\square$

Now, using the equality

$$\int_{\omega_1} \partial_\zeta(S\mathsf{v})\partial_\zeta(S\mathsf{w})K_\varepsilon d\zeta d\tau = \int_{\omega_1} S^2\partial_\zeta\mathsf{v}\partial_\zeta\mathsf{w}K_\varepsilon d\zeta d\tau \\ + \frac{\pi}{4h^2}\int_{\omega_1} S^2\mathsf{v}\mathsf{w}K_\varepsilon d\zeta d\tau - \varepsilon\int_{\omega_1} SS'\mathsf{v}\mathsf{w}\varkappa(\tau)d\zeta d\tau, \tag{4.63}$$

we derive the integral identity

$$a_\varepsilon(\{V^\varepsilon,\mathsf{v}^\varepsilon\},\{W,\mathsf{w}\}) = \beta^\varepsilon(\{V^\varepsilon,\mathsf{v}^\varepsilon\},\{W,\mathsf{w}\})_\varepsilon \tag{4.64}$$

where the new spectral parameter is defined by

$$\beta^\varepsilon = \Big(\frac{\lambda^\varepsilon}{\varepsilon^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon^{3/2}}, \tag{4.65}$$

and $a_\varepsilon(\cdot,\cdot)$ and $(\cdot,\cdot)_\varepsilon$ are given by

$$a_\varepsilon(\{V,\mathsf{v}\},\{W,\mathsf{w}\}) = \varepsilon^{t-3/4}A\int_\Omega \nabla_x V\cdot\nabla_x W\,dx + \frac{a}{\varepsilon^{7/4}}\int_{\omega_1} S^2\partial_\zeta\mathsf{v}\partial_\zeta\mathsf{w}K_\varepsilon d\zeta d\tau \\ - \frac{a}{\varepsilon^{3/4}}\int_{\omega_1} SS'\mathsf{v}\mathsf{w}\varkappa(\tau)d\zeta d\tau - \frac{a}{\varepsilon^{3/4}}\int_{\omega_1}\frac{\varkappa(\tau_0)}{h}S^2\mathsf{v}\mathsf{w}K_\varepsilon d\zeta d\tau \\ + a\varepsilon^{1/4}\int_{\omega_1} S^2\partial_\tau\mathsf{v}\partial_\tau\mathsf{w}K_\varepsilon^{-1}d\zeta d\tau - \varepsilon^{t-3/4+m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\int_\Omega VW\,dx \tag{4.66}$$

and

$$(\{V,\mathsf{v}\},\{W,\mathsf{w}\})_\varepsilon = \varepsilon^{t+3/4+m-2}\int_\Omega VW\,dx + \frac{1}{\varepsilon^{1/4}}\int_{\omega_1} S^2\mathsf{v}\mathsf{w}K_\varepsilon d\zeta d\tau, \tag{4.67}$$

respectively, for any $\{V,\mathsf{v}\},\{W,\mathsf{w}\}\in\mathcal{V}$ being

$$\mathcal{V} = \{\{V,\mathsf{v}\}\,:V\in H^1(\Omega),\,\mathsf{v}\in H^1_{S,per}(\omega_1),\,V|_\Gamma = -\mathsf{v}(0,\tau)\text{ for }\tau\in\mathbb{S}_\ell\}.$$

Below, we show certain estimates for functions in $H^1_{S,per}(\omega_1)$ of further use.

Proposition 4.5. *Any function $\mathsf{v}\in H^1_{S,per}(\omega_1)$ can be written in the form*

$$\mathsf{v}(\zeta,\tau) = \mathsf{v}_0(\tau) + \mathsf{v}_\perp(\zeta,\tau) \tag{4.68}$$

where $\mathsf{v}_0 \in H^1(\Gamma)$, $\mathsf{v}_\perp \in H^1_{S,per}(\omega_1)$, *and* $\int_0^h S^2 \mathsf{v}_\perp \, d\zeta = 0$. *Moreover,*

$$\int_0^h |\mathsf{v}_\perp|^2 \, d\zeta \le C \int_0^h S^2 |\partial_\zeta \mathsf{v}_\perp|^2 \, d\zeta. \tag{4.69}$$

In addition, under the assumptions 1 *and* 2, *and for a small* ε, *the following inequality is valid:*

$$\|\mathsf{v}_0\|^2_{L^2(\Gamma)} \le C\varepsilon^{1/2}\Big(\|\partial_\tau \mathsf{v}_0\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon}\int_\Gamma (\varkappa(\tau) - \varkappa(\tau_0))|\mathsf{v}_0|^2 \, d\tau\Big). \tag{4.70}$$

Proof. Note that (4.68) is true taking $\mathsf{v}_0(\tau) = \frac{2}{h}\int_0^h S^2 \mathsf{v} \, d\zeta$ and $\mathsf{v}_\perp = \mathsf{v} - \mathsf{v}_0$. Let us prove (4.69). Due to the orthogonality condition $\int_0^h S^2 \mathsf{v}_\perp \, d\zeta = 0$, it is easy to check that

$$\int_0^h S^2 |\partial_\zeta \mathsf{v}_\perp|^2 \, d\zeta \ge \nu_1 \int_0^h S^2 |\mathsf{v}_\perp|^2 \, d\zeta \ge \frac{\nu_1}{2} \int_0^{h/2} |\mathsf{v}_\perp|^2 \, d\zeta, \tag{4.71}$$

where ν_1 is the first positive eigenvalue of problem

$$\begin{cases} \partial_\zeta(S^2 \partial_\zeta \mathsf{w}) = \nu S^2 \mathsf{w} \quad \zeta \in (0,h), \\ \partial_\zeta \mathsf{w}(0) = \partial_\zeta \mathsf{w}(h) = 0. \end{cases}$$

To estimate $\int_{h/2}^h |\mathsf{v}_\perp|^2 \, d\zeta$, we use the cut-off function (4.61) and the Hardy inequality

$$\int_0^\infty |\varphi(y)|^2 \, dy \le 4 \int_0^\infty y^2 |\varphi'(y)|^2 \, dy \qquad \forall \varphi \in C_0^\infty([0,\infty)).$$

Thus, setting $y = h - \zeta$ yields

$$\begin{aligned} \int_{h/2}^h |\mathsf{v}_\perp|^2 \, d\zeta &\le \int_{-\infty}^h |\psi \mathsf{v}_\perp|^2 \, d\zeta \le 4 \int_{-\infty}^h (h-\zeta)^2 |\partial_\zeta(\psi \mathsf{v}_\perp)|^2 \, d\zeta \\ &\le C\Big(\int_0^{h/2} (h-\zeta)^2 |\mathsf{v}_\perp|^2 \, d\zeta + \int_0^h (h-\zeta)^2 |\partial_\zeta \mathsf{v}_\perp|^2 \, d\zeta\Big). \end{aligned}$$

Now, formulas (4.62) and (4.71) give

$$\begin{aligned} \int_{h/2}^h |\mathsf{v}_\perp|^2 \, d\zeta &\le C\Big(\int_0^{h/2} S^2 |\mathsf{v}_\perp|^2 \, d\zeta + \int_0^h S^2 |\partial_\zeta \mathsf{v}_\perp|^2 \, d\zeta\Big) \\ &\le C\Big(\frac{1}{\nu_1} + 1\Big) \int_0^h S^2 |\partial_\zeta \mathsf{v}_\perp|^2 \, d\zeta, \end{aligned} \tag{4.72}$$

and, combining (4.71) and (4.72), (4.69) is proved.

As regards (4.70), under the assumption 2, fixed $d > 0$

$$\exists C_1 > 0 \quad \text{such that} \quad \varkappa(\tau) - \varkappa(\tau_0) > C_1 \qquad \forall \tau \in \mathbb{S}_\ell, |\tau - \tau_0| > d/2. \tag{4.73}$$

Then, we divide the integrals on Γ into two parts $\Gamma \cap \{|\tau - \tau_0| < d/2\}$ and $\Gamma \cap \{|\tau - \tau_0| > d/2\}$. Owing to (4.73), it is clear that, for a small ε,

$$\begin{aligned}\int_{\Gamma \cap \{|\tau-\tau_0|>d/2\}} |\partial_\tau \mathsf{v}_0|^2 \, d\tau + \frac{1}{\varepsilon} \int_{\Gamma \cap \{|\tau-\tau_0|>d/2\}} (\varkappa(\tau) - \varkappa(\tau_0)) |\mathsf{v}_0|^2 \, d\tau \\ \geq \frac{C_1}{\varepsilon^{1/2}} \int_{\Gamma \cap \{|\tau-\tau_0|>d/2\}} |\mathsf{v}_0|^2 \, d\tau.\end{aligned} \tag{4.74}$$

To estimate the integral over $\Gamma \cap \{|\tau - \tau_0| < d/2\}$, we use the cut-off function (2.40), the variable $s = \varepsilon^{-1/4}(\tau - \tau_0)$ and the Hardy inequality

$$\int_{\mathbb{R}} |\varphi|^2 \, ds \leq C\Big(\int_{\mathbb{R}} |\varphi'|^2 \, ds + \int_{\mathbb{R}} s^2 |\varphi|^2 \, ds \Big) \qquad \forall \varphi \in H^1(\mathbb{R}).$$

Thus,

$$\begin{aligned}&\frac{1}{\varepsilon^{1/2}} \int_{\Gamma \cap \{|\tau-\tau_0|<d/2\}} |\mathsf{v}_0|^2 \, d\tau \leq \frac{1}{\varepsilon^{1/2}} \int_{\mathbb{R}} |\mathsf{v}_0 \chi|^2 \, d\tau \\ &\quad \leq C\Big(\int_{\mathbb{R}} |\partial_\tau (\mathsf{v}_0 \chi)|^2 \, d\tau + \frac{1}{\varepsilon} \int_{\mathbb{R}} (\tau - \tau_0)^2 |\mathsf{v}_0 \chi|^2 \, d\tau \Big) \\ &\quad \leq C\Big(\int_{\Gamma} |\partial_\tau \mathsf{v}_0|^2 \, d\tau + \int_{\Gamma} |\mathsf{v}_0|^2 \, d\tau + \frac{1}{\varepsilon} \int_{\Gamma \cap \{|\tau-\tau_0|<d\}} (\tau - \tau_0)^2 |\mathsf{v}_0|^2 \, d\tau \Big).\end{aligned}$$

Moreover, since $\varkappa$ has a local minimum at $\tau = \tau_0$, there exists $C_2 > 0$ such that

$$\varkappa(\tau) - \varkappa(\tau_0) > C_2 (\tau - \tau_0)^2 \qquad \forall \tau \in \mathbb{S}_\ell, |\tau - \tau_0| < d, \tag{4.75}$$

and, consequently,

$$\begin{aligned}&\frac{1}{\varepsilon^{1/2}} \int_{\Gamma \cap \{|\tau-\tau_0|<d/2\}} |\mathsf{v}_0|^2 \\ &\quad \leq C\Big(\int_{\Gamma} |\partial_\tau \mathsf{v}_0|^2 \, d\tau + \int_{\Gamma} |\mathsf{v}_0|^2 \, d\tau + \frac{1}{\varepsilon} \int_{\Gamma \cap \{|\tau-\tau_0|<d\}} (\varkappa(\tau) - \varkappa(\tau_0)) |\mathsf{v}_0|^2 \, d\tau \Big).\end{aligned} \tag{4.76}$$

Now, gathering (4.74) and (4.76) we obtain (4.70), which concludes the proof. □

Proposition 4.6. *Let $t \geq 1$ and $m > 2$. Then, under the assumptions* 1 *and* 2*, and for a small ε, we have*

$$\begin{aligned} & a_\varepsilon(\{V,\mathsf{v}\},\{V,\mathsf{v}\}) \\ & \geq C\Big(\varepsilon^{t-3/4}\|\nabla_x V\|^2_{L^2(\Omega)} + \varepsilon^{-7/4}\|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)} + \varepsilon^{1/4}\|S\partial_\tau \mathsf{v}_\perp\|^2_{L^2(\omega_1)} \\ & \quad + \varepsilon^{1/4}\Big(\|\partial_\tau \mathsf{v}_0\|^2_{L^2(\Gamma)} + \varepsilon^{-1}\int_\Gamma (\varkappa(\tau)-\varkappa(\tau_0))|\mathsf{v}_0|^2\, d\tau\Big)\Big) \quad \forall\{V,\mathsf{v}\}\in\mathcal{V}, \end{aligned} \tag{4.77}$$

where C is a constant independent of ε and $\{V,\mathsf{v}\}$. Moreover,

$$\begin{aligned} & a_\varepsilon(\{V,\mathsf{v}\},\{V,\mathsf{v}\}) \\ & \geq C\Big(\varepsilon^{t-3/4}\|\nabla_x V\|^2_{L^2(\Omega)} + \varepsilon^{-7/4}\|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)} \\ & \quad + \varepsilon^{1/4}\|S\partial_\tau \mathsf{v}_\perp\|^2_{L^2(\omega_1)} + \varepsilon^{1/4}\|\mathsf{v}_0\|^2_{H^1(\Gamma)}\Big) \qquad \forall\{V,\mathsf{v}\}\in\mathcal{V}. \end{aligned} \tag{4.78}$$

Proof. From the decomposition (4.68), the definition of K_ε, the smoothness of the functions S and $\varkappa$, and (2.30), it follows that

$$\begin{aligned} & \varepsilon^{-7/4}\int_{\omega_1} S^2|\partial_\zeta \mathsf{v}|^2 K_\varepsilon d\zeta d\tau + \varepsilon^{1/4}\int_{\omega_1} S^2|\partial_\tau \mathsf{v}|^2 K_\varepsilon^{-1} d\zeta d\tau \\ & \geq C\Big(\varepsilon^{-7/4}\|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)} + \varepsilon^{1/4}\|\partial_\tau \mathsf{v}_0\|^2_{L^2(\Gamma)} + \varepsilon^{1/4}\|S\partial_\tau \mathsf{v}_\perp\|^2_{L^2(\omega_1)}\Big) \end{aligned} \tag{4.79}$$

and

$$\begin{aligned} & \frac{a}{\varepsilon^{3/4}}\int_{\omega_1} SS'\varkappa(\tau)|\mathsf{v}|^2 d\zeta d\tau + \frac{a}{\varepsilon^{3/4}}\int_{\omega_1}\frac{\varkappa(\tau_0)}{h}S^2|\mathsf{v}|^2 K_\varepsilon d\zeta d\tau \\ & \quad = \frac{a}{2\varepsilon^{3/4}}\int_\Gamma (\varkappa(\tau_0)-\varkappa(\tau))|\mathsf{v}_0|^2 d\tau + R_1 \end{aligned} \tag{4.80}$$

where

$$\begin{aligned} R_1 = & \frac{a\varkappa(\tau_0)}{h\varepsilon^{3/4}}\int_{\omega_1} S^2|\mathsf{v}|^2(K_\varepsilon-1)d\zeta d\tau + \frac{2a}{\varepsilon^{3/4}}\int_{\omega_1} SS'\varkappa(\tau)\mathsf{v}_0\mathsf{v}_\perp d\zeta d\tau \\ & + \frac{a}{\varepsilon^{3/4}}\int_{\omega_1} SS'\varkappa(\tau)|\mathsf{v}_\perp|^2 d\zeta d\tau + \frac{a\varkappa(\tau_0)}{h\varepsilon^{3/4}}\int_{\omega_1} S^2|\mathsf{v}_\perp|^2 d\zeta d\tau. \end{aligned}$$

Note that, for any $\alpha > 0$,

$$\begin{aligned} |R_1| \leq & C\varepsilon^{1/4}\Big(\|\mathsf{v}_0\|^2_{L^2(\Gamma)} + \|S\mathsf{v}_\perp\|^2_{L^2(\omega_1)}\Big) \\ & + C\varepsilon^{-3/4}\Big(\varepsilon^\alpha\|\mathsf{v}_0\|^2_{L^2(\Gamma)} + \varepsilon^{-\alpha}\|\mathsf{v}_\perp\|^2_{L^2(\omega_1)} + \|\mathsf{v}_\perp\|^2_{L^2(\omega_1)}\Big). \end{aligned}$$

Besides, taking $\alpha = 3/4$ and using estimates (4.70) and (4.69), we obtain

$$|R_1| \leq C\varepsilon^{1/4}\Big(\varepsilon^{1/4}\Big(\|\partial_\tau \mathsf{v}_0\|^2_{L^2(\Gamma)}+\frac{1}{\varepsilon}\int_\Gamma(\varkappa(\tau)-\varkappa(\tau_0))|\mathsf{v}_0|^2\,d\tau\Big)+\varepsilon^{-7/4}\|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)}\Big). \tag{4.81}$$

On the other hand, since $V|_\gamma = S(0)\mathsf{v}(0,\tau)$, $S(h) = 0$, and (4.68), we deduce

$$\|V\|^2_{L^2(\Gamma)} = h\|\partial_\zeta(S\mathsf{v})\|^2_{L^2(\omega_1)} \leq C\Big(\|\mathsf{v}_0\|^2_{L^2(\Gamma)} + \|\mathsf{v}_\perp\|^2_{L^2(\omega_1)} + \|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)}\Big). \tag{4.82}$$

Then, by Friedichs' inequality, (4.82), (4.69) and (4.70), we get

$$\begin{aligned}\|V\|^2_{L^2(\Omega)} \leq & C\Big(\|\nabla_x V\|^2_{L^2(\Omega)} + \|S\partial_\zeta \mathsf{v}_\perp\|^2_{L^2(\omega_1)} \\ & + \varepsilon^{1/2}\Big(\|\partial_\tau \mathsf{v}_0\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon}\int_\Gamma(\varkappa(\tau)-\varkappa(\tau_0))|\mathsf{v}_0|^2\,d\tau\Big)\Big).\end{aligned} \tag{4.83}$$

Therefore, using the definition of $a_\varepsilon(\cdot,\cdot)$ and gathering (4.79), (4.80), (4.81) and (4.83) gives (4.77) for a small ε.

Finally, (4.78) holds due to (4.70) and (4.77), which completes the proof. □

As a consequence of Propositions 4.5 and 4.6, for each $\varepsilon > 0$, $a_\varepsilon(\cdot,\cdot)$ defines a scalar product in the space $\mathcal{V}$; let us denote by $\mathcal{V}^\varepsilon$ the space $\mathcal{V}$ equipped with this scalar product. Thus, the new spectral problem reads: find β^ε, $\{V^\varepsilon, \mathsf{v}^\varepsilon\} \in \mathcal{V}^\varepsilon$, $\{V^\varepsilon, \mathsf{v}^\varepsilon\} \not\equiv 0$, satisfying (4.64) for any $\{W, \mathsf{w}\} \in \mathcal{V}^\varepsilon$. In addition, introducing $\mathcal{W}^\varepsilon$ the weighted space $\{\{V, \mathsf{v}\} : V \in L^2(\Omega),\, \mathsf{v} \in L^2_{S,per}(\omega_1)\}$ with the norm defined by (4.67), we show that the embedding $\mathcal{V}^\varepsilon \subset \mathcal{W}^\varepsilon$ is compact, and the spectral problem (4.64) in $\mathcal{V}^\varepsilon$ has the monotone unbounded positive sequence of eigenvalues

$$0 < \beta_1^\varepsilon \leq \beta_2^\varepsilon \leq \cdots \leq \beta_k^\varepsilon \leq \cdots \xrightarrow{k\to\infty} \infty,$$

and the corresponding eigenfunctions $\{\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}\}_{k=1}^\infty$ can be subject to the orthonormalization condition

$$a_\varepsilon(\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}, \{V_l^\varepsilon, \mathsf{v}_l^\varepsilon\}) = \delta_{k,l} \tag{4.84}$$

where $a_\varepsilon(\cdot,\cdot)$ is defined by (4.66). In the next section, we study the asymptotic behavior, $\varepsilon \to 0$, of the eigenvalues β_k^ε for fixed k.

4.3. Convergence of the rescaled eigenvalues

First, we obtain a result that gives us estimates for the eigenvalues of (4.64) (cf. Proposition 1.1 to compare).

Proposition 4.7. *Let $t \geq 1$ and $m > 2$. Let $\{\beta_k^\varepsilon\}_{k=1}^\infty$ be eigenvalues of (4.64). Under the assumptions 1 and 2, for each fixed $k = 1, 2, \dots$ and a small ε, we have*

$$C \leq \beta_k^\varepsilon \leq C_k \tag{4.85}$$

where the positive constants C and C_k do not depend on ε, but $C_k \to \infty$ as $k \to \infty$.

Proof. The lower bound holds as a direct consequence of (4.64), (4.77), (4.70), (4.67), Friedrichs' inequality, (4.82) and (4.69), namely,

$$\begin{aligned}\beta_k^\varepsilon &= \frac{a_\varepsilon(\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}, \{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\})}{(\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}, \{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\})_\varepsilon} \\ &\geq \frac{C_1^*\Big(\varepsilon^{t-3/4}\|\nabla_x V_k^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-7/4}\|S\partial_\zeta \mathsf{v}^\varepsilon_{k\perp}\|^2_{L^2(\omega_1)} + \varepsilon^{-1/4}\|\mathsf{v}^\varepsilon_{k0}\|^2_{L^2(\Gamma)}\Big)}{C_2^*\Big(\varepsilon^{t-3/4}\|\nabla_x V_k^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-1/4}\|\mathsf{v}^\varepsilon_{k0}\|^2_{L^2(\Gamma)} + \varepsilon^{-1/4}\|S\partial_\zeta \mathsf{v}^\varepsilon_{k\perp}\|^2_{L^2(\omega_1)}\Big)}.\end{aligned}$$

As regards the upper bound, the minimax principle gives the equalities

$$\beta_k^\varepsilon = \min_{\substack{E_k \subset \mathcal{V}^\varepsilon \\ \dim E_k = k}} \max_{\substack{\{V, \mathsf{v}\} \in E_k \\ \{V, \mathsf{v}\} \not\equiv 0}} \frac{a_\varepsilon(\{V, \mathsf{v}\}, \{V, \mathsf{v}\})}{(\{V, \mathsf{v}\}, \{V, \mathsf{v}\})_\varepsilon}, \tag{4.86}$$

where the minimum is taken over all the subspaces $E_k \subset \mathcal{V}^\varepsilon$ with $\dim E_k = k$.

Let $\{\lambda_{3/2,k}\}_{k=1}^\infty$ be the eigenvalues of the harmonic oscillator equation (2.37), and $\{v^k\}_{k=1}^\infty$ the corresponding eigenfunctions which are assumed to be normalized in $L^2(\mathbb{R})$. For each fixed k, let E_k^ε be the linear space $E_k^\varepsilon = [\{\mathsf{V}_1^\varepsilon, \mathsf{v}_1^\varepsilon\}, \dots, \{\mathsf{V}_k^\varepsilon, \mathsf{v}_k^\varepsilon\}] \subset \mathcal{V}^\varepsilon$, where V_r^ε denotes the solution of (3.45) for χ the cut-off function defined by (2.40), $g = v_r$ and $\gamma = 1/4$, and $\mathsf{v}_r^\varepsilon(\zeta, \tau) = -\chi(\tau - \tau_0)v^r((\tau - \tau_0)/\varepsilon^{1/4})$ for $(\zeta, \tau) \in \omega_1$, $r = 1, 2 \dots, k$. Note that $\mathsf{v}^\varepsilon_{r\perp} = 0$ and, by Lemma 3.2, $\nabla \mathsf{V}_r^\varepsilon$ is bounded in $L^2(\Omega)$ for $r = 1, 2 \dots, k$. Then, from (4.86), (4.66), (4.80), and (4.67), we derive

$$\beta_k^\varepsilon \leq C_1^* \max_{\substack{\{V, \mathsf{v}\} \in E_k^\varepsilon \\ \{V, \mathsf{v}\} \not\equiv 0}} \frac{\varepsilon^{t-3/4}\|\nabla V\|^2_{L^2(\Omega)} + \varepsilon^{1/4}\|\mathsf{v}_0\|^2_{H^1(\Gamma_d)} + \varepsilon^{-3/4}\displaystyle\int_{\Gamma_d}(\varkappa(\tau) - \varkappa(\tau_0))|\mathsf{v}_0|^2 d\tau}{\varepsilon^{-1/4}\|\mathsf{v}_0\|^2_{L^2(\Gamma_{d/2})}},$$

where Γ_d denotes $\Gamma \cap \{|\tau - \tau_0| < d\}$. Besides, by assumption 1, there exists $C_2^* > 0$ such that $\varkappa(\tau) - \varkappa(\tau_0) < C_2^*(\tau - \tau_0)^2$ for $\tau \in \Gamma_d$, and introducing the change of variable $\eta = (\tau - \tau_0)/\varepsilon^{1/4}$ and taking into account that $\{\lambda_{3/2,k}, v_k\}$ is an eigenpair of (2.37), we obtain

$$\beta_k^\varepsilon \leq C_k^* + C_4^* \lambda_{3/2,k},$$

which completes the proof. □

As a consequence of Proposition 4.7, the sequence β_k^ε is bounded and there are converging subsequences ε, still denoted by ε, $\beta_k^\varepsilon \to \beta_k^*$ as $\varepsilon \to 0$, for certain $\beta_k^* > 0$. Moreover, by Theorem 4.8 below, this limit must be an eigenvalue of the harmonic oscillator operator (2.37). Later on, in (4.4), we identify β_k^* with the kth eigenvalue of (2.37), namely, we prove the convergence, as $\varepsilon \to 0$, of the eigenvalues of (4.64) towards the eigenvalues of (2.37) with conservation of the multiplicity (cf. Corollary 4.10).

Theorem 4.8. *Let $t \geq 1$ and $m > 2$. Let $\{\beta^\varepsilon\}_\varepsilon$ be any sequence of eigenvalues of* (4.64) *such that β^ε converges when $\varepsilon \to 0$ towards some β^*. Then, under the assumptions* 1 *and* 2*, β^* is an eigenvalue of* (2.37).

Proof. Let $\{V^\varepsilon, \mathsf{v}^\varepsilon\}$ be eigenfunction of (4.64) corresponding to β^ε satisfying the normalization condition (4.84). Then, choosing $\{W, \mathsf{w}\} = \{V^\varepsilon, \mathsf{v}^\varepsilon\}$ in (4.64) and taking into account the boundedness of β^ε, (4.77), (4.69), and (4.70) we get

$$\begin{aligned}\varepsilon^{t-3/4}\|\nabla_x V^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-7/4}\|\partial_\zeta \mathsf{v}_\perp^\varepsilon\|^2_{L^2(\omega_1)} + \varepsilon^{1/4}\|\partial_\tau \mathsf{v}_\perp^\varepsilon\|^2_{L^2(\omega_1)} + \varepsilon^{-7/4}\|\mathsf{v}_\perp^\varepsilon\|^2_{L^2(\omega_1)} \leq C\\ \varepsilon^{-1/4}\|\mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \varepsilon^{1/4}\Big(\|\partial_\tau \mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon}\int_\Gamma (\varkappa(\tau) - \varkappa(\tau_0))|\mathsf{v}_0|^2\, d\tau\Big) \leq C.\end{aligned} \tag{4.87}$$

Set $\phi^\varepsilon(\eta) = \mathsf{v}_0^\varepsilon(\tau_0 + \varepsilon^{1/4}\eta)\chi(\varepsilon^{1/4}\eta)$ for $\eta \in \mathbb{R}$, where χ is the cut-off function (2.40). Owing to the change of variable $\tau = \tau_0 + \varepsilon^{1/4}\eta$, (4.75), and (4.87), we have

$$\begin{aligned}&\|\phi^\varepsilon\|^2_{L^2(\mathbb{R})} + \|\phi^{\varepsilon\prime}\|^2_{L^2(\mathbb{R})} + \|\eta\phi^\varepsilon\|^2_{L^2(\mathbb{R})}\\ &\quad\leq \varepsilon^{-1/4}C_1\|\mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \varepsilon^{1/4}\|\partial_\tau \mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon^{3/4}}\int_{\Gamma_d}(\tau - \tau_0)^2|\mathsf{v}_0|^2 d\tau\\ &\quad\leq \varepsilon^{-1/4}C_1\|\mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \varepsilon^{1/4}C_2\Big(\|\partial_\tau \mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon}\int_\Gamma (\varkappa(\tau) - \varkappa(\tau_0))|\mathsf{v}_0|^2\, d\tau\Big) \leq C,\end{aligned}$$

and, consequently, there is a subsequence of ε, still denoted by ε, satisfying

$$\phi^\varepsilon \rightharpoonup \phi^*, \quad \eta\phi^\varepsilon \rightharpoonup \varphi^*, \quad \phi^{\varepsilon\prime} \rightharpoonup \varrho^* \quad \text{weakly in } L^2(\mathbb{R}),$$

as ε tends to zero, for certain functions $\phi^*, \varphi^*, \varrho^* \in L^2(\mathbb{R})$. Obviously, $\varphi^* = \eta\phi^*$ and $\varrho^* = \phi^{*\prime}$ in $\mathcal{D}'(\mathbb{R})$.

In order to identify the pair (β^*, ϕ^*), we consider (4.64) for the test functions $W = \mathtt{V}^\varepsilon$ and $\mathtt{w} = z((\tau - \tau_0)/\varepsilon^{1/4})$, where $z \in C_0^\infty(\mathbb{R})$ and $\mathtt{V}^\varepsilon$ is the solution of (3.45) for $g = -z$ and $\gamma = 1/4$. Note that for a small ε we can assume that $\mathtt{w} = \mathtt{w}_0(\tau) = 0$ for $|\tau - \tau_0| > d/2$ and $\{\mathtt{V}^\varepsilon, \mathtt{w}\} \in \mathcal{V}^\varepsilon$. Besides, $\{\mathtt{V}^\varepsilon, \mathtt{w}\} = \{\mathtt{V}^\varepsilon, \mathtt{w}_0\}$ verify (3.51) and

$$\varepsilon^{-1/4}\|\mathtt{w}_0\|^2_{L^2(\Gamma)} + \varepsilon^{1/4}\|\partial_\tau \mathtt{w}_0\|^2_{L^2(\Gamma)} \leq C. \tag{4.88}$$

Therefore, due to the decomposition (4.68) and (2.30), we obtain

$$\begin{aligned} &\varepsilon^{1/4}\frac{ah}{2}\int_{\Gamma_{d/2}} \partial_\tau \mathtt{v}_0^\varepsilon \partial_\tau \mathtt{w}_0 \, d\tau + \frac{a}{2\varepsilon^{3/4}}\int_{\Gamma_{d/2}} (\varkappa(\tau) - \varkappa(\tau_0))\mathtt{v}_0^\varepsilon \mathtt{w}_0 \, d\tau \\ &\quad = \beta^\varepsilon \frac{h}{2\varepsilon^{1/4}}\int_{\Gamma_{d/2}} \mathtt{v}_0^\varepsilon \mathtt{w}_0 \, d\tau + R^\varepsilon \end{aligned} \tag{4.89}$$

where

$$\begin{aligned} R^\varepsilon =& \beta^\varepsilon \varepsilon^{t+3/4+m-2}\int_\Omega V^\varepsilon \mathtt{V}^\varepsilon \, dx + \frac{\beta^\varepsilon}{\varepsilon^{1/4}}\int_{\omega_1} S^2 \mathtt{v}^\varepsilon \mathtt{w}_0 (K_\varepsilon - 1) d\zeta d\tau \\ &- \varepsilon^{t-3/4} A\int_\Omega \nabla_x V^\varepsilon \cdot \nabla_x \mathtt{V}^\varepsilon \, dx + \frac{a}{\varepsilon^{3/4}}\int_{\omega_1} SS'\varkappa(\tau)\mathtt{v}_\perp^\varepsilon \mathtt{w}_0 \, d\zeta d\tau \\ &+ \frac{a\varkappa(\tau_0)}{h\varepsilon^{3/4}}\int_{\omega_1} S^2 \mathtt{v}^\varepsilon \mathtt{w}_0 (K_\varepsilon - 1) \, d\zeta d\tau - a\varepsilon^{1/4}\int_{\omega_1} S^2 \partial_\tau \mathtt{v}^\varepsilon \partial_\tau \mathtt{w}_0 (K_\varepsilon^{-1} - 1) \, d\zeta d\tau \\ &+ \varepsilon^{t-3/4+m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\int_\Omega V^\varepsilon \mathtt{V}^\varepsilon dx. \end{aligned}$$

Using (3.51), (4.87), (4.88) and Friedrichs' inequality, we verify that R^ε tends to zero as $\varepsilon \to 0$. Moreover, by assumption 1,

$$\Big| \int_{\Gamma_{d/2}} (\varkappa(\tau) - \varkappa(\tau_0))\mathtt{v}_0^\varepsilon \mathtt{w}_0 \, d\tau - \int_{\Gamma_{d/2}} \frac{\varkappa''(\tau_0)}{2}\mathtt{v}_0^\varepsilon \mathtt{w}_0 \, d\tau \Big| \leq C \Big| \int_{\Gamma_{d/2}} (\tau - \tau_0)^3 \mathtt{v}_0^\varepsilon \mathtt{w}_0 \, d\tau \Big|.$$

Thus, introducing the variable $\eta = \varepsilon^{-1/4}(\tau - \tau_0)$ in (4.89) and passing to the limit as ε to 0, we get the integral identity

$$\frac{ah}{2}\int_{\mathbb{R}} \partial_\eta \phi^* \partial_\eta z \, d\eta + \frac{a\varkappa''(\tau_0)}{4}\int_{\mathbb{R}} \eta^2 \phi^* z \, d\eta = \beta^* \frac{h}{2}\int_{\mathbb{R}} \phi^* z \, d\eta \qquad \forall z \in C_0^\infty(\mathbb{R}).$$

To conclude the proof, it suffices to prove that $\phi^* \not\equiv 0$.

To this end, we take limits as $\varepsilon \to 0$ in the normalization condition for the eigenfunction $\{V^\varepsilon, \mathsf{v}^\varepsilon\}$ and use the Friedrichs' inequality, (4.74), (4.87), and (2.30). Then,

$$1 = \beta^* \lim_{\varepsilon\to 0} \|\{V^\varepsilon, \mathsf{v}^\varepsilon\}\|_\varepsilon^2 = \beta^* \lim_{\varepsilon\to 0} \varepsilon^{-1/4} \|S\mathsf{v}_0^\varepsilon\|^2_{L^2(\omega_1)} = \frac{h\beta^*}{2} \lim_{\varepsilon\to 0} \varepsilon^{-1/4} \|\mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma)};$$

in fact, due to (4.73) and (4.87), we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/4} \|\mathsf{v}_0^\varepsilon\|^2_{L^2(\Gamma_{d/2})} = \frac{2}{h\beta^*} \quad \text{and consequently} \quad \lim_{\varepsilon\to 0} \|\phi^\varepsilon\|^2_{L^2(\mathbb{R})} \geq \frac{2}{h\beta^*}.$$

Note that $\beta^* > 0$ because of (4.85). Moreover, since $\eta\phi^\varepsilon$ is bounded in $L^2(\mathbb{R})$, for any $R > 0$, $\|\phi^\varepsilon\|^2_{L^2(|\eta|>R)} < C/R^2$, C being a constant independent of R. Let us choose $R_0 > 0$ satisfying $C/R_0^2 < 2/h\beta^*$; thus,

$$\|\phi^*\|^2_{L^2(|\eta|<R_0)} = \lim_{\varepsilon\to 0} \|\phi^\varepsilon\|^2_{L^2(|\eta|<R_0)} > \frac{2}{h\beta^*} - \frac{C}{R_0^2} > 0,$$

and hence $\phi^* \not\equiv 0$, which completes the proof. $\square$

4.4. The equivalence of the spectral problems

In this section we show the relation between the eigenvalues of problems (1.3) and (4.64) (cf. (4.9) below). This relation along with Theorem 3.3 and Theorem 4.8 allow us to show the convergence, as $\varepsilon \to 0$ of the rescaled eigenvalues of (1.3) in the way stated by Theorem 4.1. As a consequence, we also prove the convergence, as $\varepsilon \to 0$, of the eigenvalues of (4.64) towards the eigenvalues of (2.37) with conservation of the multiplicity (cf. Corollary 4.10).

Theorem 4.9. *Let $t \geq 1$ and $m > 2$. Let $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ and $\{\beta_k^\varepsilon\}_{k=1}^\infty$ be eigenvalues of* (1.3) *and* (4.64)*, respectively. Then, under the assumptions* 1 *and* 2*,*

$$\beta_k^\varepsilon = \Big(\frac{\lambda_k^\varepsilon}{\varepsilon^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon^{3/2}}, \qquad \textit{for } k \in \mathbb{N}.$$

Proof. We divide the proof into two parts. First, we state that any eigenpair $(\lambda_k^\varepsilon, \{U_k^\varepsilon, u_k^\varepsilon\})$ of (1.3) gives rise to an eigenpair $(\beta_K^\varepsilon, \{V_K^\varepsilon, \mathsf{v}_K^\varepsilon\})$ of (4.64) defined by

$$\beta_K^\varepsilon = \Big(\frac{\lambda_k^\varepsilon}{\varepsilon^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon^{3/2}} \quad \text{and} \quad \{V_K^\varepsilon, \mathsf{v}_K^\varepsilon\} = \{U_k^\varepsilon, \mathsf{u}_k^\varepsilon/S\}, \tag{4.90}$$

where u_k^ε denotes the function u_k^ε in the local variable (1.5). From Proposition 4.4 it is clear that the function $\{V_K^\varepsilon, \mathsf{v}_K^\varepsilon\}$ defined by (4.90) belongs to $\mathcal{V}^\varepsilon$. Moreover, if we set

$$G = W \text{ in } \Omega \text{ and } g(\nu,\tau) = S(\nu/\varepsilon)\mathsf{w}(\nu/\varepsilon,\tau) \text{ in } \omega_\varepsilon, \text{ for any } \{W,\mathsf{w}\} \in \mathcal{V}^\varepsilon,$$

it is easy to check that $\{G, g\} \in H_0^1(\Omega_\varepsilon)$. Thus, taking $\{G, g\}$ as a test function in the integral identity (1.3) written in the local variable, namely (1.10), and using that

$$\lambda_k^\varepsilon = \varepsilon^{m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\beta_K^\varepsilon\Big), \quad U_k^\varepsilon = V_K^\varepsilon, \quad \mathsf{u}_k^\varepsilon = S\mathsf{v}_K^\varepsilon,$$

and (4.63), we obtain that the pair $(\beta_K^\varepsilon, \{V_K^\varepsilon, \mathsf{v}_K^\varepsilon\})$, defined by (4.90), verifies (4.64) for any $\{W,\mathsf{w}\} \in \mathcal{V}^\varepsilon$. Hence, any eigenpair $(\lambda_k^\varepsilon, \{U_k^\varepsilon, u_k^\varepsilon\})$ of (1.3) generates an eigenpair of (4.64).

Secondly, we state that any eigenpair $(\beta_k^\varepsilon, \{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\})$ of (4.64) gives rise to an eigenpair $(\lambda_K^\varepsilon, \{U_K^\varepsilon, u_K^\varepsilon\})$ of (1.3) defined by

$$\lambda_K^\varepsilon = \varepsilon^{m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h} + \varepsilon^{3/2}\beta_k^\varepsilon\Big) \quad \text{and} \quad \{U_K^\varepsilon, u_K^\varepsilon\} = \{V_k^\varepsilon, S^\varepsilon v_k^\varepsilon\},$$

where $S^\varepsilon(\nu) = S(\nu/\varepsilon)$ and $v_k^\varepsilon(\nu,\tau) = \mathsf{v}_k^\varepsilon(\nu/\varepsilon,\tau)$ for $(\nu,\tau) \in [0,\varepsilon h) \times \mathbb{S}_\ell$. Clearly, $\{U_K^\varepsilon, u_K^\varepsilon\} \in H_0^1(\Omega_\varepsilon)$. Besides, setting

$$W = G \text{ in } \Omega \text{ and } \mathsf{w}(\zeta,\tau) = S(\zeta)^{-1}g(\varepsilon\zeta,\tau) \text{ in } \omega_1, \text{ for any } \{G,g\} \in H_0^1(\Omega_\varepsilon),$$

we can check that $\{W,\mathsf{w}\} \in \mathcal{V}^\varepsilon$ and

$$\begin{aligned} A\int_\Omega \nabla_x U_K^\varepsilon \cdot \nabla_x G\, dx &+ \frac{a}{\varepsilon^t}\int_{\omega_\varepsilon} \nabla_x u_K^\varepsilon \cdot \nabla_x g\, dx = \varepsilon^{3/4-t}\Big(a_\varepsilon(\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}, \{W,\mathsf{w}\}) \\ &+ \frac{a\pi^2}{\varepsilon^{7/4}4h^2}\int_{\omega_1} S^2\mathsf{v}_k^\varepsilon \mathsf{w} K_\varepsilon\, d\zeta d\tau + \frac{a\varkappa(\tau_0)}{\varepsilon^{3/4}h}\int_{\omega_1} S^2\mathsf{v}_k^\varepsilon \mathsf{w}\, K_\varepsilon\, d\zeta d\tau \\ &+ \varepsilon^{t-3/4+m-2}\Big(\frac{a\pi^2}{4h^2} + \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\int_\Omega V_k^\varepsilon W\, dx\Big). \end{aligned}$$

Now, taking into account that $(\beta_k^\varepsilon, \{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\})$ is an eigenpair of (4.64) and that

$$\beta_k^\varepsilon = \Big(\frac{\lambda_K^\varepsilon}{\varepsilon^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon\frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon^{3/2}} \qquad \text{and} \qquad \{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\} = \{U_K^\varepsilon, \mathsf{u}_K^\varepsilon/S\},$$

we see that $(\lambda_K^\varepsilon, \{U_K^\varepsilon, u_K^\varepsilon\})$ verifies (1.3) for any $\{G,g\} \in H_0^1(\Omega_\varepsilon)$, which completes the proof. $\square$

Proof of Theorem 4.1. First, let $k=1$. By Theorem 4.9, Proposition 4.7, and Theorem 4.8, we can extract a subsequence $\varepsilon_n \to 0$ such that

$$\Big(\frac{\lambda_1^{\varepsilon_n}}{\varepsilon_n^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon_n \frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon_n^{3/2}} = \beta_1^{\varepsilon_n} \to \beta_1^*$$

for some β_1^* eigenvalue of (2.37). Besides, owing to Theorem 3.3 with $\lambda_0 = \frac{a\pi^2}{4h^2}$ and $\lambda_{3/2} = a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}$, there is at least one eigenvalue of (1.3) $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ satisfying (3.47), namely, using again Theorem 4.9, there is at least one eigenvalue of (4.64) $\beta^\varepsilon = \beta^\varepsilon_{k(\varepsilon)}$ satisfying $|\beta^\varepsilon_{k(\varepsilon)} - a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}| \leq C\varepsilon^{q-3/2}$ where $q = \min\{(4t+4m-3-2\delta)/8, 13/8\}$ with $\delta \in (0,1/8)$. Now, taking limits as $\varepsilon_n \to 0$ in $\beta_1^{\varepsilon_n} \leq \beta^{\varepsilon_n}_{k(\varepsilon_n)}$ yields $\beta_1^* = a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}$, and convergence (4.56) holds for $k=1$.

Without loss of generality, we prove the convergence for $k=2$ and the result for any k holds by induction. From Theorem 4.9, Proposition 4.7, and Theorem 4.8, we can extract a subsequence $\varepsilon_n \to 0$ such that

$$\Big(\frac{\lambda_2^{\varepsilon_n}}{\varepsilon_n^{m-2}} - \frac{a\pi^2}{4h^2} - \varepsilon_n \frac{a\varkappa(\tau_0)}{h}\Big)\frac{1}{\varepsilon_n^{3/2}} = \beta_2^{\varepsilon_n} \to \beta_2^*$$

for some β_2^* eigenvalue of (2.37). Also, on account of Theorem 3.3 with $\lambda_0 = \frac{a\pi^2}{4h^2}$ and $\lambda_{3/2} = 3a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}$, for sufficiently small ε, there is at least one eigenvalue of (1.3) $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ satisfying (3.47), namely, using again Theorem 4.9, there is at least one eigenvalue of (4.64) $\beta^\varepsilon = \beta^\varepsilon_{k(\varepsilon)}$ satisfying $|\beta^\varepsilon_{k(\varepsilon)} - 3a(\frac{\varkappa''(\tau_0)}{2h})^{1/2}| \leq C\varepsilon^{q-3/2}$ where $q = \min\{(4t+4m-3-2\delta)/8, 13/8\}$ with $\delta \in (0,1/8)$. It is clear that, for ε small enough and $t+m > 15/4$, $k(\varepsilon) \geq 2$. Thus, taking limits as $\varepsilon_n \to 0$ in $\beta_1^{\varepsilon_n} \leq \beta_2^{\varepsilon_n} \leq \beta^{\varepsilon_n}_{k(\varepsilon_n)}$ gives

$$a\Big(\frac{\varkappa''(\tau_0)}{2h}\Big)^{1/2} = \beta_1^* \leq \beta_2^* \leq 3a\Big(\frac{\varkappa''(\tau_0)}{2h}\Big)^{1/2}.$$

Since β_2^* is an eigenvalue of (2.37), convergence (4.56) will be proved for $k=2$ once we show that $\beta_2^* \neq \beta_1^*$.

Indeed, we set $\phi_k^\varepsilon(\eta) = \mathsf{v}^\varepsilon_{k0}(\tau_0 + \varepsilon^{1/4}\eta)\chi(\varepsilon^{1/4}\eta)$ for $\eta \in \mathbb{R}$ and $k \in \mathbb{N}$, where $\{\{V_k^\varepsilon, \mathsf{v}_k^\varepsilon\}\}_{k=1}^\infty$ eigenfunctions corresponding to $\{\beta_k^\varepsilon\}_{k=1}^\infty$ subject to the orthonormalization condition (4.84), and χ is the cut-off function (2.40). Similar reasonings to those used for the proof of Theorem 4.8 lead us to prove that the weak limit in $L^2(\mathbb{R})$ of ϕ_1^ε and ϕ_2^ε, ϕ_i^* and ϕ_2^* respectively, are eigenfunctions of (2.37) corresponding to β_1^* and β_2^* respectively, and $\int_{\mathbb{R}} \phi_1^* \phi_2^* \, d\eta = 0$. Hence, since the eigenvalues of (2.37) are simple, $\beta_1^* \neq \beta_2^*$, which concludes the proof.

Corollary 4.10. *Let $t \geq 1$ and $m > 2$ such that $t + m > 15/4$. Let $\{\beta_k^\varepsilon\}_{k=1}^\infty$ be eigenvalues of* (4.64). *Then, under the assumptions* 1 *and* 2*, for each k fixed, β_k^ε converges, as $\varepsilon \to 0$, towards the kth eigenvalue of* (2.37).

5. High frequencies

As occurs in many singularly perturbed problems (see, for instance, [22, 13, 12, 20, 16]), there are sequences of eigenvalues of (1.1), $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ with $k(\varepsilon) \to \infty$, of order ε^β for some $\beta < m - 2$, whose corresponding eigenfunctions suitably normalized do not vanish asymptotically. Here, we focus our attention on the eigenvalues of (1.1) of order 1, the so-called high frequencies.

Throughout this section we consider the case where $m > 0$. We first obtain the limiting problem associated with the eigenvalues λ^ε of (1.1) of order 1 by means of asymptotic expansions. Later on, we show that the eigenvalues λ^ε asymptotically close to eigenvalues of the Dirichlet problem in Ω give rise to global vibrations in the way stated by Theorem 5.1 and Theorem 5.2: roughly speaking, *only the eigenfunctions corresponding to eigenvalues λ^ε asymptotically near an eigenvalue of the Dirichlet problem* (5.95) *can be asymptotically different from zero in $H^1(\Omega)$*. It should be noted that convergence results hold for all $m > 0$, while some restrictions and extensions for the asymptotic expansions for certain values of m are in Remark 5.4.

For $m > 2$ (see Remark 5.4 for $m \in (0, 2]$), we assume an asymptotic expansion for the eigenvalues λ^ε and for the corresponding eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ in Ω and ω_ε of the form:

$$\lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \tag{5.91}$$

$$U^\varepsilon(x) = V(x) + \varepsilon \mathsf{V}_1(x) + \varepsilon^2 \mathsf{V}_2(x) + \cdots, \qquad x \in \Omega, \tag{5.92}$$

$$u^\varepsilon(\zeta,\tau) = v_0(\zeta,\tau) + \varepsilon v_1(\zeta,\tau) + \varepsilon^2 v_2(\zeta,\tau) + \cdots, \qquad \zeta \in [0,h), \tau \in \mathbb{S}_\ell, \tag{5.93}$$

respectively, where (ζ, τ) are the local coordinates given by (1.5), and v_i are ℓ–periodic functions in τ. Besides, we suppose that at least one of the functions V or v_0 in (5.92)–(5.93) are different from zero.

We introduce the local coordinates (1.5) in the Laplacian written in the curvilinear coordinates, namely, in (2.14) and we gather the coefficients of the different powers of ε. Thus, we write

$$\Delta_{\zeta,\tau} = \varepsilon^{-2}\,\partial_\zeta^2 + \varepsilon^{-1}\,\varkappa(\tau)\partial_\zeta - \varkappa(\tau)^2\zeta\partial_\zeta + \partial_\tau^2 + \cdots\,. \tag{5.94}$$

By replacing (5.91), (5.92) and (5.93) in (1.1), on account of (5.94), we have that the leading terms in the asymptotic expansions satisfy the equations

$$\begin{gathered} -A\Delta_x V = \lambda_0 V \quad \text{ in } \Omega, \\ 0 = \lambda_0 v_0, \quad \zeta \in (0,h), \tau \in \mathbb{S}_\ell, \\ V = v_0 \quad \text{on } \Gamma. \end{gathered}$$

Hence, $\lambda_0 = 0$ or $v_0 \equiv 0$. Since we are dealing with the eigenvalues of order 1, we consider the case where $\lambda_0 \neq 0$, and consequently we have that (λ_0, V) is an eigenpair of the Dirichlet problem

$$\begin{cases} -A\Delta_x V = \lambda_0 V & \text{in } \Omega, \\ V = 0 & \text{on } \Gamma. \end{cases} \tag{5.95}$$

As outlined for the asymptotics of the eigenfunctions corresponding to the low frequencies, an appropriate normalization for the eigenfunctions must be prescribed to obtain convergence for the high frequencies. We denote by $\mathfrak{H}^\varepsilon$ the space $H_0^1(\Omega_\varepsilon)$ with the scalar product

$$\begin{aligned} (\{W,w\},\{G,g\})_{\mathfrak{H}^\varepsilon} = A\int_\Omega \nabla_x W \cdot \nabla_x G\,dx + \frac{a}{\varepsilon^t}\int_{\omega_\varepsilon} \nabla_x w \cdot \nabla_x g\,dx \\ \forall \{W,w\},\{G,g\} \in H_0^1(\Omega_\varepsilon). \end{aligned} \tag{5.96}$$

Next, we use Lemma 3.1 to show the convergence of sequences of eigenvalues of (1.1) towards those of (5.95) and to obtain bounds for the convergence rates for the eigenvalues and eigenfunctions stated in Theorem 5.1 (cf. (5.97)). Theorem 5.2 shows that this result for the high frequencies is optimal, since, on account that any real λ^* is a limit point of sequences of eigenvalues $\lambda^\varepsilon = O(1)$ of (1.1) (cf. [16], for instance, for the technique), the normalization for the corresponding eigenfunctions (or linear combination of eigenfunctions) $\{U^\varepsilon, u^\varepsilon\}$ in $\mathfrak{H}^\varepsilon$ (see (5.96)), lead to possible limits being $(\lambda^*, 0)$ in $\mathbb{R} \times H^1(\Omega) - weak$ in the case where λ^* is not an eigenvalue (5.95). For brevity, below we state the main results and outline the proofs which follow the arguments in [16].

Theorem 5.1. *Let (λ_0, V) be an eigenpair of the Dirichlet problem* (5.95) *such that $\|V\|_{L^2(\Omega)} = 1$. Then, for $m > 0$, there are eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of problem* (1.1) *such that*

$$|\lambda^\varepsilon_{k(\varepsilon)} - \lambda_0| \leq C\varepsilon$$

where C is a constant independent of ε. In addition, there is a linear combination of eigenfunctions $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\} \in H^1_0(\Omega_\varepsilon)$, $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}$ corresponding to the eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1.1) *in the interval $[\lambda_0 - K\varepsilon^\theta, \lambda_0 + K\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 1$, $\|\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}\|_{\mathfrak{H}^\varepsilon} = 1$, such that*

$$\|\widetilde{U}^\varepsilon - \lambda_0^{-1/2} V\|_{H^1(\Omega)} \leq C\varepsilon^{1-\theta}. \tag{5.97}$$

Proof. We apply Lemma 3.1 for $H = \mathfrak{H}^\varepsilon$ in (5.96), $A = \mathfrak{A}^\varepsilon$ the compact and symmetric operator on $\mathfrak{H}^\varepsilon$ defined by

$$(\mathfrak{A}^\varepsilon\{W, w\}, \{G, g\})_{\mathfrak{A}^\varepsilon} = \int_\Omega W G\,dx + \frac{1}{\varepsilon^{t+m}} \int_{\omega_\varepsilon} wg\,dx \quad \forall \{W, w\}, \{G, g\} \in H^1_0(\Omega_\varepsilon);$$

$\lambda = \lambda_0^{-1}$ and $u = \{V, 0\}\|V\|^{-1}_{H^1(\Omega)} \in H^1_0(\Omega_\varepsilon)$ where (λ_0, V) is as the theorem states. Then, we rewrite the proof of Theorem 3.3 with the suitable simplifications, and the theorem holds. □

Theorem 5.2. *Let λ^* be any positive real number which is not an eigenvalue of the Dirichlet problem* (5.95)*. Let $m > 0$, and let δ_ε denote any positive infinitesimal sequence. Assuming that there are eigenvalues λ^ε of problem* (1.1) *in the interval $[\lambda^* - \delta^\varepsilon, \lambda^* + \delta^\varepsilon]$, let $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\} \in H^1(\Omega_\varepsilon)$ be any linear combination of eigenfunctions of* (1.1) *corresponding to the eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ in the above interval, $\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}$ satisfying $\|\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\}\|_{\mathfrak{H}^\varepsilon} = 1$. Then, $\widetilde{U}^\varepsilon$ converge weakly in $H^1(\Omega)$ towards zero as $\varepsilon \to 0$.*

Proof. We employ the technique in [16]. First, we consider the case when the interval contains only one eigenvalue $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$. This amounts to taking $\delta_\varepsilon = |\lambda^* - \lambda^\varepsilon| \to 0$ as $\varepsilon \to 0$. Let $\{U^\varepsilon, u^\varepsilon\}$ be the corresponding eigenfunction of norm 1 in $\mathfrak{H}^\varepsilon$ (see (5.96)). Thus, $\|\{U^\varepsilon, u^\varepsilon\}\|_{H^1(\Omega_\varepsilon)}$ is bounded by a constant independent of ε, and we can extract a subsequence (still denoted by ε) such that U^ε converges weakly in $H^1(\Omega)$ towards U^*. Taking into that $U^\varepsilon = u^\varepsilon$ on Γ, $u^\varepsilon = 0$ on Γ^ε, and the normalization yields

$$\|U^\varepsilon\|^2_{L^2(\Gamma)} \leq C\varepsilon \|\nabla_x u^\varepsilon\|^2_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{(1+t)},$$

and hence $U^* = 0$ on Γ. In order to identify U^*, we consider (1.3) for $G \in \mathcal{D}(\Omega)$ extended by zero to Ω_ε, we take limits as $\varepsilon \to 0$ and we obtain that (λ^*, U^*) satisfies

$$\int_\Omega \nabla_x U^* \cdot \nabla_x G \, dx = \lambda^* \int_\Omega U^* G \, dx \quad \forall G \in H_0^1(\Omega) \,,$$

which is the weak formulation of (5.95). Consequently, if λ^* is not an eigenvalue of (5.95), then $U^* \equiv 0$.

Finally, we rewrite the above arguments with minor modifications in the general case where there are several eigenvalues of (1.1) in the interval $[\lambda^* - \delta^\varepsilon, \lambda^* + \delta^\varepsilon]$. Indeed, let $\{\lambda^\varepsilon_{k(\varepsilon)+j}\}_{j=0}^J$ denote the set of eigenvalues $[\lambda^* - \delta^\varepsilon, \lambda^* + \delta^\varepsilon]$, and $\{\{U^\varepsilon_{k(\varepsilon)+j}, u^\varepsilon_{k(\varepsilon)+j}\}\}_{j=0}^J$ the set of the corresponding eigenfunctions; J being a certain natural that can depend on ε. Let us assume that

$$\{\widetilde{U}^\varepsilon, \widetilde{u}^\varepsilon\} = \sum_{j=0}^J \alpha_j^\varepsilon \{U^\varepsilon_{k(\varepsilon)+j}, u^\varepsilon_{k(\varepsilon)+j}\}$$

for certain constants α_j^ε. We write the equation (1.3) for each eigenvalue and the corresponding eigenfunction of the set, and for $G \in H_0^1(\Omega)$, $g = 0$. Then, we take the sum after multiplying each equation by α_j^ε, j ranging from 0 to J. We take into account the convergence

$$\sum_{j=0}^J \alpha_j^\varepsilon (\lambda^\varepsilon_{k(\varepsilon)+j} - \lambda^*) \int_\Omega U^\varepsilon_{k(\varepsilon)+j} G \, dx \to 0 \quad \text{as } \varepsilon \to 0,$$

and the result of the theorem holds. $\square$

Remark 5.3. There can be different points where $\varkappa(\tau_0)$ has a local minimum, and even several different points with the same value for the second derivative $\varkappa''(\tau_0)$. Thus, without stronger restrictions for $\varkappa(\tau_0)$, the type of results in Theorem 5.2, which would complement those in Theorem 3.3, cannot be obtained.

Remark 5.4. It should be noted that the technique of asymptotic expansions throughout this section also applies in the case where $m \in (0, 2)$ and we obtain the same limit problem (5.95). In this case we need to use further terms of the asymptotic expansions of u^ε in ω_ε. As a matter of fact, for $m \neq 1$ the expansion (5.93) must be suitably modified by introducing other terms for different powers of ε, namely of the order ε^p, with $p > 0$, $p \notin \mathbb{N}$, depending on the particular value of m. Moreover, for $m \in (0, 2)$, the converge of the kth eigenvalue of (1.3), when $\varepsilon \to 0$, towards the kth eigenvalue of (5.95) holds following the technique in [16] (see also [12]).

In the case where $m = 2$, the asymptotic expansions (5.91)–(5.93) and (2.17)–(2.19) provide two possibilities for λ_0 that we state here without a proof. One is λ_0 to be an eigenvalue of (5.95) and the other is λ_0 to be an eigenvalue of (2.22). Now, it remains to identify the eigenfunctions in (5.92)–(5.93) and (2.18)–(2.19) which involve different normalization (see norms (3.44) and (5.96) to compare). This case remains as an open problem.

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