Z. Angew. Math. Phys. (2022) 73:234

https://doi.org/10.1007/s00033-022-01869-8

Zeitschrift für angewandte
Mathematik und Physik ZAMP

Boundary homogenization with large reaction terms on a strainer-type wall

D. Gómez and M.-E. Pérez-Martínez

Abstract. We consider a homogenization problem for the Laplace operator posed in a bounded domain of the upper half-space, a part of its boundary being in contact with the plane $\{x_3 = 0\}$. On this part, the boundary conditions alternate from Neumann to nonlinear-Robin, being of Dirichlet type outside. The nonlinear-Robin boundary conditions are imposed on small regions periodically placed along the plane and contain a *Robin parameter* that can be very large. We provide all the possible homogenized problems, depending on the relations between the three parameters: period ε, size of the small regions r_ε and Robin parameter $\beta(\varepsilon)$. In particular, we address the convergence, as ε tends to zero, of the solutions for the critical size of the small regions $r_\varepsilon = O(\varepsilon^2)$. For certain $\beta(\varepsilon)$, a *nonlinear capacity term* arises in the *strange term* which depends on the macroscopic variable and allows us to extend the usual capacity definition to semilinear boundary conditions.

Mathematics Subject Classification. 35B27, 35J05, 47J20, 35B40, 35B25, 74Q05, 35Q74.

Keywords. Boundary homogenization, Local problems, Extended capacity, Critical relations, Laplace operator.

1. Introduction

We consider a boundary homogenization problem for the Laplace operator posed in a bounded domain Ω of the upper half-space $\mathbb{R}^{3+} = \{x \in \mathbb{R}^3 : x_3 > 0\}$, a part of its boundary Σ being in contact with the plane $\{x_3 = 0\}$. A Dirichlet boundary condition is imposed out of Σ. On Σ the boundary conditions alternate periodically from Neumann to nonlinear-Robin. The nonlinear-Robin conditions contain the so-called *Robin parameter* $\beta(\varepsilon)$ multiplied by a nonlinear function σ of the solution u^ε, $\sigma = \sigma(x, u^\varepsilon)$. The reaction term $\beta(\varepsilon)\sigma$ concentrates on small regions T^ε, *the reaction regions*, placed along Σ while $\beta(\varepsilon)$ can be very large. These conditions recall the elastic response or the reaction of the media; cf. [17] and [20] in this connection. The small parameter ε measures the periodicity, and we address the asymptotic behavior of the solution when $\varepsilon \to 0$.

The problem for the same operator and geometrical configuration here considered but with alternating boundary conditions of Neumann (or linear-Robin) and Dirichlet type has been addressed, for instance, in [4] and [31]; cf. [2] and [26] for the elasticity system. Alternating Neumann and linear-Robin boundary conditions have been considered in [32] for the Laplacian, and [12] and [13] for the elasticity operator. The model under consideration (2.5) may represent the scalar version of a nonlinear Winkler bed, namely a block of an elastic material which has a part of its boundary ($\partial\Omega \setminus \Sigma$) clamped to a rigid profile, while the other part (Σ) rests partially on a nonlinear Winkler foundation along the small region T^ε; see [12], [13] and [17] for linear models.

From the geometrical viewpoint, the problem belongs to a large class of boundary homogenization problems studied for a long time in the literature of applied mathematics for different operators. We mention some of the first works in which keywords such as critical sizes and critical relations between parameters have been introduced [7, 29, 30] and [35], also [8] for nonhomogeneous boundary conditions. Let us refer to [5, 6] and references therein for rapidly alternating Dirichlet–Steklov boundary conditions and

Published online: 21 October 2022

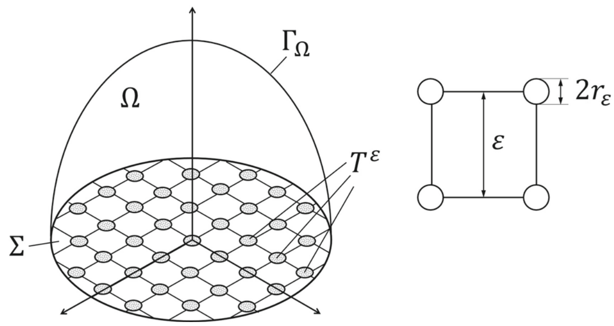

Fig. 1. Geometrical configuration of the problem

[11,18,28] for further references and possible applications in the framework of Geophysics and Winkler beds (foundations). See [9–15] and [32] for an extensive and updated bibliography on different boundary homogenization problems with Robin-type boundary conditions. Finally, we also mention the first works [16] and [19] where different strange terms in the homogenization of volume perforated media with nonlinear-Robin boundary conditions have been introduced.

The small regions T^ε mentioned above have a diameter $O(r_\varepsilon)$ and are placed along the plane at a distance $O(\varepsilon)$ between them, see Fig. 1. Here, ε and r_ε are two parameters that converge toward zero, $r_\varepsilon \ll \varepsilon$, while $\beta(\varepsilon)$ can range from very small to very large as $\varepsilon \to 0$. The nonlinear function $\sigma := \sigma(x,u)$ is a continuous function in $\overline{\Omega} \times \mathbb{R}$, globally Lipschitz and monotonic in the u variable, cf. (2.2)–(2.4).

Three different relations between the parameters play an important role when describing the asymptotic behavior of the solution. As a matter of fact, there appear *critical relations* between parameters for which different *strange terms* arise in the homogenized Robin boundary conditions. These conditions are intermediate between the Dirichlet and Neumann ones which appear asymptotically for the extreme cases. Let us describe these relations in further detail.

Setting

$$\lim_{\varepsilon\to 0} \beta(\varepsilon) r_\varepsilon^2 \varepsilon^{-2} = \beta^*, \tag{1.1}$$

a critical relation between ε, r_ε and $\beta(\varepsilon)$ appears when $\beta^* > 0$. Other key relations between parameters are given by

$$\lim_{\varepsilon\to 0} r_\varepsilon \varepsilon^{-2} = r_0 \tag{1.2}$$

and

$$\lim_{\varepsilon\to 0} r_\varepsilon \beta(\varepsilon) = \beta^0. \tag{1.3}$$

In the case where $r_0 > 0$ we deal with the *classical critical size* of the regions T^ε. We call it *classical* since it was obtained a long time ago in the case of a Dirichlet condition on T^ε instead of a Robin one (cf., e.g., [4] and [31] for the same operator and geometrical configuration here considered).

Using matched asymptotic expansions, we obtain the homogenized problems depending on whether these limits β^*, r_0 and β_0 take the value zero, positive or infinity (cf. Sect. 2.1). For the sake of brevity, we avoid introducing the method here and we refer to [10] for the technique for a perforate media (with volume perforations) along a wall and to [32] for a linear problem. Also for brevity, we show the convergence of the solutions in the most troubled situations, namely the cases in which the so-called *microscopic* or *local problems* are crucial to describe the macroscopic behavior of the media. Below, we summarize the whole limit situations, the state of the art and the structure of the paper.

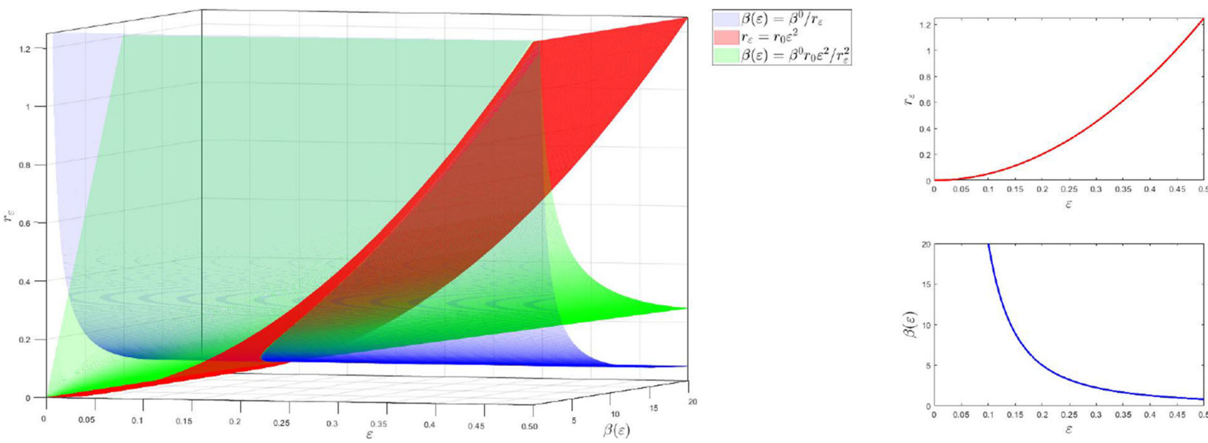

FIG. 2. Example of the most critical relation when $r_0 > 0$ and $\beta^0 > 0$: see the intersecting line of the three surfaces and its projection over the planes

The most critical situation happens when $r_0 > 0$ and $\beta^0 > 0$ which also amounts to $r_0 > 0$ and $\beta^* > 0$, cf. the intersecting line in the 3D graphic in Fig. 2. In this case, the strange term contains a nonlinear function of the solution u^0, $\mathcal{C}^e(x, u^0)$ which is referred to as *extended capacity*, cf. (2.12), and depends on the function σ in a non-trivial way. This dependence involves the solutions of a bi-parametric family of *nonlinear-Robin local problems* posed in the upper half-space $\mathbb{R}^{3+}$ (cf. problem (2.13) and Fig. 4), the parameters dealing with the macroscopic variable and the unknown solution of the homogenized problem. The capacity also depends on the shape of the unit region T and, as a matter of fact, we show that the function $\mathcal{C}^e(x, u^0)u^0$, has similar properties to the given nonlinear function σ (cf. Proposition 3.3). The proof of the convergence compulsorily implies introducing suitable test functions in variational inequalities. We construct the test functions from the solutions of the local problems, and after proving a certain smoothness of these solutions in the *macroscopic parameters* of these problems, we are led to a well-known result on convergence of surface measures (cf. Lemma 4.1). All of this involves some technical restrictions on the nonlinear dependence of σ in the u variable, see Remarks 2.1 and 2.2 in this connection. The homogenized problem reads (2.11).

When $r_0 > 0$, the other critical case arises when $\beta^0 = +\infty$ (equivalently, $\beta^* = +\infty$). It implies a Dirichlet condition on T for the solution of the microscopic problem which becomes linear (cf. problem (2.17) and Fig. 4). This case asymptotically amounts to Dirichlet conditions on T^ε and, consequently, the same capacity term appears in the strange term. It is a constant capacity, which seemingly ignores the nonlinear function σ but also depends on the shape of T, cf. (2.16). However, it is defined through a product of duality in $H^{-1/2}(T) \times H^{1/2}(T)$ which adds unforeseen difficulties in justifications. Due to the fact that the u^ε does not vanish on the T^ε, now the proof of the convergence requires introducing new results on integrals of potential type and on the convergence of the traces of the solutions u^ε on the reaction regions. It also involves further restrictions on the nonlinear function σ, see Proposition 5.2 and Remark 5.1. Also the local problem (2.17) and the test functions from its solution become essential. The homogenized problem reads (2.15).

In the case where $r_0 = +\infty$ with $\beta^* > 0$, the homogenized Robin condition contains the same nonlinear function σ multiplied by the somewhat *averaged Robin parameter*, $\beta^*|T|$, which only takes into account the area of the unit region T for any shape. Notice that $\beta^* > 0$ is obtained when the total area of the regions T^ε multiplied by *the Robin parameter* $\beta(\varepsilon)$ is of order 1, in such a way that a critical size of T^ε corresponds to each Robin parameter $\beta(\varepsilon)$, namely $r_\varepsilon = O(\beta(\varepsilon)^{-1/2}\varepsilon)$, while a critical Robin parameter $\beta(\varepsilon) = O(\varepsilon^2 r_\varepsilon^{-2})$ corresponds to each size r_ε, cf. Fig. 3. The homogenized problem is (2.18).

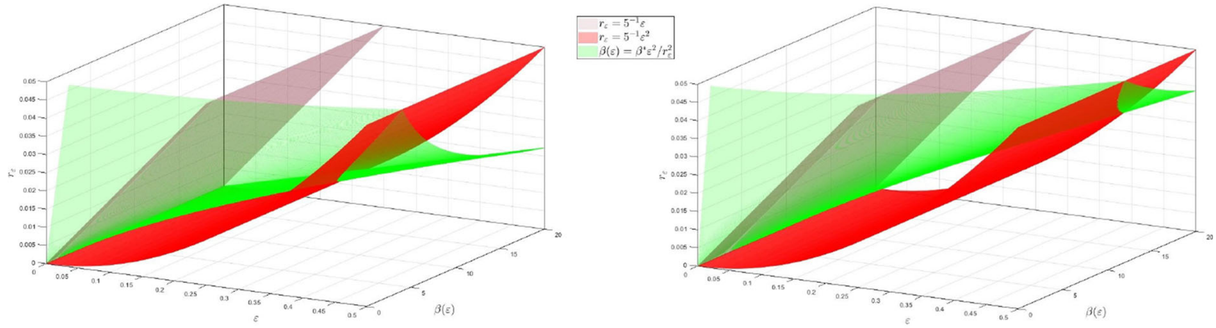

FIG. 3. Examples of critical relations when $r_0 = +\infty$ with two different $\beta^* > 0$: there is a wide variety of possible surfaces (green color) between the two surfaces $r_\varepsilon = 5^{-1}\varepsilon$ and $r_\varepsilon = 5^{-1}\varepsilon^2$ (Color figure online)

The rest of the possible values of the limits in (1.1)–(1.3) lead to extreme cases when either a Neumann condition or a Dirichlet one is asymptotically imposed on Σ, the homogenized problems being (2.19) or (2.20).

In this paper, we show the convergence for $r_0 > 0$; see Remark 5.2 related to proofs in the rest of the cases and possible extensions.

As regards closer works in the literature, it should be emphasized that the justification scheme here developed applies to the homogenization of the nonlinear diffusion problems in porous media addressed in [10]. This justification was left as an open problem. Currently, in [10], it may get simplified due to the geometry of the local problem in porous media. Also the technique extends that for the linear problem in [32] when $r_0 > 0$ and $\beta^0 > 0$, while it justifies the case where $r_0 > 0$ and $\beta^0 = +\infty$ which also was left as an open problem in [32]. Moreover, although we deal with a scalar problem, the technique developed in [2,13] and [31] for the linear elasticity operator, based on projections over spaces of linear functions, does not work for the nonlinear problem here considered.

Finally, the structure of the paper is as follows. Section 2 contains the setting of the homogenization problem and the list of homogenized problems with the corresponding local problems (cf. Sect. 2.1). Sections 3 and 4 deal with the convergence when $r_0 > 0$ and $\beta^0 > 0$. Section 3 contains the setting of the bi-parametric family of local problems, and some properties for solutions which are key points to show the convergence in Sect. 4. Section 5 addresses the convergence when $r_0 > 0$ and $\beta^0 = +\infty$; the study of the local problem is in Sect. 5.1.

2. Setting of the problem and limit problems

Let Ω be an open bounded domain of $\mathbb{R}^3$ situated in the upper half-space $\mathbb{R}^{3+}$, with a Lipschitz boundary $\partial\Omega$. Let Σ be the part of $\partial\Omega$ in contact with the plane $\{x_3 = 0\}$ which is assumed to be non-empty and let Γ_Ω be the rest of the boundary: $\partial\Omega = \overline{\Gamma}_\Omega \cup \overline{\Sigma}$. Let T denote an open bounded domain of the plane $\{x_3 = 0\}$ with a smooth boundary. Without any restriction, we can assume that both Σ and T contain the origin of coordinates.

Let ε be a small parameter $\varepsilon \ll 1$. We consider r_ε an order function such that $r_\varepsilon \ll \varepsilon$. For $\mathrm{k} = (k_1, k_2) \in \mathbb{Z}^2$, we denote by $\widetilde{x}^\varepsilon_\mathrm{k}$ the point of the plane $\{x_3 = 0\}$ of coordinates $\widetilde{x}^\varepsilon_\mathrm{k} = (k_1\varepsilon, k_2\varepsilon, 0)$, and by $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$ the homothetic domain of T of ratio r_ε after translation to the point $\widetilde{x}^\varepsilon_\mathrm{k}$, namely the set

$$T^\varepsilon_{\widetilde{x}_\mathrm{k}} = \widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon T\,.$$

If there is no ambiguity, we shall write $\widetilde{x}_\mathrm{k}$ instead of $\widetilde{x}^\varepsilon_\mathrm{k}$, and T^ε instead of $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$.

In this way, for a fixed ε, we construct a grid of squares in the plane $\{x_3 = 0\}$ whose vertices are in the regions T^ε. Let the set $\mathcal{J}^\varepsilon$ denote $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 : T^\varepsilon_{\tilde{x}_\mathrm{k}} \subset \Sigma\}$, while N_ε denotes the number of elements of $\mathcal{J}^\varepsilon$:

$$N_\varepsilon \approxeq \frac{|\Sigma|}{\varepsilon^2} = O(\varepsilon^{-2}). \tag{2.1}$$

Finally, if no confusion arises, we denote by $\bigcup T^\varepsilon$ the union of all the T^ε contained in Σ. Also, in what follows $x = (x_1, x_2, x_3)$ denotes the usual Cartesian coordinates, while by $\hat{x} = (x_1, x_2)$ we refer to the two first components of $x \in \mathbb{R}^3$.

Let us consider the function $\sigma \equiv \sigma(x, u)$, a continuous function in $\overline{\Omega} \times \mathbb{R}$, globally Lipschitz in the following sense:

$$|\sigma(x,u) - \sigma(x',u')| \leq K_1\Big(|x - x'| + |u - u'|\big(1 + |u|^\tau + |u'|^\tau\big)\Big) \quad \forall x, x' \in \overline{\Omega},\ \forall u, u' \in \mathbb{R}, \tag{2.2}$$

which is also monotonic in the variable u, and satisfying

$$\sigma(x, 0) = 0 \quad \text{and} \quad (\sigma(x,u) - \sigma(x,u'))(u - u') \geq 0 \quad \forall x \in \overline{\Omega},\ \forall u, u' \in \mathbb{R}, \tag{2.3}$$

and

$$K_2|u| \leq |\sigma(x,u)| \quad \forall x \in \overline{\Omega},\ u \in \mathbb{R}. \tag{2.4}$$

Above, K_1 and K_2 are certain positive constants and $\tau \in [0, 2]$.

For different technical reasons further restrictions on the constant τ will be imposed throughout the paper in order to obtain the desired convergence (cf. Theorems 4.1 and 5.2). See Remark 2.1 for less and more restrictive conditions and see Remark 2.2 for the above-mentioned reasons.

Let $f \in L^2(\Omega)$ and u^ε be the solution of the following homogenization problem:

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } \Omega\,, \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega\,, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma \setminus \bigcup T^\varepsilon, \\ \dfrac{\partial u^\varepsilon}{\partial n} + \beta(\varepsilon)\sigma(x, u^\varepsilon) = 0 & \text{on } \bigcup T^\varepsilon\,, \end{cases} \tag{2.5}$$

where n stands for the unit outer normal to Ω along Σ, and $\beta(\varepsilon)$ is a positive parameter.

The weak formulation of (2.5) reads: Find $u^\varepsilon \in \mathbf{V}$ satisfying

$$\int\limits_\Omega \nabla u^\varepsilon . \nabla v \,\mathrm{d}x + \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} \sigma(\hat{x}, u^\varepsilon) v \,\mathrm{d}\hat{x} = \int\limits_\Omega f v \,\mathrm{d}x,\ \forall v \in \mathbf{V}, \tag{2.6}$$

where the space $\mathbf{V}$ is obtained by completion of $\{v \in C^1(\overline{\Omega}) : v = 0 \text{ on } \Gamma_\Omega\}$ with respect to the Dirichlet norm.

The existence and uniqueness of solution u^ε of (2.6) holds from that of the variational inequality

$$\langle A^\varepsilon u^\varepsilon, v - u^\varepsilon\rangle \geq (f, v - u^\varepsilon)_{L^2(\Omega)}, \quad \forall v \in \mathbf{V},$$

where $A^\varepsilon : \mathbf{V} \longmapsto \mathbf{V}'$ is the monotonic hemicontinuous operator defined by

$$\langle A^\varepsilon u, v\rangle = \int\limits_\Omega \nabla u . \nabla v \,\mathrm{d}x + \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} \sigma(\hat{x}, u) v \,\mathrm{d}\hat{x}, \quad \text{for } u, v \in \mathbf{V}$$

(see Theorems 8.2-8.4 in Sections II.8.2 and II.8.3 of [24], and, also, see Section I.2 in [3] and Theorem 2.1 in [15] for a detailed application of these results), and this amounts to: Find $u^\varepsilon \in \mathbf{V}$ satisfying

$$\int\limits_\Omega \nabla v . \nabla(v - u^\varepsilon) \,\mathrm{d}x + \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} \sigma(\hat{x}, v)(v - u^\varepsilon) \,\mathrm{d}\hat{x} \geq \int\limits_\Omega f(v - u^\varepsilon) \,\mathrm{d}x,\ \forall v \in \mathbf{V}. \tag{2.7}$$

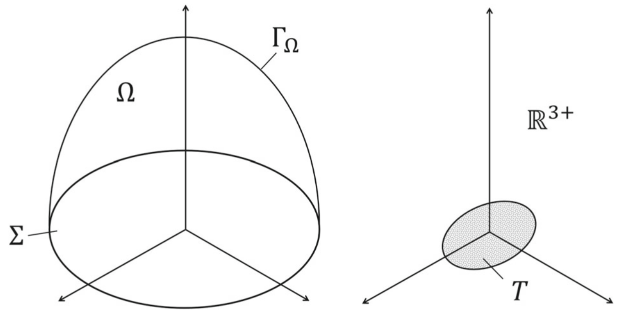

FIG. 4. The domain of setting for homogenized and local problems

On account of the Poincaré inequality (2.3) and (2.4), u^ε satisfies and

$$\int_\Omega |\nabla u^\varepsilon|^2 \, dx \leq C, \quad \beta(\varepsilon) \int_{\bigcup T^\varepsilon} \sigma(\hat{x}, u^\varepsilon) u^\varepsilon \, d\hat{x} \leq C \quad \text{and} \quad \beta(\varepsilon) \int_{\bigcup T^\varepsilon} |u^\varepsilon|^2 \, d\hat{x} \leq C, \tag{2.8}$$

where C is a constant independent of ε. Hence, for any sequence, we can extract a subsequence, still denoted by ε such that

$$u^\varepsilon \to u^0 \text{ in } H^1(\Omega) - \textit{weak}, \quad \text{as } \varepsilon \to 0, \tag{2.9}$$

for some $u^0 \in \mathbf{V}$. The aim of this work is to identify u^0 with the unique solution of a certain homogenized problem which depends on the different relations for the parameters ε, r_ε and $\beta(\varepsilon)$.

Depending on the values β^*, r_0 and β^0 in (1.1), (1.2) and (1.3), respectively, in Sect. 2.1, we state all the possible homogenized problems: (2.11), (2.15), (2.18), (2.19) and (2.20). In Sects. 3–5, we provide the proof of the convergence in the critical cases where $r_0 > 0$ and $\beta^0 > 0$ or $\beta^0 = +\infty$. The corresponding homogenized problems are (2.11) and (2.15), respectively; see Remark 5.2 for other cases.

Remark 2.1. As outlined in [10], the homogenization problem (2.5) is a well-posed problem under less restrictive conditions for σ. Indeed, it suffices to guarantee that the boundary integral

$$\int_{\bigcup T^\varepsilon} \sigma(\hat{x}, u^\varepsilon(\hat{x})) v(\hat{x}) \, d\hat{x}$$

is well defined for u^ε, $v \in H^1(\Omega)$ and the left-hand side of (2.6) defines a monotonic operator.

Several papers in the literature consider the case of a smooth σ which satisfies $\sigma \in C^1(\overline{\Omega} \times \mathbb{R})$, $\sigma(x, 0) = 0$ and, for instance,

$$0 < K_2 \leq \frac{\partial \sigma}{\partial u}(x, u) \leq K_1(1 + |u|^\tau) \quad \forall x \in \overline{\Omega},\ u \in \mathbb{R}, \qquad \text{for a } \tau \in [0, 2]. \tag{2.10}$$

These hypotheses on smoothness and boundedness for σ are weakened in our hypotheses (2.2)–(2.4). However, it should be noted that (2.10) already allows a certain nonlinear increasing of σ. Many models arising in hydrology and ecology use nonlinear functions which fall in the framework of these hypotheses or even of more restrictive hypotheses on σ which somehow imply a linear increasing; namely, when $0 < K_2 \leq \frac{\partial \sigma}{\partial u}(x, u) \leq K_1$ (see [1] and [10] in this connection). □

2.1. The homogenized problems and the local problems

The technique of matched asymptotic expansions, which follows from that in [9,10] and [12], with the suitable modifications, leads us to the homogenized problems listed below:

- In the most critical situation when $\beta^0 > 0$ and $r_0 > 0$, the homogenized problem reads

$$\begin{cases} -\Delta_x u^0 = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \Gamma_\Omega, \\ \dfrac{\partial u^0}{\partial n_x} + r_0\, \mathcal{C}^e(x, u^0) u^0 = 0 \text{ on } \Sigma, \end{cases} \tag{2.11}$$

where $\mathcal{C}^e$ is the function defined as:

$$\mathcal{C}^e(x, u) = \int_T \frac{\partial W^{x,u}}{\partial n_y} \mathrm{d}\hat{y}, \tag{2.12}$$

$W^{x,u}$ being the solution of the (x, u)-*dependent local problem*

$$\begin{cases} -\Delta_y W^{x,u} = 0 & \text{in } \mathbb{R}^{3+}, \\ \dfrac{\partial W^{x,u}}{\partial n_y} = 0 & \text{on } \{y_3 = 0\} \setminus T, \\ u \dfrac{\partial W^{x,u}}{\partial n_y} - \beta^0 \sigma(x, (1 - W^{x,u})u) = 0 & \text{on } T, \\ W^{x,u}(y) \to 0 & \text{as } |y| \to \infty,\ y_3 > 0. \end{cases} \tag{2.13}$$

Above, and in what follows, the variable y denotes an auxiliary variable in $\mathbb{R}^3$ (cf. (2.14)), and the lower indexes x or y indicate the variable for derivatives, while the upper indexes x, u refer to the parameter arising in the equation on T, which deals with the macroscopic variable x. Note that we have indeed a biparametric family of local problems, x, u being the two parameters. As is well known, macroscopic and local variables are related by

$$y = \frac{x - \widetilde{x}_{\mathrm{k}}}{r_\varepsilon}. \tag{2.14}$$

It is self-evident that for $u = 0$, the trivial equality of the boundary condition on T in (2.13) gives nothing, and we set $W^{x,0} = 0$.

- For the critical size $r_0 > 0$, when $\beta^0 = +\infty$, the homogenized problem reads

$$\begin{cases} -\Delta_x u^0 = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \Gamma_\Omega, \\ \dfrac{\partial u^0}{\partial n_x} + r_0\, \mathcal{C} u^0 = 0 \text{ on } \Sigma, \end{cases} \tag{2.15}$$

where $\mathcal{C}$ is now a constant defined as:

$$\mathcal{C} = \left\langle \frac{\partial W}{\partial n_y}, 1 \right\rangle_{H^{-1/2}(T) \times H^{1/2}(T)}, \tag{2.16}$$

W being the solution of the *local problem*

$$\begin{cases} -\Delta_y W = 0 & \text{in } \mathbb{R}^{3+}, \\ \dfrac{\partial W}{\partial n_y} = 0 & \text{on } \{y_3 = 0\} \setminus T, \\ W = 1 & \text{on } T, \\ W(y) \to 0 & \text{as } |y| \to \infty,\ y_3 > 0. \end{cases} \tag{2.17}$$

- For $\beta^* > 0$ and large sizes $r_0 = +\infty$, the homogenized problem reads

$$\begin{cases} -\Delta_x u^0 = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \Gamma_\Omega, \\ \dfrac{\partial u^0}{\partial n_x} + \beta^* |T| \sigma(x, u^0) = 0 \text{ on } \Sigma. \end{cases} \tag{2.18}$$

- For the extreme cases where $\beta^* = 0$ or $r_0 = 0$, the homogenized problem is

$$\begin{cases} -\Delta_x u^0 = f \text{ in } \Omega, \\ u^0 = 0 \qquad \text{on } \Gamma_\Omega, \\ \dfrac{\partial u^0}{\partial n_x} = 0 \qquad \text{on } \Sigma. \end{cases} \tag{2.19}$$

- For the extreme cases where $r_0 = +\infty$ and, $\beta^0 > 0$, or $\beta^0 = +\infty$, or $\beta^0 = 0$ and $\beta^* = +\infty$, the homogenized problem is the Dirichlet problem

$$\begin{cases} -\Delta_x u^0 = f \text{ in } \Omega, \\ u^0 = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

The variational formulations of (2.20) in $H_0^1(\Omega)$, and of (2.19) in $\mathbf{V}$ are classical in the literature. The existence and uniqueness of a weak solution of (2.18) in $\mathbf{V}$ holds as that of (2.5). The existence of a unique solution in $\mathbf{V}$ of the linear problem (2.15) is a consequence of the fact that the capacity constant $\mathcal{C}$ is positive (see (5.8)). Its variational formulation reads: find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \nabla_x u^0 . \nabla_x v \, \mathrm{d}x + r_0 \, \mathcal{C} \int_\Sigma u^0 v \, \mathrm{d}\hat{x} = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in \mathbf{V}. \tag{2.21}$$

Regarding problem (2.11), we show that $\mathcal{C}^e(x,u) > 0$ in (2.12) from the properties (see (3.5)) of the solution of the local problem (2.13). Let us define the function Ξ as:

$$\Xi(x,u) := \mathcal{C}^e(x,u)u \quad \text{ for } u \neq 0, \text{ and } \Xi(x,0) = 0. \tag{2.22}$$

It is clear that Ξ depends on the nonlinear function σ, on the parameter β^0, and on the shape of the unit reaction region T. For $\tau \in [0, \sqrt{3} - 1]$, in Sect. 3, we show that Ξ satisfies analogous properties to σ (cf. Proposition 3.3 and Remark 2.2) and, therefore, the existence and uniqueness of a weak solution of (2.11) holds true.

The weak formulation of problem (2.11) reads: Find $u^0 \in \mathbf{V}$ satisfying

$$\int_\Omega \nabla_x u^0 . \nabla_x v \, \mathrm{d}x + r_0 \int_\Sigma \mathcal{C}^e(\hat{x}, u^0) u^0 v \, \mathrm{d}\hat{x} = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in \mathbf{V},$$

or equivalently,

$$\int_\Omega \nabla_x u^0 . \nabla_x v \, \mathrm{d}x + r_0 \int_\Sigma \Xi(\hat{x}, u^0) v \, \mathrm{d}\hat{x} = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in \mathbf{V}. \tag{2.23}$$

Rewriting the reasoning for (2.6) and (2.7), we show that (2.23) has a unique solution which coincides the unique solution $u^0 \in \mathbf{V}$ of the variational inequality

$$\int_\Omega \nabla_x v . \nabla_x (v - u^0) \, \mathrm{d}x + r_0 \int_\Sigma \Xi(\hat{x}, v)(v - u^0) \, \mathrm{d}\hat{x} \geq \int_\Omega f(v - u^0) \, \mathrm{d}x, \quad \forall v \in \mathbf{V}. \tag{2.24}$$

Note that in the case where $\Xi(\hat{x}, v) \equiv \mathcal{C}v$, also the solution of (2.21) is the unique solution of (2.24).

Remark 2.2. In connection with Remark 2.1, it should be emphasized that the Lipschitz condition (2.2) becomes essential in order to show both the continuity on the *macroscopic parameters* of the solutions of the local problem (2.13) and the correct position of the homogenized problem (2.11). The further restrictions on τ, which we perform throughout the paper, seem to be technical questions to be overcome. As a matter of fact, the restriction $\tau \in [0, \sqrt{3} - 1]$ has been obtained in [10] in connection with the correct setting of the homogenized problem for the case of perforated media. More specifically, related to the local problem, in Proposition 4.2 of [10] we prove the estimate

$$|\Xi(x,u)| \leq C(|u| + |u|^{(\tau+1)^2}) \quad \forall x \in \overline{\Omega}, u \in \mathbb{R}, \tau \in [0,1), \tag{2.25}$$

which allows us to define correctly the surface integral, cf. (2.23),

$$\int_{\Sigma} \Xi(\hat{x}, u^0) v \, \mathrm{d}\hat{x}$$

for $u^0, v \in \mathbf{V}$ and a $\tau \in [0, \sqrt{3} - 1]$.

On account of this restriction, even for a larger τ, but $\tau \in [0, 1)$, we obtain the bounds arising in (3.9)–(3.14) and the limit (4.7), namely the necessary bounds to get (4.8). Finally, the last restriction $\tau \in [0, \frac{\sqrt{5}-1}{2}] \subset [0, \sqrt{3} - 1] \subset [0, 1)$, is needed when applying a density argument in the integral on Σ in which the extended capacity appears (cf. (4.4) and (4.5)). □

3. Solutions of the parametric family of local problems

In this section, we deal with the abstract framework of the parameter family of local problems (2.13) and the properties of their solutions in their dependence on the parameters $x \in \overline{\Omega}$ and $u \in \mathbb{R}$. These properties become essential to show the correct setting of (2.23) and (2.24) and to derive the convergence of solutions of the homogenization problem (2.5) toward that of (2.11), as $\varepsilon \to 0$.

Let $\mathcal{D}(\overline{\mathbb{R}^{3+}})$ be the space of functions which are the restrictions to $\overline{\mathbb{R}^{3+}}$ of the elements of $\mathcal{D}(\mathbb{R}^3)$. Consider the space $\mathfrak{V}$, completion of $\mathcal{D}(\overline{\mathbb{R}^{3+}})$ with respect to the Dirichlet norm

$$\|U\|_{\mathfrak{V}} = \|\nabla_y U\|_{L^2(\mathbb{R}^{3+})} \,. \tag{3.1}$$

As it is well known, the elements of $\mathfrak{V}$ belong to $L^6(\mathbb{R}^{3+})$ and to $H^1_{loc}(\mathbb{R}^{3+})$, and the continuous embedding $\mathfrak{V} \subset L^2(T)$ holds, namely

$$\|U\|_{L^2(T)} \le C \|U\|_{\mathfrak{V}} \quad \forall U \in \mathfrak{V}, \tag{3.2}$$

with C a constant independent of U. This ensures that the integrals arising in (3.3) are well defined and also that the elements of $\mathfrak{V}$ somehow converge toward zero as $|y| \to \infty$ (see, e.g., Section I.4 of [21] and Section IV.8 of [34]).

Problem (2.13) has a weak formulation: Find $W^{x,u} \in \mathfrak{V}$ satisfying

$$u \int_{\mathbb{R}^{3+}} \nabla_y W^{x,u} . \nabla_y V \, \mathrm{d}y - \beta_0 \int_T \sigma(x, (1 - W^{x,u}(\hat{y}))u) V \, \mathrm{d}\hat{y} = 0, \qquad \forall V \in \mathfrak{V}. \tag{3.3}$$

In the next theorem, we show the existence and uniqueness of solution of (3.3) in $\mathfrak{V}$, as well as its precise behavior at infinity (cf. (2.13)).

Theorem 3.1. *Problem (3.3) has a unique solution $W^{x,u} \in \mathfrak{V}$ and it satisfies*

$$W^{x,u}(y) = \frac{\mathcal{K}(x,u)}{|y|} + O(\frac{1}{|y|^2}) \quad \text{as} \quad |y| \to \infty, \tag{3.4}$$

where $\mathcal{K}(x,u)$ is a constant, independent of y, but dependent on the parameters of the problem x and u. In addition, we have the following chain of equalities defining $\mathcal{K}(x,u)$:

$$\begin{aligned} 2\pi\mathcal{K}(x,u) &= \int_T \frac{\partial W^{x,u}}{\partial n_y} \mathrm{d}\hat{y} = \frac{\beta_0}{u} \int_T \sigma(x, (1 - W^{x,u}(y))u) \mathrm{d}\hat{y} \\ &= \int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2 \mathrm{d}y + \frac{\beta_0}{u^2} \int_T \sigma(x, (1 - W^{x,u}(\hat{y}))u)(1 - W^{x,u}(\hat{y}))u \, \mathrm{d}\hat{y}. \end{aligned} \tag{3.5}$$

Also, the function (2.22) reads

$$\Xi(x, 0) = 0 \quad \text{and} \quad \Xi(x,u) = 2\pi\mathcal{K}(x,u)u, \quad \forall u \in \mathbb{R},\ u \ne 0. \tag{3.6}$$

Proof. To show the existence and uniqueness of the solution of (3.3), we use the theory of monotonic operators (cf., e.g., Theorem 8.2 in Section II.8.2 in [24]) and rewrite the proof in Theorem 4.1 of [10], with the suitable modifications.

Now, let us deduce the representation (3.4), for a certain constant $\mathcal{K}(x,u)$. Since $W^{x,u}$ is harmonic in $\mathbb{R}^{3+}$ and satisfies a Neumann condition on the plane outside the circle $B(0,R_0)\cap\{y_3=0\}$, with R_0 such that $\overline{T}\subset B(0,R_0)$, we can extend it to a harmonic function in $\mathbb{R}^3\setminus B(0,R_0)$, denoted by $\mathrm{W}^{x,u}$, satisfying $\nabla_y \mathrm{W}^{x,u}\in (L^2(\mathbb{R}^3\setminus B(0,R_0)))^3$. The representation of $\mathrm{W}^{x,u}$ in $L^2(\partial B(0,R_0))$ in terms of the spherical harmonics leads to (3.4) for a certain $\mathcal{K}(x,u)$ independent of the y variable: see, for instance, Section II in [23] and Section IV.8 in [34]. In fact, formula (3.5) which we show below gives a characterization of this constant $\mathcal{K}(x,u)$ in terms of the parameters x and u.

Now, taking $V=W^{x,u}$ in (3.3) gives

$$\int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2 \mathrm{d}y = \frac{\beta_0}{u}\int_T \sigma(x,(1-W^{x,u})u)W^{x,u}\mathrm{d}\hat{y},$$

when $u\neq 0$, and straightforward computations provide the last equality in (3.5). Moreover, the equation on T in (2.13) gives the second equality in (3.5). Finally, to show the first equality, we apply again the Green formula in $B^+(0,R)$ and consider (3.4). Thus,

$$\int_T \frac{\partial W^{x,u}}{\partial n_y}\mathrm{d}\hat{y} = -\lim_{R\to\infty}\int_{\Gamma_R^+} \frac{\partial W^{x,u}}{\partial \nu_y}\mathrm{d}s_y = 2\pi\mathcal{K}(x,u).$$

Since $\mathcal{C}^e(x,u)$ defined in (2.12) is also defined by any term of (3.5), the formula (3.6) for Ξ, is a consequence of the definition (2.22). Note that this is in good agreement with the fact that all the integrals in (3.5) vanish when $u=0$. Thus, the theorem holds. □

Under the hypotheses (2.2)–(2.4), the following results provide further properties of the functions $W^{x,u}$ and $\Xi(u,v)$ which are useful in the proof of the convergence.

As a consequence of Theorem 3.1, in Proposition 3.2 we show that the solution $W^{x,u}$ of (2.13) as well as certain other related functions are continuous functions of the parameters $x\in\overline{\Omega}$ and $u\in\mathbb{R}$. First, we obtain:

Proposition 3.1. *Let σ satisfy (2.2)–(2.4) with $\tau\in[0,2]$. Then, for all $(x,u)\in\overline{\Omega}\times\mathbb{R}$, the solution $W^{x,u}$ of (2.13) verifies estimates*

$$\|W^{x,u}\|_{L^2(T)}\leq C(1+|u|^\tau) \quad \text{and} \quad \|\nabla_y W^{x,u}\|_{L^2(\mathbb{R}^{3+})}\leq C(1+|u|^\tau) \tag{3.7}$$

where C is a constant independent of x and u.

Proof. First, let us note that for $u=0$ the above estimates hold since the function $W^{x,0}=0$ and all the norms above vanish in this case.

For $u\neq 0$, using the monotonicity (2.3) of σ and the weak formulation of (2.13), cf. (3.3), with $V=W^{x,u}/u$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2\mathrm{d}y\\
&\leq \int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2\mathrm{d}y + \frac{\beta_0}{u^2}\int_T \big(\sigma(x,(1-W^{x,u})u)-\sigma(x,u)\big)\big((1-W^{x,u})u-u\big)\,\mathrm{d}\hat{y}\\
&= \int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2\mathrm{d}y - \frac{\beta_0}{u}\int_T \sigma(x,(1-W^{x,u})u)W^{x,u}\,\mathrm{d}\hat{y} + \frac{\beta_0}{u}\int_T \sigma(x,u)W^{x,u}\mathrm{d}\hat{y}
\end{aligned}$$

$$= \frac{\beta_0}{u} \int_T \sigma(x,u) W^{x,u} \mathrm{d}\hat{y}.$$

Now, applying (2.2), the Cauchy–Bunyakovsky–Schwarz inequality and (3.2), we get

$$\int_{\mathbb{R}^{3+}} |\nabla_y W^{x,u}|^2 \mathrm{d}y \le C_1 \int_T (1+|u|^\tau)|W^{x,u}| \mathrm{d}\hat{y} \le C_2(1+|u|^\tau)\|\nabla_y W^{x,u}\|_{L^2(\mathbb{R}^{3+})},$$

and consequently, cf. (3.2), the estimates in (3.7) hold with C a constant independent of x and u. In particular, for $\tau = 0$,

$$\|W^{x,u}\|_{L^2(T)} \le C \qquad \text{and} \qquad \|\nabla_y W^{x,u}\|_{L^2(\mathbb{R}^{3+})} \le C. \tag{3.8}$$

□

Proposition 3.2. *Under the hypotheses of Proposition 3.1, with $\tau \in [0,1)$, the function $W^{x,u}u$ depends continuously on $(x,u) \in \overline{\Omega} \times \mathbb{R}$ in the topology of $L^2(T)$ and $\mathfrak{V}$. Also, the functions*

$$\int_T \sigma(x,(1-W^{x,u})u)\mathrm{d}\hat{y} \qquad \textit{and} \qquad \mathcal{K}(x,u)u$$

depend continuously on $(x,u) \in \overline{\Omega} \times \mathbb{R}$, and the following estimates hold:

$$\|W^{x,u}u - W^{x',u'}u'\|_{L^2(T)} \le C\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big], \tag{3.9}$$

$$\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|_{L^2(\mathbb{R}^{3+})} \le C\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big], \tag{3.10}$$

$$\|W^{x,u} - W^{x',u'}\|_{L^2(T)} \le C\frac{1}{|u|}\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big], \quad \textit{with } u \ne 0, \tag{3.11}$$

$$\|\nabla_y(W^{x,u} - W^{x',u'})\|_{L^2(\mathbb{R}^{3+})} \le C\frac{1}{|u|}\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big], \quad \textit{with } u \ne 0, \tag{3.12}$$

$$\begin{aligned} &\Big|\int_T \sigma(x,(1-W^{x,u}(\hat{y}))u) - \sigma(x',(1-W^{x',u'}(\hat{y}))u')\,\mathrm{d}\hat{y}\Big| \\ &\le C\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big]\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big), \end{aligned} \tag{3.13}$$

$$\big|\mathcal{K}(x,u)u - \mathcal{K}(x',u')u'\big| \le C\big[|x-x'| + |u-u'|\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big)\big]\big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\big), \tag{3.14}$$

$\forall (x,u),\ (x',u') \in \overline{\Omega} \times \mathbb{R}$.

In addition, for each $\phi \in C^1(\overline{\Omega})$ with $\phi = 0$ on Γ_Ω, the function $\Theta(x) := \phi(x)\mathcal{K}(x,\phi(x))$ is a continuous function whose distributional partial derivatives satisfy

$$\frac{\partial \Theta}{\partial x_i} \in L^\infty(\Omega), \quad i = 1,2,3. \tag{3.15}$$

Proof. For $x, x' \in \overline{\Omega}$, $u, u' \in \mathbb{R}$, let us consider (3.3) for $V = W^{x,u}u - W^{x',u'}u'$, and the weak formulation of problem (2.13), with parameters (x',u') instead of (x,u), for $V = W^{x,u}u - W^{x',u'}u'$, and subtract both expression. Thus,

$$\begin{aligned} &\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|^2_{L^2(\mathbb{R}^{3+})} \\ &= u\int_{\mathbb{R}^{3+}} \nabla_y W^{x,u}.\nabla_y(W^{x,u}u - W^{x',u'}u')\,\mathrm{d}y - u'\int_{\mathbb{R}^{3+}} \nabla_y W^{x',u'}.\nabla_y(W^{x,u}u - W^{x',u'}u')\,\mathrm{d}y \\ &= \beta_0 \int_T \big(\sigma(x,(1-W^{x,u})u) - \sigma(x',(1-W^{x',u'})u')\big)\big(W^{x,u}u - W^{x',u'}u'\big)\,\mathrm{d}\hat{y} \end{aligned}$$

and

$$\beta_0 \int_T \left(\sigma(x,(1-W^{x,u})u) - \sigma(x',(1-W^{x',u'})u')\right)\left((1-W^{x,u})u - (1-W^{x',u'})u'\right) \mathrm{d}\hat{y}$$
$$+\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|^2_{L^2(\mathbb{R}^{3+})}$$
$$= \beta_0 \int_T \left(\sigma(x,(1-W^{x,u})u) - \sigma(x',(1-W^{x',u'})u')\right)\left(u-u'\right) \mathrm{d}\hat{y}. \tag{3.16}$$

Now, using (2.2), the Cauchy–Bunyakovsky–Schwarz inequality and, for $0<\tau<1$, the Hölder inequality with $p=1/\tau$ and $q=1/(1-\tau)$, we obtain

$$\beta_0 \int_T \left(\sigma(x,(1-W^{x,u})u) - \sigma(x',(1-W^{x',u'})u')\right)\left(u-u'\right) \mathrm{d}\hat{y}$$
$$\leq C_1|x-x'||u-u'|$$
$$+C_2|u-u'|\int_T \left(|u-u'| + |W^{x,u}u - W^{x',u'}u'|\right)\left(1+|u|^\tau|1-W^{x,u}|^\tau + |u'|^\tau|1-W^{x',u'}|^\tau\right) \mathrm{d}\hat{y}$$
$$\leq C_1|x-x'||u-u'| + C_3\Big(|u-u'|^2 + |u-u'|\|W^{x,u}u - W^{x',u'}u'\|_{L^2(T)}\Big)$$
$$\Big(1+|u|^\tau\|1-W^{x,u}\|^\tau_{L^2(T)} + |u'|^\tau\|1-W^{x',u'}\|^\tau_{L^2(T)}\Big). \tag{3.17}$$

Therefore, gathering (2.3), (3.16), (3.17), (3.2) and (3.7) yields

$$\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|^2_{L^2(\mathbb{R}^{3+})} \leq C_1|x-x'||u-u'|$$
$$+C_3\Big(|u-u'|^2 + |u-u'|\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|_{L^2(\mathbb{R}^{3+})}\Big)\Big(1+|u|^{\tau+\tau^2}+|u'|^{\tau+\tau^2}\Big). \tag{3.18}$$

Here, we have also used that

$$1+|u|^\tau(1+|u|^\tau)^\tau \leq 2(1+|u|^{\tau+\tau^2}) \qquad \text{for any } u\in\mathbb{R} \text{ and } 0\leq\tau<1. \tag{3.19}$$

Consequently, (3.10) is proved because either $\|\nabla_y(W^{x,u}u - W^{x',u'}u')\|_{L^2(\mathbb{R}^{3+})} \leq |x-x'|+|u-u'|$ and (3.10) holds or $|u-u'| \leq |x-x'|+|u-u'| \leq \|\nabla_y(W^{x,u}u - W^{x',u'}u')\|_{L^2(\mathbb{R}^{3+})}$ and, due to (3.18), (3.10) holds.

Estimate (3.12) follows from (3.10), (3.7), and the fact that

$$W^{x,u} - W^{x',u'} = \frac{1}{u}(W^{x,u}u - W^{x',u'}u') + \frac{u'-u}{u}W^{x',u'} \qquad \text{for } u\neq 0.$$

Estimates (3.9) and (3.11) are a direct consequence of the continuous embedding $\mathfrak{V}\subset L^2(T)$ (see (3.2)) and estimates (3.10) and (3.12), respectively.

As regards (3.13) and (3.14), we use again (2.2) and the Hölder inequality to obtain

$$\int_T \left(\sigma(x,(1-W^{x,u})u) - \sigma(x',(1-W^{x',u'})u')\right) \mathrm{d}\hat{y}$$
$$\leq C_1|x-x'| + C_3\Big(|u-u'| + \|W^{x,u}u - W^{x',u'}u'\|_{L^2(T)}\Big)$$
$$\Big(1+|u|^\tau\|1-W^{x,u}\|^\tau_{L^2(T)} + |u'|^\tau\|1-W^{x',u'}\|^\tau_{L^2(T)}\Big)$$

(cf. (3.17)). Now, taking into account (3.7), (3.9) and (3.19) yields (3.13). Estimate (3.14) is a consequence of (3.13) and (3.5).

Let us show the last assertion in the statement of the theorem. The continuity of Θ in $\overline{\Omega}$ follows from that of ϕ and $\mathcal{K}(x,\phi(x))$. The assertion on the derivative is obtained from (3.14), which taking $u=\phi(x)$ and $u'=\phi(x')$ reads

$$\begin{aligned}&\big|\phi(x)\mathcal{K}(x,\phi(x))-\phi(x')\mathcal{K}(x',\phi(x'))\big|\\&\le C\big[|x-x'|+|\phi(x)-\phi(x')|\big(1+|\phi(x)|^{\tau+\tau^2}+|\phi(x')|^{\tau+\tau^2}\big)\big]\big(1+|\phi(x)|^{\tau+\tau^2}+|\phi(x')|^{\tau+\tau^2}\big).\end{aligned}$$

From the smoothness for ϕ and its partial derivatives, we write the global Lipschitz condition of Θ,

$$|\Theta(x)-\Theta(x')|\le C_\phi|x-x'|, \tag{3.20}$$

with a constant independent of x and x', and (3.15) also holds, see Sections III.24 and III.28 in [38]. Thus, the proposition is proved. □

Proposition 3.3. *Under the assumptions* (2.2)–(2.3) *with* $\tau\in[0,1)$*, the function* Ξ *defined by (2.22) satisfies* $\Xi\in C(\overline{\Omega}\times\mathbb{R})$*,*

$$\Xi(x,0)=0,\quad (\Xi(x,u)-\Xi(x,v))(u-v)\ge 0, \tag{3.21}$$

$$|\Xi(x,u)-\Xi(x,v)|\le C|u-v|(1+|u|^{\tau+\tau^2}+|v|^{\tau+\tau^2})^2, \tag{3.22}$$

$$|\Xi(x,u)u-\Xi(x,v)v|\le C|u-v|(|u|+|v|)(1+|u|^{\tau+\tau^2}+|v|^{\tau+\tau^2})^2 \tag{3.23}$$

$\forall x\in\overline{\Omega}$, $u,v\in\mathbb{R}$.

Proof. The continuity of function (2.22) in $\overline{\Omega}\times\mathbb{R}$ is a consequence of the continuity of $\mathcal{K}(x,u)u$ stated in Proposition 3.2 and equation (3.6). To prove (3.21), we rewrite the proof in Proposition 4.2 in [10] with minor modifications.

Inequality (3.22) follows from (3.6) and (3.14). Besides, taking $v=0$ in (3.22), we obtain

$$|\Xi(x,u)|\le C|u|(1+|u|^{\tau+\tau^2})^2,\qquad \forall x\in\overline{\Omega}, u\in\mathbb{R} \tag{3.24}$$

(cf. also (2.25)). Finally, to prove (3.23) we write

$$\Xi(x,u)u-\Xi(x,v)v=\Xi(x,u)(u-v)+(\Xi(x,u)-\Xi(x,v))v$$

and use (3.24) and (3.22). Thus, the proposition is proved. □

Proposition 3.4. *For* $x\in\overline{\Omega}$*,* $u\in\mathbb{R}$*, the solution* $W^{x,u}$ *of (3.3) satisfies*

$$|W^{x,u}(y)|\le C_{u,\tau}\frac{C}{d(y,\overline{T})}\quad\text{and}\quad\left|\frac{\partial W^{x,u}}{\partial y_j}(y)\right|\le C_{u,\tau}\frac{C}{d(y,\overline{T})^2},\quad \forall y\in\mathbb{R}^{3+}, \tag{3.25}$$

for $j=1,2,3$*, where* $d(y,\overline{T})$ *denotes the distance from the point* $y\in\mathbb{R}^{3+}$ *to* $\overline{T}$*,*

$$C_{u,\tau}=\big(1+|u|^{\tau+\tau^2}\big)^2 \text{ when } u\ne 0,\qquad C_{0,\tau}=0,$$

and C *is a constant independent of* x*,* u*,* τ *and* y*.*

Proof. Considering

$$q^{x,u}=\frac{\partial W^{x,u}}{\partial n_y}\Big|_T,$$

then a solution of (2.13) reads

$$W^{x,u}(y)=-\frac{1}{2\pi}\int_T\frac{1}{\sqrt{(y_1-\xi_1)^2+(y_2-\xi_2)^2+y_3^2}}\,q^{x,u}(\xi_1,\xi_2)\mathrm{d}\hat{\xi},\ \forall y\in\mathbb{R}^{3+}$$

(cf., e.g., [22,31] and [35]). To show that this function belongs to $\mathfrak{V}$, we follow the technique in Theorem 4.1 in [26] with minor modifications.

As a consequence, we have (3.25) for $C_{u,\tau} \equiv C_{x,u,\tau}$ a certain constant, which in principle can depend on x, u and τ. However, considering the equation on T in (2.13), (2.2), (2.3) and Proposition 3.2 (cf., e.g., (3.13) with $x = x'$ and $u' = 0$), we can take:

$$|W^{x,u}(y)| \leq \frac{C_{x,u,\tau}}{d(y,\overline{T})}$$

with

$$C_{x,u,\tau} = \frac{1}{2\pi} \int\limits_T |q^{x,u}(\hat{\xi})| \, \mathrm{d}\hat{\xi} = \frac{1}{2\pi} \frac{\beta^0}{|u|} \int\limits_T |\sigma(x,(1-W^{x,u})u)| \, \mathrm{d}\hat{\xi}$$
$$\leq C\big(1 + |u|^{\tau+\tau^2}\big)^2,$$

where C is a constant independent of x, u, τ and y.

The same reasoning shows (3.25) for the derivatives. Thus, the proposition holds. □

3.1. On the test functions in the most critical case

In this section, we introduce some functions which allow us to obtain the convergence of the homogenization problem when $r_0 > 0$ and $\beta^0 > 0$. These auxiliary functions are constructed from the solutions of the parametric family of local problems (2.13), $\widetilde{x}_{\mathrm{k}}$ and $u(\widetilde{x}_{\mathrm{k}})$ being the parameters. Throughout the section, we assume that u is a function such that $u(\widetilde{x}_{\mathrm{k}})$ is defined, $\widetilde{x}_{\mathrm{k}} \in \Sigma$, u bounded on $\overline{\Omega}$:

$$|u(x)| \leq C_u, \quad \forall x \in \overline{\Omega}, \tag{3.26}$$

where C_u is a constant independent of x.

Let us consider $\varphi \in C^\infty[0,1]$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0,1/8]$ and $\mathrm{Supp}(\varphi) \subset [0,1/4]$. We construct the function

$$\varphi^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} \overline{B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{8}\right)}, \\ \varphi\Big(\dfrac{|x-\widetilde{x}_{\mathrm{k}}| - r_\varepsilon}{\varepsilon}\Big) & \text{for } x \in \mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}, \quad \mathrm{k} \in \mathcal{J}^\varepsilon, \\ 0 & \text{for } x \in \Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right), \end{cases} \tag{3.27}$$

where $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 \,:\, T^\varepsilon_{\widetilde{x}_{\mathrm{k}}} \subset \Sigma\}$, $B^+(\widetilde{x}_{\mathrm{k}}, r)$ denotes the half-ball of radius r centered at the point $\widetilde{x}_{\mathrm{k}}$, namely, $B(\widetilde{x}_{\mathrm{k}}, r) \cap \{x_3 > 0\}$, and $\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}$ stands for the half-annulus

$$\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}} = B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right) \setminus B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{8}\right).$$

Let us define the functions $\widetilde{W}^{\varepsilon,u}(x)$, which we construct from the solutions of the local problems (2.13), as follows: We set

$$W^{\mathrm{k},\varepsilon,u}(x) = W^{\widetilde{x}_{\mathrm{k}},u(\widetilde{x}_{\mathrm{k}})}\left(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\right)\varphi^\varepsilon(x) \quad \text{for} \quad x \in B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right),$$

and

$$\widetilde{W}^{\mathrm{k},\varepsilon,u}(x) = 1 - W^{\mathrm{k},\varepsilon,u}(x) \quad \text{for} \quad x \in B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right),$$

that we extend by 1 in $\Omega \setminus \bigcup B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right)$. Then, we define

$$\widetilde{W}^{\varepsilon,u}(x) = \begin{cases} \widetilde{W}^{\mathrm{k},\varepsilon,u}(x) & \text{for } x \in B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right), \, \mathrm{k} \in \mathcal{J}^\varepsilon, \\ 1 & \text{for } x \in \Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_{\mathrm{k}}, r_\varepsilon + \frac{\varepsilon}{4}\right). \end{cases} \tag{3.28}$$

Proposition 3.5. *There is a constant C independent of ε such that $\forall x \in \mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}$:*

$$\left|\frac{\partial \varphi^\varepsilon}{\partial x_j}(x)\right| \leq C\frac{1}{\varepsilon}, \quad j=1,2,3, \tag{3.29}$$

$$\left|W^{\widetilde{x}_{\mathrm{k}},u(\widetilde{x}_{\mathrm{k}})}\Big(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial W^{\widetilde{x}_{\mathrm{k}},u(\widetilde{x}_{\mathrm{k}})}}{\partial x_j}\Big(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon^2}, \; j=1,2,3, \tag{3.30}$$

and

$$|W^{\mathrm{k},\varepsilon,u}(x)| \leq C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial W^{\mathrm{k},\varepsilon,u}}{\partial x_j}(x)\right| \leq C\frac{r_\varepsilon}{\varepsilon^2}, \quad j=1,2,3. \tag{3.31}$$

In addition,

$$\|\widetilde{W}^{\varepsilon,u}\|_{H^1(\Omega)} \leq C \quad \textit{and} \quad \widetilde{W}^{\varepsilon,u} \xrightarrow{\varepsilon\to 0} 1 \quad \text{in } H^1(\Omega)-\textit{weak}. \tag{3.32}$$

Proof. Bound (3.29) is a consequence of the definition (3.27), while bounds (3.30) are a consequence of (3.25). Estimates (3.29) and (3.30) give (3.31). Let us show (3.32).

First, we evaluate

$$\begin{aligned}\left\|\nabla\widetilde{W}^{\varepsilon,u}\right\|^2_{L^2(\Omega)} &= \sum_{\widetilde{x}_{\mathrm{k}}} \left\|\nabla_x W^{\mathrm{k},\varepsilon,u}\right\|^2_{L^2(\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}})} + \sum_{\widetilde{x}_{\mathrm{k}}} \left\|\nabla_x W^{\mathrm{k},\varepsilon,u}\right\|^2_{L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\varepsilon/8))} \\ &\leq C\frac{r_\varepsilon^2}{\varepsilon^4}\sum_{\widetilde{x}_{\mathrm{k}}}\int_{\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_{\mathrm{k}}}} \mathrm{d}x + r_\varepsilon \sum_{\widetilde{x}_{\mathrm{k}}} \left\|\nabla_y W^{\widetilde{x}_{\mathrm{k}},u(\widetilde{x}_{\mathrm{k}})}\right\|^2_{L^2(B^+(0,1+\varepsilon/(r_\varepsilon 8)))} \leq C \end{aligned} \tag{3.33}$$

where we have considered (3.31), (2.14), (3.7), (2.1) and $r_0 > 0$ in (1.2).

Then, we estimate

$$\begin{aligned}\left\|\widetilde{W}^{\varepsilon,u}-1\right\|^2_{L^2(\Omega)} &= \sum_{\widetilde{x}_{\mathrm{k}}} \left\|W^{\mathrm{k},\varepsilon,u}\right\|^2_{L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\varepsilon/4))} \\ &\leq C\varepsilon^2 \sum_{\widetilde{x}_{\mathrm{k}}} \left\|\nabla_x W^{\mathrm{k},\varepsilon,u}\right\|^2_{L^2(B^+(\widetilde{x}_{\mathrm{k}},r_\varepsilon+\varepsilon/4))} \\ &\leq \varepsilon^2 C \left\|\nabla_x \widetilde{W}^{\varepsilon,u}\right\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\end{aligned}$$

where we have used the definition (3.27) and the Poincaré inequality on each half-ball and (3.33). Thus, the convergence of $(\widetilde{W}^{\varepsilon,u}-1)$ toward zero in $L^2(\Omega)$ holds, as $\varepsilon \to 0$, and also the bound in (3.32) holds true. This concludes the proof of the proposition. □

On account of (3.26) and (3.25), for $\tau > 0$ in (2.2), the constants appearing in bounds (3.30)–(3.31) as well as in the proof of Propositions 3.4 and 3.5 may depend on the function u and the parameter τ, cf. (3.26). We avoid writing this dependence because, in the next section, these bounds will be applied with a fixed τ and also a fixed $u \equiv \phi \in C^1(\overline{\Omega})$ with $\phi|_{\Gamma_\Omega} = 0$.

4. The convergence for the most critical case

In this section, we consider the case where $r_0 > 0$ and $\beta^0 > 0$, and we show that the limit of u^ε in $H^1(\Omega)$-weak given by (2.9) is the weak solution of the homogenized problem (2.11). In order to do this, it proves useful to introduce here a convergence result of measures (see [25] for the proof).

Lemma 4.1. *Let $a_\varepsilon < \varepsilon$ be such that $a_\varepsilon \varepsilon^{-1} \to a_0$, as $\varepsilon \to 0$, and let $B(\widetilde{x}_{\rm k}, a_\varepsilon)$ denote the ball of radius a_ε centered at $\widetilde{x}_{\rm k}$. Then, $\forall w \in H_0^1(\Omega)$,*

$$\Big|\sum_{\widetilde{x}_{\rm k}} \int_{\partial B(\widetilde{x}_{\rm k}, a_\varepsilon)} w \,{\rm d}s_x - 4\pi a_0^2 \int_\Sigma w \,{\rm d}\hat{x}\Big| \leq C(\varepsilon^{1/2} + |a_\varepsilon \varepsilon^{-1} - a_0|)\|w\|_{H^1(\Omega)}.$$

Theorem 4.1. *For $r_0 > 0$ and $\beta^0 > 0$ in (1.2) and (1.3), and $\tau \in [0, \frac{\sqrt{5}-1}{2}]$ in (2.2), the solution of (2.6) converges in $H^1(\Omega)$-weak towards the solution of (2.23) as $\varepsilon \to 0$.*

Proof. Let us consider $\phi \in \{v \in C^1(\overline{\Omega}) \,:\, v = 0 \text{ on } \Gamma_\Omega\}$. From the definitions of the spaces $\mathfrak{V}$ and $\mathbf{V}$, and of the function $W^{\widetilde{x}_{\rm k}, \phi(\widetilde{x}_{\rm k})}$, on account of (3.28) and (3.32), we have that $\phi\widetilde{W}^{\varepsilon,\phi} \in \mathbf{V}$ and $\phi\widetilde{W}^{\varepsilon,\phi} \to \phi$ in $H^1(\Omega)$-weak as $\varepsilon \to 0$. Then, we take the test functions $v = \phi\widetilde{W}^{\varepsilon,\phi}$ in (2.7) and we write:

$$\int_\Omega \nabla(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon).\nabla(\phi\widetilde{W}^{\varepsilon,\phi}) \,{\rm d}x + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} \sigma(\hat{x}, \phi\widetilde{W}^{\varepsilon,\phi})(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon) \,{\rm d}\hat{x} \geq \int_\Omega f(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon) \,{\rm d}x.$$

This amounts to

$$\begin{aligned}\int_\Omega \nabla(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon).\nabla\phi\widetilde{W}^{\varepsilon,\phi} \,{\rm d}x + \int_\Omega \nabla(\phi^2\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon\phi).\nabla\widetilde{W}^{\varepsilon,\phi} \,{\rm d}x - \int_\Omega (\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon)\nabla\phi.\nabla\widetilde{W}^{\varepsilon,\phi} \,{\rm d}x \\ + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} \sigma(\hat{x}, \phi\widetilde{W}^{\varepsilon,\phi})(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon) \,{\rm d}\hat{x} \geq \int_\Omega f(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon) \,{\rm d}x.\end{aligned}$$

On account of (2.9) and (3.32), for subsequences, still denoted by ε, we take limits as $\varepsilon \to 0$ and obtain:

$$\begin{aligned}&\int_\Omega \nabla(\phi - u^0).\nabla\phi \,{\rm d}x - \int_\Omega f(\phi - u^0) \,{\rm d}x \\ &\geq -\lim_{\varepsilon\to 0} \left(\int_\Omega \nabla(\phi^2\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon\phi).\nabla\widetilde{W}^{\varepsilon,\phi} \,{\rm d}x + \beta(\varepsilon) \int_{\bigcup T^\varepsilon} \sigma(\hat{x}, \phi\widetilde{W}^{\varepsilon,\phi})(\phi\widetilde{W}^{\varepsilon,\phi} - u^\varepsilon) \,{\rm d}\hat{x} \right) \\ &:= -\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon.\end{aligned} \tag{4.1}$$

Below, we show that the limit on the right-hand side is

$$\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = r_0 \int_\Sigma \int_T \frac{\partial W^{\hat{x},\phi(\hat{x})}}{\partial n_y} {\rm d}\hat{y}\, \phi(\phi - u^0) {\rm d}\hat{x}. \tag{4.2}$$

Thus, (4.1) reads

$$\int_\Omega \nabla(\phi - u^0).\nabla\phi \,{\rm d}x + r_0 \int_\Sigma \int_T \frac{\partial W^{\hat{x},\phi(\hat{x})}}{\partial n_y} {\rm d}\hat{y}\, \phi(\phi - u^0) {\rm d}\hat{x} \geq \int_\Omega f(\phi - u^0) \,{\rm d}x,$$

and, considering (3.5) and (3.6), the inequality above is nothing but

$$\int_\Omega \nabla(\phi - u^0).\nabla\phi \,{\rm d}x + r_0 \int_\Sigma \Xi(\hat{x}, \phi)(\phi - u^0) {\rm d}\hat{x} \geq \int_\Omega f(\phi - u^0) \,{\rm d}x, \tag{4.3}$$

which holds for any $\phi \in \{v \in C^1(\overline{\Omega}) \,:\, v = 0 \text{ on } \Gamma_\Omega\}$. Now, taking into account the continuity of $\Xi(x, u)$ and $\Xi(x, u)u$ and using a density argument yields (2.24) for $\tau \in [0, \frac{\sqrt{5}-1}{2}]$. Let us explain the last assertion in further detail.

Here, we have applied the following inequalities:

$$\begin{aligned}\int\limits_{\Sigma}(\Xi(\hat{x},\phi)-\Xi(\hat{x},v))u\mathrm{d}\hat{x}\\ &\leq C\|\phi-v\|_{L^4(\Sigma)}\Big(\|u\|_{L^{4/3}(\Sigma)}+\|u\|_{L^4(\Sigma)}\|\phi^{\tau+\tau^2}\|^2_{L^4(\Sigma)}+\|u\|_{L^4(\Sigma)}\|v^{\tau+\tau^2}\|^2_{L^4(\Sigma)}\Big)\\ &\leq C\|\phi-v\|_{L^4(\Sigma)}\Big(\|u\|_{L^{4/3}(\Sigma)}+\|u\|_{L^4(\Sigma)}\|\phi\|^2_{L^4(\Sigma)}+\|u\|_{L^4(\Sigma)}\|v\|^2_{L^4(\Sigma)}\Big)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\int\limits_{\Sigma}(\Xi(\hat{x},\phi)\phi-\Xi(\hat{x},v)v)\mathrm{d}\hat{x}\leq C\|\phi-v\|_{L^4(\Sigma)}\Big(\|\phi\|_{L^{4/3}(\Sigma)}+\|v\|_{L^{4/3}(\Sigma)}+\|\phi^{1+2(\tau+\tau^2)}\|_{L^{4/3}(\Sigma)}\\ +\|v^{1+2(\tau+\tau^2)}\|_{L^{4/3}(\Sigma)}+\|\phi\|_{L^4(\Sigma)}\|v^{\tau+\tau^2}\|^2_{L^4(\Sigma)}+\|v\|_{L^4(\Sigma)}\|\phi^{\tau+\tau^2}\|^2_{L^4(\Sigma)}\Big)\\ \leq C\|\phi-v\|_{L^4(\Sigma)}\Big(\|\phi\|_{L^{4/3}(\Sigma)}+\|v\|_{L^{4/3}(\Sigma)}+\|\phi\|_{L^4(\Sigma)}\|v\|^2_{L^4(\Sigma)}+\|v\|_{L^4(\Sigma)}\|\phi\|^2_{L^4(\Sigma)}\Big),\end{aligned} \tag{4.5}$$

which can be obtained from (3.22), (3.23), the Hölder inequality, and the continuous embedding $\mathbf{V}\subset L^4(\Sigma)$. Indeed, we take the maximum τ such that inequalities of the type

$$\int\limits_{\Sigma}(\phi-v)w\,\mathrm{d}\hat{x}\leq\Big(\int\limits_{\Sigma}(\phi-v)^p\,\mathrm{d}\hat{x}\Big)^{1/p}\Big(\int\limits_{\Sigma}w^q\,\mathrm{d}\hat{x}\Big)^{1/q},$$

for certain p,q with $1/p+1/q=1$, provide a bound for $u,v\in\mathbf{V}$ to be in good agreement with the above mentioned embedding. This gives the maximum $\tau=(\sqrt{5}-1)/2$.

Thus, taking limits in (4.3) for $\phi\to v$ in $\mathbf{V}$ gives (2.24). By the uniqueness of solution, the whole sequence $u^\varepsilon\to u^0$ as $\varepsilon\to0$ in the weak topology of $H^1(\Omega)$ and u^0 is also the unique solution of (2.24) and (2.23).

Therefore, it remains to show (4.2) to end the proof of the theorem.

The proof of equality (4.2).

For the sake of brevity, we introduce the following notations

$$W^{\mathrm{k},r_\varepsilon}(x)\equiv W^{\widetilde{x}_\mathrm{k},\phi(\widetilde{x}_\mathrm{k})}\left(\frac{x-\widetilde{x}_\mathrm{k}}{r_\varepsilon}\right)\quad\text{and}\quad\Gamma^+_{\widetilde{x}_\mathrm{k},r_\varepsilon+\frac{\varepsilon}{8}}=\partial B(\widetilde{x}_\mathrm{k},r_\varepsilon+\frac{\varepsilon}{8})\cap\mathbb{R}^{3+}.$$

We divide the proof into three steps.

First step: The reduction of the limit in (4.1) *to the limit of integrals on half-spheres* (cf. (4.8)). For the first integral in $\mathbf{I}_\varepsilon$, cf. (4.1), on account of (3.27), we write

$$\begin{aligned}\int\limits_{\Omega}\nabla(\phi^2\widetilde{W}^{\varepsilon,\phi}-u^\varepsilon\phi).\nabla\widetilde{W}^{\varepsilon,\phi}\,\mathrm{d}x=\sum_{\widetilde{x}_\mathrm{k}}\int\limits_{B^+(\widetilde{x}_\mathrm{k},r_\varepsilon+\frac{\varepsilon}{8})}\nabla_x((1-W^{\mathrm{k},r_\varepsilon})\phi^2-u^\varepsilon\phi).\nabla_x(1-W^{\mathrm{k},r_\varepsilon})\,\mathrm{d}x\\ +\sum_{\widetilde{x}_\mathrm{k}}\int\limits_{\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}}\nabla_x((1-W^{\mathrm{k},r_\varepsilon}\varphi^\varepsilon)\phi^2-u^\varepsilon\phi).\nabla_x(1-W^{\mathrm{k},r_\varepsilon}\varphi^\varepsilon)\,\mathrm{d}x:=\mathbf{II}_\varepsilon+\mathbf{III}_\varepsilon.\end{aligned}$$

Taking into account the estimates in Proposition 3.5, (2.8), (2.1), (1.2), the volume of each $\mathfrak{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}$, and applying the Cauchy–Bunyakovsky–Schwarz inequality, we show that $\mathbf{III}_\varepsilon$ above is bounded $C\sqrt{\varepsilon}$.

Therefore, we write the limit in (4.1) as follows:

$$\lim_{\varepsilon\to0}\mathbf{I}_\varepsilon=\lim_{\varepsilon\to0}(\mathbf{II}_\varepsilon+\mathbf{IV}_\varepsilon),$$

where

$$\mathbf{IV}_\varepsilon := \beta(\varepsilon) \int\limits_{\bigcup T^\varepsilon} \sigma(\hat{x}, (1 - W^{\mathrm{k},r_\varepsilon})\phi)(\phi(1 - W^{\mathrm{k},r_\varepsilon}) - u^\varepsilon)\, \mathrm{d}\hat{x},$$

and, using the Green formula in $B^+(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8})$, we derive

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = & -\lim_{\varepsilon\to 0} \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} ((1 - W^{\mathrm{k},r_\varepsilon})\phi^2 - u^\varepsilon \phi) \frac{\partial W^{\mathrm{k},r_\varepsilon}}{\partial \nu_x}\, \mathrm{d}s_x \\
& - \lim_{\varepsilon\to 0} \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} ((1 - W^{\mathrm{k},r_\varepsilon})\phi^2 - u^\varepsilon \phi) \frac{\partial W^{\mathrm{k},r_\varepsilon}}{\partial n_x}\, \mathrm{d}\hat{x} \\
& + \lim_{\varepsilon\to 0} \beta(\varepsilon) \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \sigma(\hat{x}, (1 - W^{\mathrm{k},r_\varepsilon})\phi)(\phi(1 - W^{\mathrm{k},r_\varepsilon}) - u^\varepsilon)\, \mathrm{d}\hat{x} \\
& := \lim_{\varepsilon\to 0} (-\mathbf{L}^1_\varepsilon - \mathbf{L}^2_\varepsilon + \mathbf{L}^3_\varepsilon).
\end{aligned}$$

For $\mathbf{L}^\varepsilon_1$, we write the boundary condition for $W^{\widetilde{x}_\mathrm{k}, u(\widetilde{x}_\mathrm{k})}$ on T in the macroscopic variable, cf. (2.13) and (2.14), and we decompose it as follows:

$$\begin{aligned}
-\mathbf{L}^1_\varepsilon = & \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} ((1 - W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})})\phi^2 - u^\varepsilon \phi) \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x}\, \mathrm{d}s_x \\
= & \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} (\phi - u^\varepsilon)\phi \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x}\, \mathrm{d}s_x - \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})} \phi^2 \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x}\, \mathrm{d}s_x.
\end{aligned}$$

Then, for the last sum above, we use (3.25) to obtain an estimate in the half-spheres. Namely, for $x \in \Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}$, we have

$$\Big| \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x} \Big(\frac{x - \widetilde{x}_\mathrm{k}}{r_\varepsilon} \Big) \Big| = \frac{1}{r_\varepsilon} \Big| \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_y}(y) \Big| \leq C \frac{1}{r_\varepsilon} (r_\varepsilon)^2 \frac{1}{d(x, T^\varepsilon_{\widetilde{x}_\mathrm{k}})^2} \leq C$$

and

$$\Big| W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})} \Big(\frac{x - \widetilde{x}_\mathrm{k}}{r_\varepsilon} \Big) \Big| \leq C r_\varepsilon \frac{1}{d(x, T^\varepsilon_{\widetilde{x}_\mathrm{k}})} \leq C \frac{r_\varepsilon}{\varepsilon}.$$

Therefore,

$$\Big| \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})} \phi^2 \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x}\, \mathrm{d}s_x \Big| \leq C r_\varepsilon \varepsilon^{-1} \sum_{\widetilde{x}_\mathrm{k}} |\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}| \leq C\varepsilon.$$

Taking limits as $\varepsilon \to 0$, we get

$$-\lim_{\varepsilon\to 0} \mathbf{L}^1_\varepsilon = \lim_{\varepsilon\to 0} \sum_{\widetilde{x}_\mathrm{k}} \int\limits_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} (\phi - u^\varepsilon)\phi \frac{\partial W^{\widetilde{x}_\mathrm{k}, \phi(\widetilde{x}_\mathrm{k})}}{\partial \nu_x}\, \mathrm{d}s_x. \tag{4.6}$$

Next, let us show that

$$\lim_{\varepsilon\to 0} (-\mathbf{L}^2_\varepsilon + \mathbf{L}^3_\varepsilon) = 0, \tag{4.7}$$

which along with (4.6) provides

$$\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = \lim_{\varepsilon\to 0} \sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{\Gamma^+_{\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{8}}} (\phi - u^\varepsilon)\phi \frac{\partial W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}}{\partial \nu_x}\, \mathrm{d}s_x. \tag{4.8}$$

Second step: The proof of (4.7). Based on the properties of σ (cf. (2.2)), the properties of $W^{x,u}$ (cf. (2.13) and Proposition 3.2) and (1.3), we perform cumbersome but straightforward computations that lead us to write

$$\begin{aligned}
&(-\mathbf{L}^2_\varepsilon + \mathbf{L}^3_\varepsilon) = \\
&-\sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} ((1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi - u^\varepsilon)(\phi(\hat{x}) - \phi(\widetilde{x}_{\mathrm{k}})) \frac{\partial W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}}{\partial n_x}\, \mathrm{d}\hat{x} \\
&+\Big(\beta(\varepsilon) r_\varepsilon - \beta^0\Big)\frac{1}{r_\varepsilon} \sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} ((1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi - u^\varepsilon)\sigma(\widetilde{x}_{\mathrm{k}}, (1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi(\widetilde{x}_{\mathrm{k}}))\, \mathrm{d}\hat{x} \\
&+\beta(\varepsilon)\sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} \Big(\sigma(\hat{x}, (1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi(\widetilde{x}_{\mathrm{k}})) - \sigma(\widetilde{x}_{\mathrm{k}}, (1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi(\widetilde{x}_{\mathrm{k}}))\Big) g^{\varepsilon,\mathrm{k}}(\hat{x})\, \mathrm{d}\hat{x} \\
&+\beta(\varepsilon)\sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} \Big(\sigma(\hat{x}, (1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi(\hat{x})) - \sigma(\hat{x}, (1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})})\phi(\widetilde{x}_{\mathrm{k}}))\Big) g^{\varepsilon,\mathrm{k}}(\hat{x})\, \mathrm{d}\hat{x} \\
&:= \mathbf{J}^1_\varepsilon + \mathbf{J}^2_\varepsilon + \mathbf{J}^3_\varepsilon + \mathbf{J}^4_\varepsilon,
\end{aligned}$$

where $g^{\varepsilon,\mathrm{k}}$ denotes the function

$$g^{\varepsilon,\mathrm{k}}(\hat{x}) = \phi(\hat{x})\Big(1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}\big(\frac{\hat{x} - \widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\big)\Big) - u^\varepsilon(\hat{x}).$$

Let us show that $\mathbf{J}^i_\varepsilon \to 0$ as $\varepsilon \to 0$ for $i = 1, 2, 3, 4$. In order to simplify notations, let us denote by

$$g_\varepsilon := \Big(\int\limits_{\bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} |g^{\varepsilon,\mathrm{k}}(\hat{x})|^2 \mathrm{d}\hat{x}\Big)^{1/2}.$$

Using (2.2), the Cauchy–Bunyakovsky–Schwarz inequality, the change of variable (2.14), the continuous embedding of $\mathfrak{V}$ in $L^{p'}(T)$ for $2 \le p' \le 4$ (see, e.g., Section I.4 of [21] and Section IV.8 of [34]), (2.1), (3.7), $\beta^0 > 0$ and $r_0 > 0$ in (1.3) and (1.2) and (2.1), we get

$$\begin{aligned}
|\mathbf{J}^4_\varepsilon| &\le C\beta(\varepsilon) \sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} \big|1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}\big|\big(1 + \big|1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}\big|^\tau\big)|\phi(\hat{x}) - \phi(\widetilde{x}_{\mathrm{k}})||g^{\varepsilon,\mathrm{k}}|\, \mathrm{d}\hat{x} \\
&\le C\beta(\varepsilon) r_\varepsilon g_\varepsilon \Big(\sum_{\widetilde{x}_{\mathrm{k}}} \int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}} \big|1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}\big|^2 + \big|1 - W^{\widetilde{x}_{\mathrm{k}},\phi(\widetilde{x}_{\mathrm{k}})}\big|^{2(1+\tau)} \mathrm{d}\hat{x}\Big)^{1/2} \\
&\le C\beta(\varepsilon) r_\varepsilon g_\varepsilon \big(\varepsilon^{-2} r_\varepsilon^2\big)^{1/2} \le C g_\varepsilon \big(r_\varepsilon\big)^{1/2}.
\end{aligned}$$

Similarly, using (2.2), (2.1), $\beta^0 > 0$ and $r_0 > 0$ in (1.3) and (1.2), and (2.1), we have

$$|\mathbf{J}^3_\varepsilon| \le C g_\varepsilon \big(r_\varepsilon\big)^{3/2}.$$

The same tools used to obtain the estimate for $\mathbf{J}^4_\varepsilon$ lead us to

$$|\mathbf{J}^2_\varepsilon| \le C\Big|\beta(\varepsilon) r_\varepsilon - \beta^0\Big| \frac{1}{(r_\varepsilon)^{1/2}} g_\varepsilon$$

and

$$|\mathbf{J}^1_\varepsilon| \leq \sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \big|g^{\varepsilon,k}\big|\big|\phi(\hat{x})-\phi(\widetilde{x}_k)\big|\Big|\frac{\partial W^{\widetilde{x}_k,\phi(\widetilde{x}_k)}}{\partial n_x}\Big|\,\mathrm{d}\hat{x} \leq C g_\varepsilon.$$

Here above, we have also applied the change of variable (2.14) for the derivative in the equation on T in the local problem.

All of this along with a uniform bound for $(r_\varepsilon)^{-1/2}g_\varepsilon$, gives the convergence towards zero of $\mathbf{J}^i_\varepsilon$, $i=1,2,3,4$. To obtain this uniform bound, it suffices to show

$$\frac{1}{r_\varepsilon}\int_{\bigcup T^\varepsilon} |\phi(1-W^{\widetilde{x}_k,\phi(\widetilde{x}_k)})|^2\mathrm{d}\hat{x} \leq C \quad \text{and} \quad \frac{1}{r_\varepsilon}\int_{\bigcup T^\varepsilon} |u^\varepsilon|^2\mathrm{d}\hat{x} \leq C.$$

Indeed, using (2.1), (2.14), (3.7) and $r_0>0$ in (1.2), the first estimate above holds

$$\frac{1}{r_\varepsilon}\int_{\bigcup T^\varepsilon} |\phi(1-W^{\widetilde{x}_k,\phi(\widetilde{x}_k)})|^2\mathrm{d}\hat{x} \leq \frac{1}{r_\varepsilon}\varepsilon^{-2}r_\varepsilon^2 \leq C.$$

Also, since $\beta^0>0$ in (1.3) and (2.8), we get the second estimate above for $|u^\varepsilon|^2$.

Therefore, (4.7) and (4.8) also hold.

Third step: The application of Lemma 4.1. To end the proof of the theorem, it only remains to show that (4.8) gives (4.2), namely that

$$\lim_{\varepsilon\to 0}\sum_{\widetilde{x}_k}\int_{\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}}} (\phi-u^\varepsilon)\phi\frac{\partial W^{\widetilde{x}_k,\phi(\widetilde{x}_k)}}{\partial \nu_x}\,\mathrm{d}s_x = r_0\int_\Sigma\int_T \frac{\partial W^{\hat{x},\phi(\hat{x})}}{\partial \nu_y}\mathrm{d}\hat{y}\,\phi(\phi-u^0)\mathrm{d}\hat{x}.$$

Using (3.4) and (3.5), we can write

$$\begin{aligned}\mathbf{LL} &:= \lim_{\varepsilon\to 0}\sum_{\widetilde{x}_k}\int_{\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}}} (\phi-u^\varepsilon)\phi\frac{\partial W^{\widetilde{x}_k,\phi(\widetilde{x}_k)}}{\partial \nu_x}\,\mathrm{d}s_x\\ &= \lim_{\varepsilon\to 0}\frac{r_\varepsilon}{(r_\varepsilon+(\varepsilon/8))^2}\sum_{\widetilde{x}_k}\int_{\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}}} (\phi-u^\varepsilon)\phi\mathcal{K}(\widetilde{x}_k,\phi(\widetilde{x}_k))\,\mathrm{d}s_x,\end{aligned}$$

where, we have used $r_0>0$ in (1.2), the uniform bounds for $\mathcal{K}$, cf. Propositions 3.2 and 3.4, and the fact that

$$\sum_{\widetilde{x}_k}\int_{\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}}} |u^\varepsilon\phi|\,\mathrm{d}x \leq C, \tag{4.9}$$

which holds true taking into account that the sequence $|u^\varepsilon\phi|$ is bounded in $H^1(\Omega)$ (cf., e.g., Section 28.I in [38]) and Lemma 4.1.

The estimate (3.20) implies

$$\Big|\frac{\phi(\widetilde{x}_k)}{\beta^0}\mathcal{K}(\widetilde{x}_k,\phi(\widetilde{x}_k)) - \frac{\phi(x)}{\beta^0}\mathcal{K}(x,\phi(x))\Big| \leq C\varepsilon, \quad \forall x\in\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}},$$

and straightforward computations give

$$\mathbf{LL} = \lim_{\varepsilon\to 0}\frac{r_\varepsilon}{(r_\varepsilon+(\varepsilon/8))^2}\sum_{\widetilde{x}_k}\int_{\Gamma^+_{\widetilde{x}_k,r_\varepsilon+\frac{\varepsilon}{8}}} (\phi(x)-u^\varepsilon(x))\phi(x)\mathcal{K}(x,\phi(x))\,\mathrm{d}s_x,$$

Now, considering the function $\Theta(x)$ defined over $\overline{\Omega}$ (recall $\Theta(x) := \phi(x)\mathcal{K}(x, \phi(x))$), we use the smoothness of Θ in Proposition 3.2, cf. (3.15), which guarantees that $\Theta(\phi - u^\varepsilon) \in H^1(\Omega)$, and considering (2.8) gives

$$\|\Theta(\phi - u^\varepsilon)\|_{H^1(\Omega)} \leq C.$$

Extending by symmetry the functions $\Theta(\phi - u^\varepsilon)$ to the lower half-space $\{x_3 < 0\}$, we get a sequence of functions $\Theta(\widehat{\phi - u^\varepsilon}) \in H^1_0(\widehat{\Omega})$, satisfying

$$\Theta(\widehat{\phi - u^\varepsilon}) \to \Theta(\widehat{\phi - u^0}) \text{ in} H^1(\widehat{\Omega}) - weak, \quad \text{as } \varepsilon \to 0,$$

where by $\widehat{\Omega}$ we denote the domain Ω extended by symmetry.

Therefore, using this extension and Lemma 4.1, we have

$$\mathbf{LL} = 8^2 r_0 \frac{1}{2} \lim_{\varepsilon \to 0} \sum_{\widetilde{x}_k} \int\limits_{\partial B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{8})} \Theta(\widehat{\phi - u^\varepsilon})\, \mathrm{d}x = r_0 2\pi \int\limits_{\Sigma} \Theta(\phi - u^0)\, \mathrm{d}\hat{x}.$$

Finally, considering (4.8) and (3.5), we obtain

$$\lim_{\varepsilon \to 0} \mathbf{I}_\varepsilon = r_0 2\pi \int\limits_{\Sigma} \mathcal{K}(\hat{x}, \phi(\hat{x}))\phi(\phi - u^0) \mathrm{d}\hat{x} = r_0 \int\limits_{\Sigma} \int\limits_{T} \frac{\partial W^{\hat{x}, \phi(\hat{x})}}{\partial n_y} \mathrm{d}\hat{y}\, \phi(\phi - u^0) \mathrm{d}\hat{x}.$$

This shows (4.2) and the theorem is proved. □

Note that the constants appearing throughout the proof of Theorem 4.1 (before taking limits in (4.3) for $\phi \to u^0$) can depend on ϕ, and more specifically on the maximum for ϕ and their derivatives in $\overline{\Omega}$, and also on τ; but since they have been fixed, we avoid writing this dependence.

5. The other critical case

In this section, we address the convergence of solutions of problem (2.5) as $\varepsilon \to 0$, when $r_0 > 0$ and $\beta^0 = +\infty$ in (1.2) and (1.3), and the nonlinear function σ satisfies:

$$\sigma \in C^1(\overline{\Omega} \times \mathbb{R}), \quad \sigma(x, 0) = 0 \quad \text{and} \quad 0 < K_1 \leq \frac{\partial \sigma}{\partial u}(x, u) \leq K_2, \quad \forall x \in \overline{\Omega},\, u \in \mathbb{R}, \tag{5.1}$$

see Remark 5.1. The main result is given by Theorem 5.2. Now, the homogenized problem reads (2.15), where $\mathcal{C}$ is the *capacity constant* defined by (2.16) with W the solution of the local problem (2.17).

We follow the scheme in Sects. 3–4 with the suitable modifications that we outline in Sects. 5.1–5.2. Section 5.1 presents properties of the auxiliary functions constructed from the solution of (2.17). The convergence result is in Sect. 5.2.

5.1. The Dirichlet local problem and the test functions

We derive some properties of the function W and the positivity of $\mathcal{C}$.

Let $\mathcal{D}_1(\overline{\mathbb{R}^{3+}})$ denote the space of functions in $\mathcal{D}(\overline{\mathbb{R}^{3+}})$ which vanish in a neighborhood of $\overline{T}$. Let $\mathfrak{V}_1$ be the space obtained by completion of $\mathcal{D}_1(\overline{\mathbb{R}^{3+}})$ with respect to the Dirichlet norm (3.1). We take a function

$$\Psi \in \mathcal{D}(\overline{\mathbb{R}^{3+}}), \quad \Psi = 1 \text{ in a neighborhood of } T.$$

Then, the variational formulation of $(2.17)_1$-$(2.17)_3$ reads: Find $W \in \Psi + \mathfrak{V}_1$ satisfying

$$\int\limits_{\mathbb{R}^{3+}} \nabla_y W . \nabla_y V \,\mathrm{d}y = 0 \quad \forall V \in \mathfrak{V}_1. \tag{5.2}$$

Problem (5.2) has a unique solution which is independent of Ψ. Also, $\frac{\partial W}{\partial n_y}\big|_{y_3=0}$ is a distribution having compact support contained in $\overline{T}$ and belongs to $H^{-1/2}(T)$ (see, e.g., Appendix in [31] and Section 4 in [26]). The condition at infinity in (2.17) is a consequence of the next theorem.

Theorem 5.1. *The solution $W \in \Psi + \mathfrak{V}_1$ of problem (5.2) admits the representation*

$$W(y) = \frac{\mathcal{K}}{|y|} + O(\frac{1}{|y|^2}) \qquad as \qquad |y| \to \infty, \tag{5.3}$$

where $\mathcal{K}$ is the constant in the chain of equalities

$$2\pi\mathcal{K} = \Big\langle \frac{\partial W}{\partial n_y}, 1 \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} = \int\limits_{\mathbb{R}^{3+}} |\nabla_y W|^2 \mathrm{d}y. \tag{5.4}$$

Proof. First, let us note that the solution $W \in \Psi + \mathfrak{V}_1$ of problem (5.2) can be represented as follows:

$$W(y_1, y_2, y_3) = -\frac{1}{2\pi}\Big\langle q, \quad \frac{1}{\sqrt{(y_1 - \cdot)^2 + (y_2 - \cdot)^2 + y_3^2}} \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \tag{5.5}$$

where

$$q = \frac{\partial W}{\partial n_y}\Big|_T.$$

We refer to [26] and Appendix in [31] for this proof (cf. also the linear homogenization problems in [2,12,13,27] and [32] in this connection).

Consequently, there is a positive constant C such that for $y \in \mathbb{R}^{3+}$, with $|y|$ large enough, we have

$$|W(y)| \le C\frac{1}{|y|} \quad \text{ and } \quad \left|\frac{\partial W}{\partial y_i}(y)\right| \le C\frac{1}{|y|^2}, \quad i = 1, 2, 3. \tag{5.6}$$

Formula (5.5) also provides the representation (5.3) for a certain constant $\mathcal{K}$. To get the chain of equalities (5.4), we multiply the Laplace equation in (2.17) by $V \in \mathcal{D}(\overline{\mathbb{R}^{3+}})$ and apply the Green formula. We obtain

$$\int\limits_{\mathbb{R}^{3+}} \nabla_y W.\nabla_y V \,\mathrm{d}y = \Big\langle \frac{\partial W}{\partial n_y}, V \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \quad \forall V \in \mathcal{D}(\overline{\mathbb{R}^{3+}}). \tag{5.7}$$

By a density argument, we have that (5.7) holds for any $V \in \mathfrak{V}$, and consequently, taking $V = W$,

$$\mathcal{C} := \Big\langle \frac{\partial W}{\partial n_y}, 1 \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} = \int\limits_{\mathbb{R}^{3+}} |\nabla_y W|^2 \,\mathrm{d}y. \tag{5.8}$$

That is, we have proved the second equality in (5.4). To get the first one, we consider the Laplace equation in (2.17) and apply the Green formula in $B^+(0, R)$ (cf. also Theorem 3.1). Thus,

$$\Big\langle \frac{\partial W}{\partial n_y}, 1 \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} = -\lim_{R\to\infty} \int\limits_{\Gamma_R^+} \frac{\partial W}{\partial \nu_y} \mathrm{d}s_y = 2\pi\mathcal{K},$$

which gives the first equality in (5.4) and ends the proof of the theorem. □

Throughout this section, we consider $\widetilde{W}^\varepsilon$ constructed as in (3.28), by replacing $W^{\widetilde{x}_\mathrm{k}, u(\widetilde{x}_\mathrm{k})}$ with the function W defined by (5.5); namely, we set

$$W^{\mathrm{k},\varepsilon}(x) = W\left(\frac{x - \widetilde{x}_\mathrm{k}}{r_\varepsilon}\right)\varphi^\varepsilon(x) \text{ and } \widetilde{W}^{\mathrm{k},\varepsilon}(x) = 1 - W^{\mathrm{k},\varepsilon}(x), \text{ for } x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right),$$

and, finally,

$$\widetilde{W}^{\varepsilon}(x)=\begin{cases}\widetilde{W}^{\mathrm{k},\varepsilon}(x) & \text{for } x\in B^{+}\left(\widetilde{x}_{\mathrm{k}},r_{\varepsilon}+\frac{\varepsilon}{4}\right),\ \mathrm{k}\in\mathcal{J}^{\varepsilon},\\ 1 & \text{for } x\in\Omega\setminus\bigcup_{\mathrm{k}\in\mathcal{J}^{\varepsilon}}B^{+}\left(\widetilde{x}_{\mathrm{k}},r_{\varepsilon}+\frac{\varepsilon}{4}\right).\end{cases}\tag{5.9}$$

Taking into account (5.6), the proof of the following proposition is obtained by rewriting that of Proposition 3.5

Proposition 5.1. *For $r_0>0$ and $\beta^0=+\infty$ in (1.2) and (1.3), the properties (3.29)–(3.32) hold changing the functions $W^{\widetilde{x}_{\mathrm{k}},u(\widetilde{x}_{\mathrm{k}})}$, $W^{\mathrm{k},\varepsilon,u}$ and $\widetilde{W}^{\varepsilon,u}$ by W, $W^{\mathrm{k},\varepsilon}$ and $\widetilde{W}^{\varepsilon}$ respectively.*

5.2. The convergence result

The aim of this section is to prove Theorem 5.2. Keeping $r_0>0$, we apply the technique in Theorem 4.1 when $\beta^0>0$, with the suitable modifications, to show the convergence of u^ε, as $\varepsilon\to 0$, for $\beta^0=+\infty$. As a matter of fact, functions (3.28) are replaced by (5.9), which vanish on $\bigcup T^\varepsilon$. In addition, some integrals on T transform into dual products in $H^{-1/2}(T)\times H^{1/2}(T)$ and the corresponding proof must be changed. In this respect, in addition to Lemma 4.1 we need a new convergence result, for the trace on $\bigcup T^\varepsilon$ of the solution of (2.5), which we introduce here below.

Proposition 5.2. *Let σ satisfy (5.1), $r_0>0$ and $\beta^0=+\infty$. Then, the solution u^ε of (2.6) verifies:*

$$\sum_{\widetilde{x}_{\mathrm{k}}}\left\|u^\varepsilon\right\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})}\to 0,\qquad \text{as } \varepsilon\to 0.$$

Proof. From the definition of the $H^{1/2}$-norm, we write the sum above as:

$$\mathfrak{L}_\varepsilon:=\sum_{\widetilde{x}_{\mathrm{k}}}\left\|u^\varepsilon\right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})}+\sum_{\widetilde{x}_{\mathrm{k}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{|u^\varepsilon(\hat{x})-u^\varepsilon(\hat{x}')|^2}{|\hat{x}-\hat{x}'|^3}\mathrm{d}\hat{x}\mathrm{d}\hat{x}'.\tag{5.10}$$

For the first summation, we have

$$\sum_{\widetilde{x}_{\mathrm{k}}}\left\|u^\varepsilon\right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})}=\frac{1}{\beta(\varepsilon)}\int\limits_{\bigcup T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\beta(\varepsilon)(u^\varepsilon)^2\mathrm{d}\hat{x}=o_\varepsilon(1),\tag{5.11}$$

where, here and below, $o_\varepsilon(1)$ denotes a certain function satisfying $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. Obviously, we have used the bound (2.8) and the fact that $\beta(\varepsilon)\to\infty$ to obtain (5.11).

Then, we use the bound obtained by a simple integration in (5.1),

$$K_1^2(u-v)^2\le(\sigma(x,u)-\sigma(x,v))^2\le K_2^2(u-v)^2,\quad \forall u,v\in\mathbb{R},\ x\in\overline{\Omega}.$$

This, along with (2.2) which holds for $\tau=0$, allows us to write, for the second summation in (5.10), the following chain of inequalities:

$$\begin{aligned}
&\sum_{\widetilde{x}_{\mathrm{k}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{|u^\varepsilon(\hat{x})-u^\varepsilon(\hat{x}')|^2}{|\hat{x}-\hat{x}'|^3}\mathrm{d}\hat{x}\mathrm{d}\hat{x}'\le\widetilde{C}\sum_{\widetilde{x}_{\mathrm{k}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{|\sigma(\hat{x},u^\varepsilon(\hat{x}))-\sigma(\hat{x},u^\varepsilon(\hat{x}'))|^2}{|\hat{x}-\hat{x}'|^3}\mathrm{d}\hat{x}\mathrm{d}\hat{x}'\\
&\le 2\widetilde{C}\int\limits_{\bigcup T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{\bigcup T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{|\sigma(\hat{x},u^\varepsilon(\hat{x}))-\sigma(\hat{x}',u^\varepsilon(\hat{x}'))|^2}{|\hat{x}-\hat{x}'|^3}\mathrm{d}\hat{x}\mathrm{d}\hat{x}'+\widehat{C}\sum_{\widetilde{x}_{\mathrm{k}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{1}{|\hat{x}-\hat{x}'|}\mathrm{d}\hat{x}\mathrm{d}\hat{x}'\\
&\le 2\widetilde{C}\left\|\sigma(\cdot,u^\varepsilon(\cdot))\right\|^2_{H^{1/2}(\bigcup T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})}+\widehat{C}\sum_{\widetilde{x}_{\mathrm{k}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\int\limits_{T^\varepsilon_{\widetilde{x}_{\mathrm{k}}}}\frac{1}{|\hat{x}-\hat{x}'|}\mathrm{d}\hat{x}\mathrm{d}\hat{x}':=2\widetilde{C}\mathfrak{L}^1_\varepsilon+\widehat{C}\mathfrak{L}^2_\varepsilon,
\end{aligned}$$

where $\widetilde{C}$ and $\widehat{C}$ denote the two constants independent of ε.

In addition, we have

$$\left(\mathfrak{L}^1_\varepsilon\right)^{1/2} \leq \frac{1}{\beta(\varepsilon)}\left\|\beta(\varepsilon)\chi_{\bigcup T^\varepsilon}\sigma(\cdot,u^\varepsilon(\cdot))\right\|_{H^{1/2}(\Sigma)} \leq C\frac{1}{\beta(\varepsilon)}\left\|\frac{\partial u^\varepsilon}{\partial n}\right\|_{H^{-1/2}(\Sigma)} \leq C\frac{1}{\beta(\varepsilon)} = o_\varepsilon(1), \quad (5.12)$$

where $\chi_{\bigcup T^\varepsilon}$ denotes the characteristic function of the set $\bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_\mathrm{k}}$, and we have used the equation on Σ in (2.6), cf. (2.5), the trace embedding theorem, (2.8) and $\beta(\varepsilon)\to+\infty$, as $\varepsilon\to 0$.

As for the other term, on each $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$, we consider the function defined as:

$$U^{\varepsilon,\mathrm{k}}(\hat{x}') = \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \frac{1}{|\hat{x}-\hat{x}'|}\mathrm{d}\hat{x},$$

and apply the Theorem in Section I.6.1 of [37] for integrals of potential type (cf. also Lemma 5 in Section I.2 of [21] and Theorem 1 in Section IV.115 of [36] in this connection); we obtain

$$|U^{\varepsilon,\mathrm{k}}(\hat{x}')| \leq C|T^\varepsilon_{\widetilde{x}_\mathrm{k}}|^{1/3}(r_\varepsilon)^{1/3}, \quad \forall \hat{x}'\in T^\varepsilon_{\widetilde{x}_\mathrm{k}}.$$

Taking into account the volume of each $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$, (2.1) and $r_0>0$ (1.2), we get

$$\mathfrak{L}^2_\varepsilon := \sum_{\widetilde{x}_\mathrm{k}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}}\int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \frac{1}{|\hat{x}-\hat{x}'|}\mathrm{d}\hat{x}\mathrm{d}\hat{x}' \leq C(r_\varepsilon)^3\varepsilon^{-2} \leq C(r_\varepsilon)^2. \quad (5.13)$$

Finally, using (5.11), (5.12) and (5.13) in (5.10), we obtain the convergence stated in the proposition.

Remark 5.1. As regards the proof of Proposition 5.2, it should be noted that in the case where $r^0>0$ and $\beta^0=+\infty$, the restriction on σ to satisfy (5.1) allows the function $\sigma(\cdot,u^\varepsilon(\cdot))$ to be in $H^1(\Omega)$. Less restrictive hypotheses, such as (2.10) or (2.2)–(2.4) with a $\tau>0$, could be allowed provided that the trace of the function $\sigma(\cdot,u^\varepsilon(\cdot))$ belongs to $H^{1/2}(\bigcup T^\varepsilon_{\widetilde{x}_\mathrm{k}})$. The restriction (5.1) is also in good agreement with that in [10] to derive the homogenized model with asymptotic expansions.

Theorem 5.2. *Under the hypotheses (5.1) for σ, $r_0>0$ and $\beta^0=+\infty$, the solution u^ε of (2.6) converges in $H^1(\Omega)$-weak, as $\varepsilon\to 0$, toward the solution u^0 of (2.21).*

Proof. For $\phi\in C^1(\overline{\Omega})$, $\phi=0$ on Γ_Ω, we take the *test function* $v(x)=\phi(x)\widetilde{W}^\varepsilon(x)$ in (2.7), with $\widetilde{W}^\varepsilon$ in (5.9). Since $\widetilde{W}^\varepsilon$ vanishes on $\bigcup T^\varepsilon$, also $\sigma(\hat{x},\,\phi(\hat{x})\widetilde{W}^\varepsilon(\hat{x}))=0$ for $\hat{x}\in\bigcup T^\varepsilon$, and we have

$$\int_\Omega \nabla(\phi\widetilde{W}^\varepsilon-u^\varepsilon).\nabla\phi\widetilde{W}^\varepsilon\,\mathrm{d}x + \int_\Omega \nabla(\phi^2\widetilde{W}^\varepsilon-u^\varepsilon\phi).\nabla\widetilde{W}^\varepsilon\,\mathrm{d}x$$
$$-\int_\Omega (\phi\widetilde{W}^\varepsilon-u^\varepsilon)\nabla\phi.\nabla\widetilde{W}^\varepsilon\,\mathrm{d}x \geq \int_\Omega f(\phi\widetilde{W}^\varepsilon-u^\varepsilon)\,\mathrm{d}x.$$

On account of (2.9) for subsequences, still denoted by ε, and of (3.32) (cf. Proposition 5.1), we take limits as $\varepsilon\to 0$ and we obtain:

$$\int_\Omega \nabla(\phi-u^0).\nabla\phi\,\mathrm{d}x - \int_\Omega f(\phi-u^0)\,\mathrm{d}x \geq -\lim_{\varepsilon\to 0}\int_\Omega \nabla(\phi^2\widetilde{W}^\varepsilon-u^\varepsilon\phi).\nabla\widetilde{W}^\varepsilon\,\mathrm{d}x. \quad (5.14)$$

Setting

$$\mathbf{I}_\varepsilon := \int_\Omega \nabla(\phi^2\widetilde{W}^\varepsilon-u^\varepsilon\phi).\nabla\widetilde{W}^\varepsilon\,\mathrm{d}x, \quad (5.15)$$

below, we show that the limit on the right-hand side is

$$\lim_{\varepsilon\to 0}\mathbf{I}_\varepsilon = r_0\left\langle \frac{\partial W}{\partial n_y}, 1\right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \int_\Sigma \phi(\phi-u^0)\mathrm{d}\hat{x}. \tag{5.16}$$

Thus, cf. (2.16), (5.14) reads

$$\int_\Omega \nabla(\phi-u^0).\nabla\phi\,\mathrm{d}x + r_0\,\mathcal{C}\int_\Sigma \phi(\phi-u^0)\mathrm{d}\hat{x} \geq \int_\Omega f(\phi-u^0)\,\mathrm{d}x,$$

which holds for any $\phi\in\{v\in C^1(\overline{\Omega}) : v=0 \text{ on } \Gamma_\Omega\}$. Using a density argument, we get (2.24) for $\Xi(\hat{x},v)\equiv\mathcal{C}v$ and, consequently, (2.21).

Therefore, it remains to show (5.16) to end the proof of the theorem. Let us do this following the ideas in steps 1 and 3 of the proof of Theorem 4.1.

Below $W^{\mathrm{k},r_\varepsilon}$ denotes $W^{\mathrm{k},r_\varepsilon}(x)\equiv W\big(\frac{x-\widetilde{x}_{\mathrm{k}}}{r_\varepsilon}\big)$, W being the solution of (2.17). Following the step 1 in Theorem 4.1, and applying the Green formula in $B^+(\widetilde{x}_{\mathrm{k}}, r_\varepsilon+\frac{\varepsilon}{8})$, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbf{I}_\varepsilon &= -\lim_{\varepsilon\to 0}\sum_{\widetilde{x}_{\mathrm{k}}}\int_{\Gamma^+_{\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{8}}} ((1-W^{\mathrm{k},r_\varepsilon})\phi^2-u^\varepsilon\phi)\frac{\partial W^{\mathrm{k},r_\varepsilon}}{\partial\nu_x}\,\mathrm{d}s_x\\ &\quad+\lim_{\varepsilon\to 0}\sum_{\widetilde{x}_{\mathrm{k}}}\left\langle\frac{\partial W^{\mathrm{k},r_\varepsilon}}{\partial n_x}, u^\varepsilon\phi\right\rangle_{H^{-1/2}(T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})\times H^{1/2}(T^\varepsilon_{\widetilde{x}_{\mathrm{k}}})} := -\lim_{\varepsilon\to 0}\mathbf{L}^1_\varepsilon+\lim_{\varepsilon\to 0}\mathbf{L}^2_\varepsilon.\end{aligned} \tag{5.17}$$

Rewriting the proof for the limit of $\mathbf{L}^1_\varepsilon$ in Theorem 4.1, with the suitable modifications, and taking limits, for the first term in (5.17) we show

$$-\lim_{\varepsilon\to 0}\mathbf{L}^1_\varepsilon = \lim_{\varepsilon\to 0}\sum_{\widetilde{x}_{\mathrm{k}}}\int_{\Gamma^+_{\widetilde{x}_{\mathrm{k}},r_\varepsilon+\frac{\varepsilon}{8}}}(\phi-u^\varepsilon)\phi\frac{\partial W^{\mathrm{k},r_\varepsilon}}{\partial\nu_x}\,\mathrm{d}s_x, \tag{5.18}$$

while for the second term, below we prove

$$\lim_{\varepsilon\to 0}\mathbf{L}^2_\varepsilon = 0. \tag{5.19}$$

In order to do this, we note that we can decompose the integral in (5.15) in sums of integrals on each $B^+(\widetilde{x}_{\mathrm{k}}, r_\varepsilon+\frac{\varepsilon}{4})$, and perform the change $x\mapsto y$, cf. (2.14), to get

$$\mathbf{I}_\varepsilon = r_\varepsilon\sum_{\widetilde{x}_{\mathrm{k}}}\int_{B^+(0,1+\frac{\varepsilon}{4r_\varepsilon})}\nabla_y(\phi^2\widetilde{W}^\varepsilon-u^\varepsilon\phi).\nabla_y\widetilde{W}^\varepsilon\,\mathrm{d}y.$$

Rewriting the considerations above and some straightforward computations lead us to

$$\mathbf{L}^2_\varepsilon := r_\varepsilon\sum_{\widetilde{x}_{\mathrm{k}}}\left\langle\frac{\partial W}{\partial n_y}, \flat_y(u^\varepsilon\phi)\right\rangle_{H^{-1/2}(T)\times H^{1/2}(T)},$$

where $\flat_y$ denotes the above mentioned change (2.14). Then, applying the Cauchy–Schwarz inequality and (2.1), we write

$$|\mathbf{L}^2_\varepsilon| \leq r_\varepsilon\sum_{\widetilde{x}_{\mathrm{k}}}\left\|\frac{\partial W}{\partial n_y}\right\|_{H^{-1/2}(T)}\|\flat_y(u^\varepsilon\phi)\|_{H^{1/2}(T)} \leq Cr_\varepsilon\varepsilon^{-1}\Big(\sum_{\widetilde{x}_{\mathrm{k}}}\left\|\flat_y u^\varepsilon\right\|^2_{H^{1/2}(T)}\Big)^{1/2}.$$

Hence, applying again (2.14), we have

$$\left\|\flat_y u^\varepsilon\right\|^2_{H^{1/2}(T)} = \int_T |\flat_y u^\varepsilon|^2 \mathrm{d}\hat{y} + \int_T \int_T \frac{|\flat_y u^\varepsilon(\hat{y}) - \flat_y u^\varepsilon(\hat{y}')|^2}{|\hat{y}-\hat{y}'|^3} \mathrm{d}\hat{y}\mathrm{d}\hat{y}'$$
$$= \frac{1}{r_\varepsilon^2} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} |u^\varepsilon|^2 \mathrm{d}\hat{x} + \frac{1}{r_\varepsilon} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \frac{|u^\varepsilon(\hat{x}) - u^\varepsilon(\hat{x}')|^2}{|\hat{x}-\hat{x}'|^3} \mathrm{d}\hat{x}\mathrm{d}\hat{x}'$$
$$\le \frac{1}{r_\varepsilon^2} \left\|u^\varepsilon\right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_\mathrm{k}})} + \frac{1}{r_\varepsilon} \left\|u^\varepsilon\right\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_\mathrm{k}})}.$$

Therefore, using (2.4),

$$|\mathbf{L}^2_\varepsilon| \le C\varepsilon^{-1} \Big(\sum_{\widetilde{x}_\mathrm{k}} \left\|u^\varepsilon\right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_\mathrm{k}})} \Big)^{1/2} + C\varepsilon^{-1}\sqrt{r_\varepsilon} \Big(\sum_{\widetilde{x}_\mathrm{k}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \frac{|u^\varepsilon(\hat{x}) - u^\varepsilon(\hat{x}')|^2}{|\hat{x}-\hat{x}'|^3} \mathrm{d}\hat{x}\mathrm{d}\hat{x}' \Big)^{1/2}$$
$$\le C \frac{\sqrt{r}_\varepsilon}{\varepsilon\sqrt{r_\varepsilon \beta(\varepsilon)}} \Big(\sum_{\widetilde{x}_\mathrm{k}} \beta(\varepsilon) \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} (u^\varepsilon)^2 \mathrm{d}\hat{x} \Big)^{1/2} + C \Big(\mathfrak{L}_\varepsilon \Big)^{1/2},$$

where $\mathfrak{L}_\varepsilon$ is defined in (5.10). Consequently, because of (2.8), $\beta^0 = +\infty$ in (1.3) and $r_0 > 0$ in (1.2), and the convergence in Proposition 5.2, we have that the two terms on the right-hand side of the last inequality tend to 0 as $\varepsilon \to 0$. This ends the proof of (5.18).

Finally, on account of (5.17), (5.18) and (5.19), we have that

$$\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = \lim_{\varepsilon\to 0} \sum_{\widetilde{x}_\mathrm{k}} \int_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} (\phi - u^\varepsilon)\phi \frac{\partial W}{\partial \nu_x} \, \mathrm{d}s_x, \tag{5.20}$$

and we proceed as in the proof of the third step of Theorem 4.1 with the suitable modifications which imply using (5.3), (5.4), (4.9), the extension of $\phi(\phi - u^\varepsilon)$ by symmetry to the lower half-space $\{x_3 < 0\}$ and Lemma 4.1. Thus, the limit in (5.20) reads

$$\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{(r_\varepsilon + (\varepsilon/8))^2} \sum_{\widetilde{x}_\mathrm{k}} \int_{\Gamma^+_{\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}}} (\phi - u^\varepsilon)\phi \mathcal{K} \, \mathrm{d}s_x$$
$$= r_0 \Big\langle \frac{\partial W}{\partial n_y}, 1 \Big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \int_\Sigma (\phi - u^0)\phi \mathrm{d}\hat{x}.$$

This shows (5.16) and providing the limit in (5.14), and the theorem is proved. □

Remark 5.2. As regards the convergence of solutions in the rest of the cases stated in Sect. 2.1, we note that when $r_0 = 0$, the convergence (3.32) takes place in $H^1(\Omega)$ (cf. (3.33)), and the proof above simplifies providing that u^0 in (2.9) is the solution of (2.19). Let us refer to the analysis in [33] when $r_0 = +\infty$.

Also, it should be mentioned that combining the technique here developed with that in [13] will likely allow us to broach the vector problem arising in the nonlinear homogenization on a strainer Winkler-type foundation, which describes the interaction of an elastic body with a nonlinear elastic foundation. This remains as an open problem to be considered by the authors in a forthcoming research.

Acknowledgements

Supported by Grant PGC2018-098178-BBI00 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe".

Funding Information Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

References

[1] Brillard, A., Gómez, D., Lobo, M., Pérez, E., Shaposhnikova, T.A.: Boundary homogenization in perforated domains for adsorption problems with an advection term. Appl. Anal. **95**, 218–237 (2016)

[2] Brillard, A., Lobo, M., Pérez, E.: Homogénéisation de frontières par epi-convergence en élasticité linéare. RAIRO Modél. Math. Anal. Numér. **24**, 5–26 (1990)

[3] Brezis, H.: Problèmes unilatéraux. J. Math. Pures Appl. **51**, 1–168 (1972)

[4] Chechkin, G.A., Gadyl'shin, R.R.: On boundary-value problems for the Laplacian in bounded domains with micro inhomogeneous structure of the boundaries. Acta Math. Sin. (Engl. Ser.) **23**, 237–248 (2007)

[5] Chechkina, A.G.: Convergence of solutions and eigenelements of Steklov type boundary value problems with boundary conditions of rapidly varying type. J. Math. Sci. (N.Y.) **162**, 443–458 (2009)

[6] Chechkina, A.G.: On the behavior of the spectrum of a perturbed Steklov boundary value problem with a weak singularity. Differ. Equ. **57**, 1382–1395 (2021)

[7] Cioranescu, D., Murat, F.: Un terme étrange venu dáilleurs I & II. In: Brezis, H., Lions, J.L. (eds) Nonlinear Partial Differential Equations and their Applications, Collège de France Séminar, Vols II and III, Pitman Research Notes in Mathematics, Vols. 60 and 70, pp. 98–138, 154–178. Pitman, London (1982)

[8] Damlamian, A., Li, T.T.: Homogénéisation sur le bord pour des problèmes elliptiques. C.R. Acad. Sci. Paris Sér. I Math. **299**, 859–862 (1984)

[9] Gómez, D., Lobo, M., Pérez-Martínez, M.-E.: Asymptotics for models of non-stationary diffusion in domains with a surface distribution of obstacles. Math. Methods Appl. Sci. **42**, 403–413 (2019)

[10] Gómez, D., Lobo, M., Pérez, E., Sanchez-Palencia, E.: Homogenization in perforated domains: a Stokes grill and an adsorption process. Appl. Anal. **97**, 2893–2919 (2018)

[11] Gómez, D., Nazarov, S.A., Pérez, M.-E. Homogenization of Winkler-Steklov spectral conditions in three-dimensional linear elasticity. Z. Angew. Math. Phys. 69, Paper 35, 23 pp (2018)

[12] Gómez, D., Nazarov, S.A., Pérez-Martínez, M.-E.: Spectral homogenization problems in linear elasticity with large reaction terms concentrated in small regions of the boundary. In: Constanda C, (eds) Computational and Analytic Methods in Science and Engineering, pp. 119–141. Birkäuser, Springer, N.Y. (2020)

[13] Gómez, D., Nazarov, S.A., Pérez-Martínez, M.-E.: Asymptotics for spectral problems with rapidly alternating boundary conditions on a strainer Winkler foundation. J. Elast. **142**, 89–120 (2020)

[14] Gómez, D., Pérez, E., Podolskiy, A.V., Shaposhnikova, T.A.: Homogenization of variational inequalities for the p-Laplace operator in perforated media along manifolds. Appl. Math. Optim. **79**, 695–713 (2019)

[15] Gómez, D., Pérez, E., Shaposhnikova, T.A.: On homogenization of nonlinear Robin type boundary conditions for cavities along manifolds and associated spectral problems. Asymptot. Anal. **80**, 289–322 (2012)

[16] Goncharenko, M.: The asymptotic behaviour of the third boundary-value problem solutions in domains with fine-grained boundaries. In: Homogenization and Applications to Material Sciences, GAKUTO Internat. Ser. Math. Sci. Appl., Vol. 9, pp. 203–213. Gakkotosho, Tokyo (1995)

[17] Gustafson, K., Abe, T.: The third boundary condition—Was it Robin's? Math. Intell. **20**, 63–71 (1998)

[18] Ionescu, I., Onofrei, D., Vernescu, B.: Γ-convergence for a fault model with slip-weakening friction and periodic barriers. Q. Appl. Math. **63**, 747–778 (2005)

[19] Kaizu, S.: The Poisson equation with semilinear boundary conditions in domains with many tiny holes. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36**, 43–86 (1989)

[20] Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: a Study of Variational Inequalities and Finite Element Methods. SIAM Studies in Applied Mathematics, 8. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1988)
[21] Ladyzhenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Flow, Mathematics and Its Applications, vol. 2. Gordon and Breach. Science Publishers, New York (1969)
[22] Landau, L., Lifchitz, E.: Physique Théorique. Tome 7. Théorie de l'Élasticité. Mir, Moscow (1990)
[23] Leguillon, D., Sanchez-Palencia, E.: Computation of Singular Solutions in Elliptic Problems and Elasticity. Masson, Paris (1987)
[24] Lions, J.L.: Quelques Méthodes de Résolution des Problémes aux Limites non Linéaires. Dunod, Paris (1969)
[25] Lobo, M., Oleinik, O.A., Pérez, M.E., Shaposhnikova, T.A.: On homogenization of solutions of boundary value problems in domains, perforated along manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4^e série, **25**, 611–629 (1997)
[26] Lobo, M., Pérez, E.: Asymptotic behaviour of an elastic body with a surface having small stuck regions. RAIRO Modél. Math. Anal. Numér. **22**, 609–624 (1988)
[27] Lobo, M., Pérez, E.: The skin effect in vibrating systems with many concentrated masses. Math. Methods Appl. Sci. **24**, 59–80 (2001)
[28] Lobo, M., Pérez, E.: Long time approximations for solutions of wave equations associated with the Steklov spectral homogenization problems. Math. Methods Appl. Sci. **33**, 1356–1371 (2010)
[29] Marchenko, V.A., Khruslov, E.Ya.: Boundary Value Problems in Domains with a Fine-Grained Boundary. Izdat. Naukova Dumka, Kiev (1974) (**in Russian**)
[30] Murat, F.: The Neumann sieve. In: Nonlinear Variational Problems (Isola d'Elba, 1983). Res. Notes in Math. 127, pp. 24–32. Pitman, Boston, MA (1985)
[31] Pérez-Martínez, M.-E.: Problemas de homogeneización de fronteras en elasticidad lineal. PhD Thesis. Universidad de Cantabria, Santander (1987)
[32] Pérez-Martínez, M.-E.: Homogenization for alternating boundary conditions with large reaction terms concentrated in small regions. In: Donato, P., Luna-Laynez, M. (eds) Emerging Problems in the Homogenization of Partial Differetial Equations. ICIAM2019 SEMA SIMAI Springer Ser. 10, pp. 37–57. Springer (2021)
[33] Pérez, E., Shaposhnikova, T.A.: Boundary homogenization of a variational inequality with nonlinear restrictions for the flux on small regions lying on a part of the boundary. Dokl. Math. **85**, 198–203 (2012)
[34] Sanchez-Hubert, J., Sanchez-Palencia, E.: Vibration and Coupling of Continuous Systems. Asymptotic Methods. Springer, Heidelberg (1989)
[35] Sanchez-Palencia, E.: Boundary value problems in domains containing perforated walls. In: Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar, Vol. III. Res. Notes in Math. Vol. 70, pp. 309–325. Pitman, Boston (1982)
[36] Smirnov, V.I.: A Course of Higher Mathematics, vol. V. Pergamon Press, Oxford-New York (1964)
[37] Sobolev, S.L.: Some Applications of Functional Analysis in Mathematical Physics. Leningrad University Press (1950). Translations of Mathematical Monographs Vol. 90, American Mathematical Society, Providence, RI (1991)
[38] Trèves, F.: Basic Linear Partial Differential Equations. Academic Press, New York (1975)

D. Gómez
Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria
Avenida de los Castros s/n
39005 Santander
Spain
e-mail: gomezdel@unican.es

M.-E. Pérez-Martínez
Departamento de Matemática Aplicada y Ciencias de la Computación
Universidad de Cantabria
Av. de los Castros s/n
39005 Santander
Spain
e-mail: meperez@unican.es

(Received: March 4, 2022; revised: July 17, 2022; accepted: September 26, 2022)