Comment on "The Winfree model with non-infinitesimal phase-response curve: Ott-Antonsen theory" [Chaos 30, 073139 (2020)]

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In a recent paper [Chaos **30**, 073139 (2020)] we analyzed an extension of the Winfree model with nonlinear interactions. The nonlinear coupling function Q was mistakenly identified with the non-infinitesimal phase-response curve (PRC). Here, we asses to what extent Q and the actual PRC differ in practice. By means of numerical simulations, we compute the PRCs corresponding to the Q functions previously considered. The results confirm a qualitative similarity between the PRC and the coupling function Q in all cases.

In Ref.[1] we studied this generalization of the Winfree model of globally coupled phase oscillators:

$$\dot{\theta}_i = \omega_i + Q(\theta_i, A), \tag{1a}$$

$$A = \frac{\varepsilon}{N} \sum_{j=1}^{N} P(\theta_j). \tag{1b}$$

Here, A is proportional to the sum over the pulses emitted by the N oscillators of the population. In contrast to the original model[2,3], function Q in Eq. (1a) has a nonlinear dependence on the mean field A. The motivation for this is the fact that nonlinearity is an unavoidable consequence of applying phase reduction beyond the first order to oscillator ensembles[4]. Note that a Taylor expansion of Q to nth order in A yields up to $(n+1)$-body phase interactions, similarly to Ref.[5].

We mistakenly called Q 'non-infinitesimal phase-response curve' in Ref.[1]. Properly speaking, function Q is a non-linear 'coupling function'[4]. The aim of this comment is to clarify to what extent the coupling function Q determines the actual phase-response curve (PRC). The PRC quantifies the phase shift gained by an oscillator in response to an external stimulus[6]. There is no analytic relation between Q and the PRC beyond the small ε limit; in that case $Q(\theta, A) \simeq \tilde{Q}(\theta)A$, where $\tilde{Q}$ turns out to be so-called infinitesimal PRC (iPRC). In consequence, we rely here on numerical simulations to compute the PRC empirically.

The family of functions $Q(\theta, A)$ considered in[1] was:

$$Q(\theta, A) = f_1(A)(1 - \cos\theta) - f_2(A)\sin\theta. \tag{2}$$

Four representative pairs of functions $f_{1,2}(A)$ were studied in detail in[1] and the corresponding coupling functions $Q(\theta, A)$ were depicted in Fig. 2 of Ref.[1]. With the aim of comparing them, we obtain the PRC for each of the four coupling functions Q considered in[1].

The PRC value depends on the timing as well as on the specific shape of the stimulus, which is not necessarily weak or brief [6]. Numerically, we obtain the PRC measuring the effect on one oscillator's phase of a pulse generated by another oscillator. This means that the two oscillators are unidirectionally coupled (i.e., a master-slave configuration). We adopt $\omega = 1$ as the natural frequency for both, perturbed and perturbing oscillators, which is the obvious choice as it is the central frequency of the distribution in[1]. Moreover, we follow[1] and use the same 2π-periodic symmetric unimodal pulse function $P(\theta)$. It vanishes at $\theta = \pm\pi$, and a free parameter $r < 1$ controls the narrowness of P: The height of the pulse is $P(0) = 2/(1-r)$, and $\lim_{r\to 1} P(\theta) = 2\pi\delta(\theta)$. In this comment we consider two different pulse widths: $r = 0.9$ (the value selected in[1]), and $r = 0.99$ corresponding to an extremely narrow pulse.

The simulation starts at time $t = 0$ with the (slave) oscillator at an initial phase θ_{in}. Then, we let it to evolve under the influence of the forcing oscillator. The phase of this one grows linearly, such that the input felt by the first oscillator is $A(t) = \varepsilon P(t - \pi)$. Parameter ε determines the strength of the stimulus. The simulation runs from $t = 0$ to $t = 2\pi$, since A exactly vanishes at these times. Note that we do not need to run the simulation further since phase oscillators are governed by first-order differential equations. For a given ε value, we measure the phase shift at $t = 2\pi$ such that $\text{PRC}(\theta_0, \varepsilon) = \theta(t = 2\pi) - (\theta_{\text{in}} + 2\pi)$. The phase θ_0 in the argument of the PRC is the phase value when A attains its maximum, assuming no input exists: $\theta_0 = \theta_{\text{in}} + \pi$. The results are shown in Figs. 1 and 2 for a set of ε values; in each panel for one particular coupling function $Q(\theta, A)$ already adopted in[1]. In all panels, the corresponding iPRC is shown as a reference. Note that the normalization of the y-axis in Figs. 1 and 2 includes a 2π factor —in addition to ε— because this is the integral of the pulse over an interval of length 2π. Figures 1 and 2 are quite similar, though for $r = 0.9$ (Fig. 1) the PRCs remain closer to their iPRCs up to a larger ε value.

The comparison of the PRCs in this comment with the corresponding Q functions in Fig. 2 of Ref.[1] evidences that $Q(\theta, A)$ is not simply the PRC. Indeed, Q in (2) has only the first harmonic in θ, whereas the non-infinitesimal PRCs in the figures display additional Fourier components. In spite of these dissimilarities, simple visual inspection indicates that the PRC strongly resembles the coupling function Q in all four cases. For example, we observe the same loss of non-

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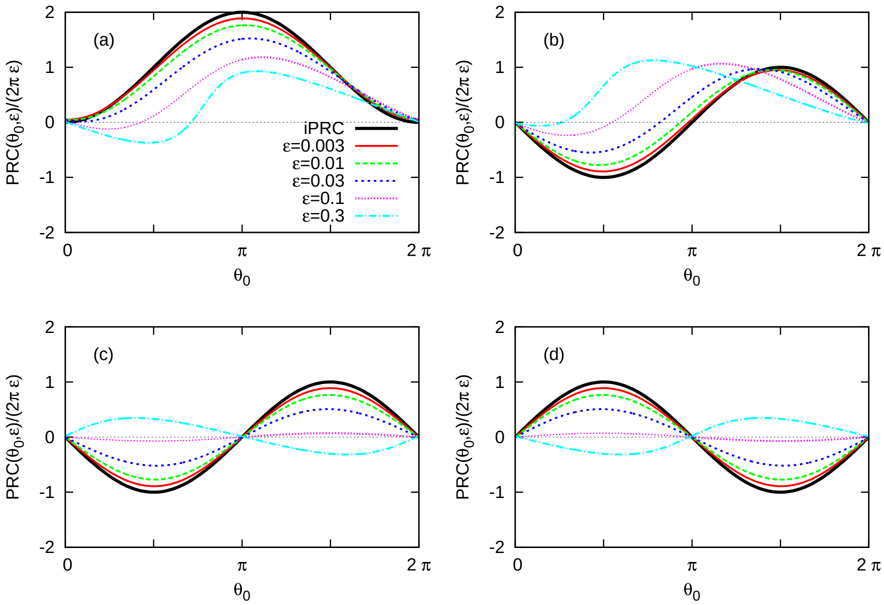

FIG. 1. PRCs for cases (a-d) in Ref.[1]. The set of five ε values used in indicated in panel (a). The coupling function (2) in each panel is: (a) $f_1 = A/(1+A) = f_2/A$; (b) $f_1 = A^2/(1+A) = Af_2$; (c) $f_1 = 0$, $f_2 = A(1-A)/(1+A)$; (d) $f_1 = 0$, $f_2 = A(A-1)/(1+A)$. The pulse acting on the oscillator is $P(t-\pi) = \frac{(1-r)[1+\cos(t-\pi)]}{1-2r\cos(t-\pi)+r^2}$, with $r = 0.9$.

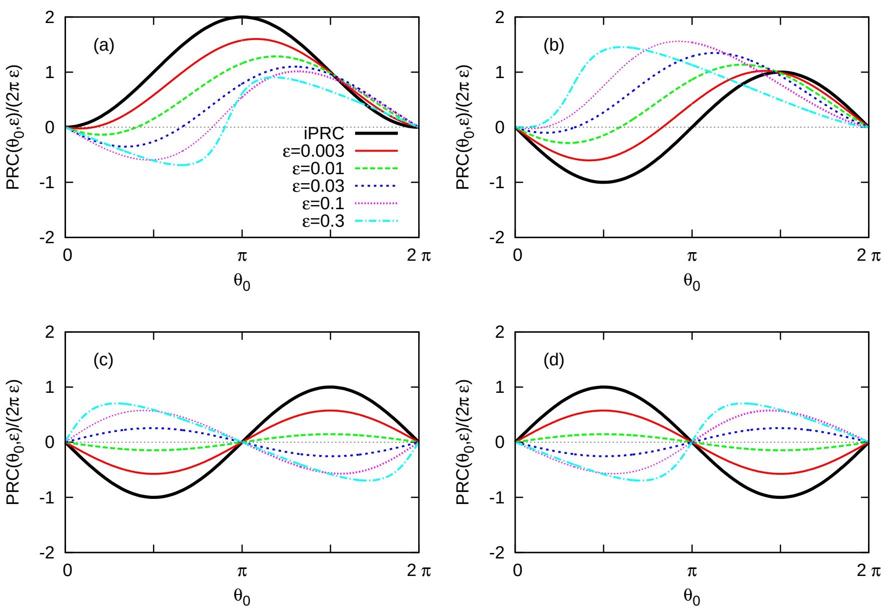

FIG. 2. The same as Fig. 1 with $r = 0.99$.

negativeness of the (type-I) iPRC as A increases in panel (a), or the transition from a synchronizing iPRC to a desynchronizing PRC for large enough A in panel (c). Summarizing, our simulations confirm that the main attributes of the coupling function Q are shared by the non-infinitesimal PRC.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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