International Journal of Structural Stability and Dynamics
EQUIVALENT STATIC FORCE IN HEAVY MASS IMPACTS ON STRUCTURES
--Manuscript Draft--

Manuscript Number:	IJSSD-D-21-00549R1
Full Title:	EQUIVALENT STATIC FORCE IN HEAVY MASS IMPACTS ON STRUCTURES
Article Type:	Research Paper
Section/Category:	
Keywords:	Impact; Equivalent force; Mass ratio; Energy distribution; Modified frequency; Computational cost
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Response to Reviewers:	

Electronic version of an article published as International Journal of Structural Stability and Dynamics 2022, 22, 02, 2250025, DOI 10.1142/S0219455422500250 © World Scientific Publishing Company, https://www.worldscientific.com/

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EQUIVALENT STATIC FORCE IN HEAVY MASS IMPACTS ON STRUCTURES

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ABSTRACT

Structural engineers usually make simple hand calculations to pre-design structures before performing further complex analysis. However, in the case of structures subjected to impact loads, the problem is quite difficult to cope with, due to a lack of knowledge about simplified methods of analysis that can reproduce with sufficient accuracy the structural behaviour, in order to establish an adequate pre-dimensioning, prior to a computer calculation.

In this paper, a new formulation is presented to solve, the problem of large mass impact on any kind of structure in a simplified way. The method enables a force equivalent to impact load to be obtained that enables analysis as a static load case.

KEYWORDS: Impact; Equivalent force; Mass ratio; Energy distribution; Modified frequency; Computational cost.

HIGHLIGHTS

The proposed formulation estimates the equivalent static force under impact loads

This paper shows that the projectile mass behaves as a structural mass in impacts

The mass ratio defines the energy distribution between the modes in impacts

The proposed formulation enables FEM results to be verified and to structures to be pre-designed

1. INTRODUCTION

In the late XIX century, classical elastic impact theories [1][2] could be divided into two trends: Saint-Venant's theory [3] and Hertz's theory [4]. Saint-Venant's theory considers only the effect of elastic vibration under an impact load and Hertz's theory considers just the local deformation. Thimoshenko [5] proposed a combined method, which became the reference, resulting in a non-linear equation that is solved by step-by-step numerical integration. Although Timoshenko does not include the mass of the projectile as part of the mass of the structure (an important issue as authors show in this paper), his proposal was an accurate solution but was very tedious to solve. To avoid this problem, Lennertz's simplification [6], the Galerkin method and the Collation method by Eringen [7] assume a force function, but all of them obtain clear discrepancies compared with some real impact tests.

Engineers need to understand the physical problem within the tedious mathematical solution to design with confidence and to take control of the solution to check the numerical results (the restitution coefficient is mainly used nowadays [8] due to its simplicity). For these reasons, previously, with Timoshenko's solution and nowadays with FEM solutions (a long time is needed for calculations) simplified methods have been developed to check and to try to understand impact problems.

In 1940, Lee [9] proposed a simplified method based on the spring-mass model, with one degree-of-freedom (1-DoF), to solve impacts on beams. The spring idealizes the stiffness of the fundamental mode and the concentrated mass represents the effective mass of the beam corresponding with that mode. To solve the problem, Lee assumed a contact force function (here this is not assumed, but deduced) and presented a complex energetic method to know in which cases the proposed simplified model works.

To extend the casuistry, in 1983, Lee, Hamilton and Sullivan [10] proposed a spring-mass model based on the previous energetic method, but it needed computational integration again.

In 1981, Suaris and Shah [11] suggested a model with 2-DoF to study the inertia effects on impact tests of cementitious composites. This model includes a spring for local contact rigidity and a striker mass in addition to the general spring and mass described in a 1-DoF model.

Even though a complex model with 2-DoF was developed by Shivakumar [12] in 1985 to take into account shearing, and membrane forces, in addition to bending and local contact, until now 2-DoF models have been developed [13][14] only to predict the elastic bending behaviour of structures, in order to maintain the simplicity while improving the accuracy and the applicability.

The latest complex analytical [15] models developed, or even mixed models [16] (analytical + local FEM model), aim to find another way to solve impact problems, while reducing the time of calculation of any commercial software, but not to simplify the understanding of the physical concept, check FEM results (it is hard to define all the parameters properly) or facilitate the pre-design of structures. There is a specific disadvantage with these kinds of models, which are actually different models (not a general one) depending on the type of the structure, longitudinal bars [17][18][19], circular slabs [20], elliptic slabs [21], domes [22], etc.

Thus, models with 2-DoF are currently the most useful tool to simplify impacts. However, these models display the following disadvantages. Although they are simple models with 2-DoF, they need a spreadsheet to calculate eigenvalues, eigenvectors and the matrix solution [23]. It is not clear when these models can be used to solve impacts with sufficient accuracy; some authors suggest that they can only be used when a high ratio exists between periods of projectile and structure (soft impacts), but with different limit values depending on whether it is a beam [14] or a slab [13]. Impact velocity [24] or mass ratios [15] are also parameters that authors limit on the use of their models. In addition, it is not always easy to establish the local stiffness [25] or the striker's dynamic properties [26] to use the 2-DoF model with confidence. Even the general dynamic properties of the structure are not always clear, especially in the case of slabs [13].

Impact problems are still under study, some of them with experimental tests in classical materials [27][28][29], but also with new materials [30][31][32][33] because they are important for structural design. However, structural design engineers still need an easy tool to deal with impact loads and to explain the structural behaviour under impact loads (such as mass striker influence [34] or support conditions influence [35]) in the same way as other structural areas [36][37]. Like in static analysis, "big numbers" are necessary to allow engineers to pre-design a structure accurately enough before making complex calculations and spending a long time waiting for results.

In this regard, this article presents a simplified theory that is particularized in an analytical formulation applicable to solve the problem of impact of a mass on any structure as a static force. The objective is to enable the verification of the results from the structural software and pre-design structures subjected to impact by using pre-calculations with equivalent static force. To calculate a structure with this article's formulation, it is necessary to transform the structure into a spring-mass system (1-DoF), corresponding to the fundamental mode of vibration. This system is specific for the point of impact. There are several ways to calculate the mass and the equivalent stiffness of a structure for one mode of vibration and a given point of contact. In this article, a simple method to calculate these values is presented. Assuming the projectile is infinitely rigid and the structure displays elastic behaviour, the formulation enables the calculation of the worst possible scenario from the structural collapse point of view, and hence provides results on the safe side. Plastic behaviour of the structure could also be predicted with the adequate parameters. The impacts analysed in this article are central impacts, that is, the point of contact and the centre of gravity of the impacting bodies are aligned. The results obtained by the proposed theory are compared to the results from the elastic and geometry non-linear calculations by means of Midas NFX software.

This article is organised as follows: the introduction sets out the general background and the main objectives. In the material and methods section, firstly, the seven scenarios (case studies) analysed are presented. Secondly, FEM models are described to provide sufficient details to allow the seven cases to be reproduced by independent researchers. In the theory section the stiffness and equivalent mass parameters of the structure are obtained. After that a general impact case is proposed, obtaining the proposed formulation to calculate the impact force and the displacement produced in the structure. In the next section, the results are validated by means of finite element models and then discussed. Finally, the main conclusions reached in the investigation are highlighted.

2. MATERIAL AND METHODS

2.1. Case studies

In order to confer the method a general character, the impact is analysed on three different structures (a simple supported steel beam, a steel cantilever, and a concrete slab) with different energy absorption mechanisms (flexural for structures 1 and 3, axial for structure 2). Fig. 1 shows the definition of the structures.

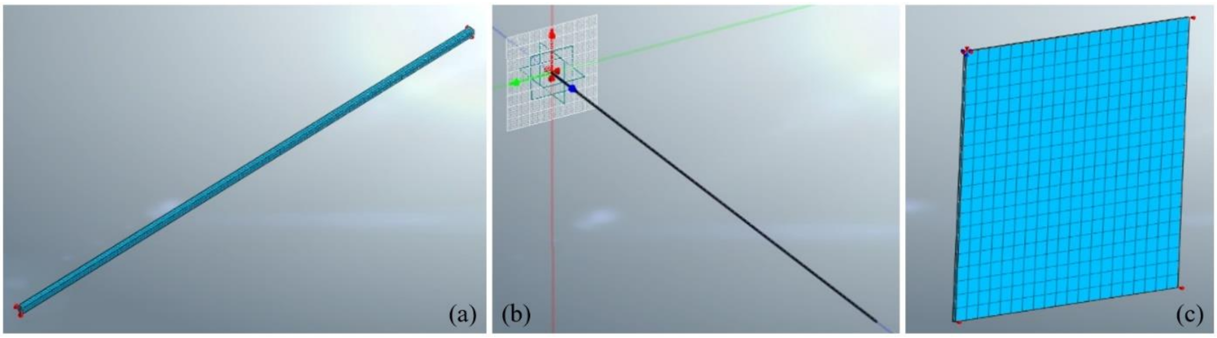

Fig. 1: Definition of the structures studied. **(a)** Structure 1: Simple supported steel (BS 355) beam of 10m long and 0.1x0.1 m^2 cross section, E=2.1·e11 N/m^2, ρ=7,850 Kg/m^3. **(b)** Structure 2: Steel (BS 275) cantilever of 5m long and 0.01x0.01 m^2 cross section, clamped at one end, E=2.1e11 N/m^2, ρ=7,850 Kg/m^3. **(c)** Structure 3: Concrete slab C 45 of 10x10 m^2 section, 0.10 m width and simply supported at its 4 corners, E=3.2·e10 N/m^2, ρ=2,549 Kg/m^3.

In order to prove that the formulation adapts to different impacts on the same structure, five cases will be analysed on structure 1 (case 1 to 5), varying the projectile mass, keeping the velocity prior to impact (V_{p0}) constant. In addition, to generalise for all kinds of structures and impact velocities, two more impacts are analysed on structure 2 and 3 respectively to show that the method works in any structure (case 6 and 7), Table 1.

Case	**Structure**	**Projectile mass (kg)**	**V_{p0} (m/s)**	**Type of impact**	**Impact point**
1	**1**	3,869.0	2.0	Transversal	Centre beam
2	**1**	1,934.0	2.0	Transversal	Centre beam
3	**1**	1,161.0	2.0	Transversal	Centre beam
4	**1**	774.0	2.0	Transversal	Centre beam
5	**1**	387.0	2.0	Transversal	Centre beam
6	**2**	16.7	3.0	Longitudinal	End cantilever
7	**3**	103,665.0	0.5	Transversal	Centre plate

Table 1: Definition of analysed impacts

2.2. Description of the FEM models

The proposed formulation is compared with all the data obtained from the FE software Midas NFX. For each case, the structure was modelled through solid elements, defining a contact without friction between bodies and using a nonlinear explicit transient analysis type. All cases were calculated with the explicit method of Midas NFX, Fig. 2.

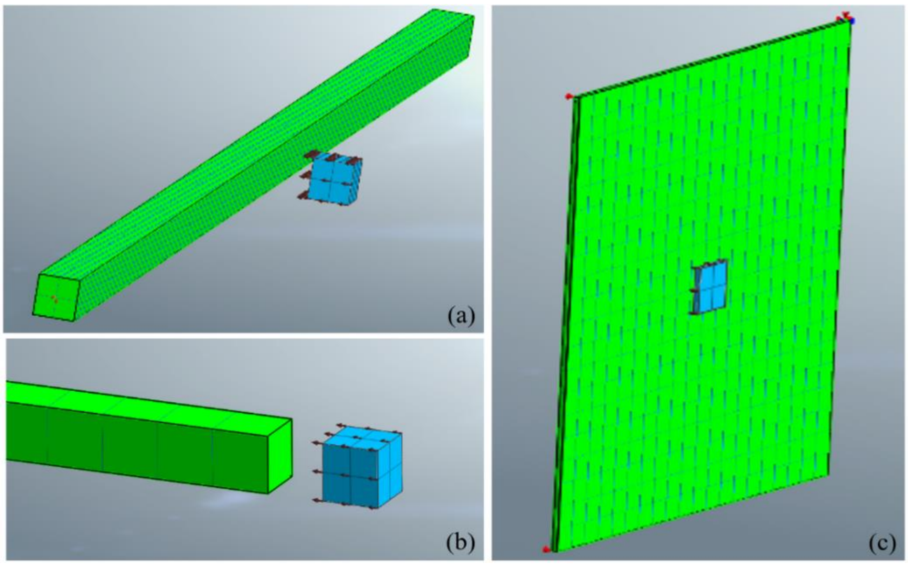

Fig. 2: Initial velocity of each projectile to direct it towards each structure. **(a)** Structure 1 (simple supported beam). **(b)** Structure 2 (Cantilever beam). **(c)** Structure 3 (slab supported at 4 corners).

The projectile was given small dimensions to act as a point mass. The mesh size and the total mass for each structure, the projectile dimensions and its density for each case, and the mass ratio α are compiled in Table 2.

Structure		Length (m)	Width (m)	Height (m)	Case	Mass (kg)	α	Density (kg/m^3)
1	Beam mesh size	0.05	0.05	0.05	1	3,869.0	10	1,934,491
					2	1,934.0	5	967,246
					3	1,1610	3	580,347
	Projectile dimensions	0.10	0.10	0.10	4	774.0	2	386,898
					5	387.0	1	193,449
2	Cantilever mesh size	0.01	0.01	0.01	6	16.7	10	1,670,111
	Projectile dimensions	0.01	0.01	0.01				
3	Slab mesh size	0.50	0.50	0.10	7	103,665.0	8	1,036,653
	Projectile dimensions	1.00	1.00	0.10				

Table 2 Case definition

In order to reproduce the hypothesis of an infinitely rigid projectile, its elastic modulus has been set to 5e15 N/m^2.

The time increment used for the numerical integration in Midas NFX is 5e.-4s in all cases for structure 1, 5e-6s for structure 2 and 2e-4s for structure 3.

The initial velocity of the projectile was set to the corresponding values in each case (2m/s, 3m/s and 0.5m/s for structures 1, 2 and 3 respectively) at the nodes of the element.

3. THEORY

3.1. Structure as spring-mass system

To use the formulation suggested in this article, it is necessary to transform the structure into a spring-mass system, enabling the dynamic behaviour of the structure's fundamental mode to be reproduced.

There are several ways to perform this transformation [38][39]. Given that this is not the main objective of the article, a sufficiently precise but simple approximation will be used to determine the equivalent stiffness and equivalent mass of the structure's associated system.

This approximation can only be applied when the displacement shape of a punctual static load at the point of impact and the displacement shape of the mode of vibration are similar, so that the projectile mass is included as part of the structure's mass at the point of impact. The more similar the two displacement shapes are, the less error in the values of mass and stiffness will be produced, being correct in the case of matching displacement shape.

In order to calculate the *Equivalent Stiffness* (K_s) at the contact point, a generic single static load must be applied at the point of contact with the structure, and the equivalent stiffness will be the ratio between the load and the displacement produced at that point. So, in general, the point at which the equivalent stiffness must be calculated differs for each structure. Equivalent Stiffness should be calculated considering the dynamic properties of the material and that concrete's elastic modulus is usually stiffer in dynamic behavior. In concrete, the dynamic modulus of elasticity is often considered as the initial tangent modulus of the static test, and this is a good approximation. However, the origin of the difference between static and dynamic modulus is more to do with the nature of material than strain level [40]. In fact, the relationship of these parameters could be considered almost linear [41], independently of the strain level. There are many tests used to estimate the dynamic modulus of elasticity of concrete, E_d, but with different results depending on the kind of the test [42]. There are also some expressions to relate the static and dynamic moduli of concrete [43][44] but in this paper, in order to continue with the aim of simple calculations, the relation proposed by Lyndon is recommended [45]; that is to say the dynamic modulus of elasticity could be considered 20% higher than the static one. The dynamic modulus of concrete in case 7 is shown in the caption of Fig.1 and it is used both in the proposed formulation and the FEM model. Equivalent stiffness values are shown in Table 3.

From the equation $M_s = K_s(T/2\Pi)^2$ related to the single degree of freedom systems, the *Equivalent Mass* (M_s) is estimated. M_s might be calculated directly without using the FEM model but it is not the aim of this paper. A modal analysis by finite elements is performed for each of the aforementioned structures to obtain the natural period of vibration values, Fig. 3.

Structure	Type	Impact point location	Static load location	Fundamental mode	Equivalent stiffness (K_s) (N/m)	
1	Simple supported steel beam	Central point	Centred on the central span of the beam	Flexural	$K_s = \frac{P}{f} = \frac{48EI}{L^3}$	84,000
2	Steel cantilever	Extreme point	At the end of the cantilever	Axial	$K_s = \frac{P}{f} = \frac{EA}{L}$	4.2e6
3	Concrete slab*	Central point	Centred at the centre of the slab	Flexural	$K_s = \frac{P}{f} = \frac{100{,}000}{0.141}$	709,894.5

*Since there is no analytical formula to calculate stiffness as in the previous structures, a specific case will be applied in this particular one. A generic static force (100 KN) is applied at the centre of the plate and the deflection is calculated at the same point (0.141 m) by means of a finite element model.

Table 3: Impact point and static load location to estimate equivalent stiffness (K_s) in each structure

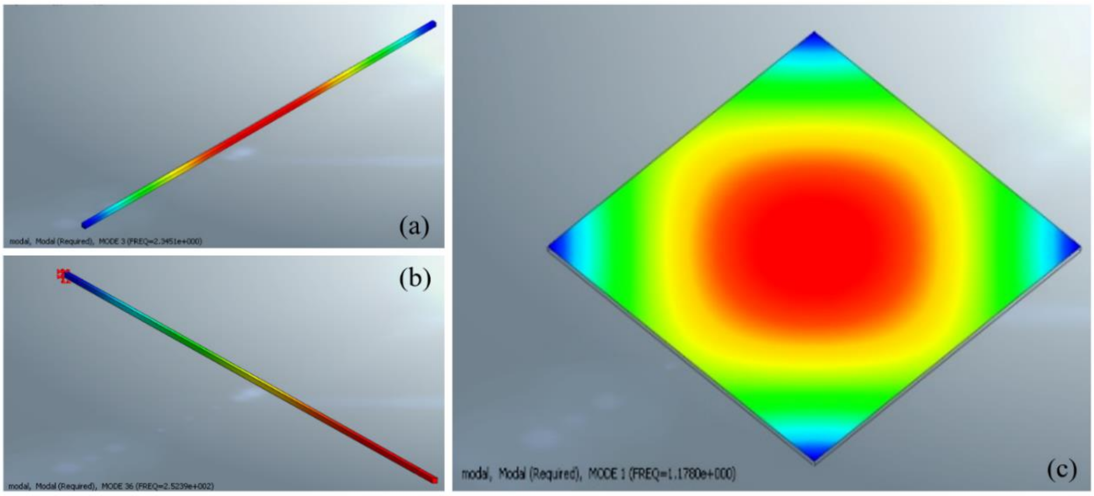

Fig. 3: Fundamental modes. **(a)** Fundamental mode for flexural load in structure 1 (simple supported beam). Frequency 2.345 Hz. **(b)** Fundamental mode for axial load in structure 2 (Cantilever beam). Frequency 252.390 Hz. **(c)** Fundamental mode for flexural load in structure 3 (slab simply supported at 4 corners). Frequency 1.178 Hz.

From the frequencies obtained, and the equivalent stiffness, Table 3, the equivalent mass (M_s) is inferred by considering the previous equation. In all three cases, the equivalent mass accounts for approximately 50% of the total mass of the structure, Table 4.

Structure	Frequency (Hz)	Period (sec)	Stiffness K_s (N/m)	M_s (kg)	M_s/ M_{total} (%)
1	2.345	4.26e-01	8.40e+04	386.9	49.3
2	252.390	3.96e-03	4.20e+08	167.0	42.6
3	1.178	8.49e-01	7.10e+05	12,958.2	50.8

Table 4. Equivalent mass (M_e) calculation through approximate method.

At this point it is worth mentioning that an important parameter to determine the structure's behaviour under impact loads is the ratio $\alpha = {M_p}/{M_s}$, M_p being the projectile mass and M_s the structure's equivalent mass. This will be addressed later in the article and its influence will be analysed.

3.2. Problem statement and resolution

3.2.1. Initial hypotheses

Initially, some assumptions must be defined in order to state and solve the problem of impact on structures. The closer the initial assumptions are to the actual impact conditions, the closer the results of the formulation are to the exact ones. The following is a description of the assumptions adopted:

The *structure* is initially in idle state, i.e. has no movement. Gravity influence, shear and membrane forces and local deformation have not been considered in the simplified model (however, they have been considered in the FEM carried out). Membrane forces could only have an influence with big displacements but this is not expected in this kind of structures. Shear forces and local deformation increase the total displacement, so they have some influence depending on the case. However, the aim of the paper is to analyse the impact to facilitate pre-design of structures and to check FEM results. Thus, it does not make sense to include them in the formulation if the contribution is limited and they do not modify the general behaviour of structures under impact loads. Local and shear deformation are not important from a pre-design or checking point of view. In any case, if these hypotheses are adequate, they will be analysed later.

The initial velocity of the *projectile* is perpendicular to the structure. The projectile is made of an infinitely stiff material and has no dimensions; it is assumed to be punctual. The mass of the projectile is heavier than the effective mass of the structure.

The *impact* is considered of low velocity, that is, viscous behaviour of the materials are excluded. Structural damping has not been considered due to the fact that impact duration is too short to allow its development. There are no energy losses by heat, noise or structural damping during the impact. The impact is centred, that is, the point of contact and the centre of gravity are aligned. The impact begins when the projectile contacts the structure (t_0), and ends when the projectile stops (t_f). This contact is continuous throughout the impact. Although this assumption will be discussed later, the fact that the mass of the projectile is greater than the effective mass of the structure means the assumption is reasonable, since, according to classical impact equations, a light mass cannot stop a heavy one, Fig. 4.

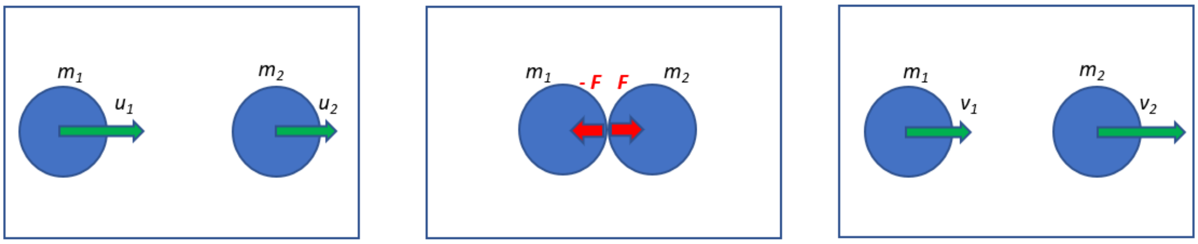

Fig. 4: Classic impact of co-linear masses.

3.2.2. Problem description

A projectile mass M_p is assumed with velocity $V_p(t)$ in a perpendicular direction to the structure. The fundamental mode of the structure has an effective mass M_s and an effective stiffness K_s. Note that dissipative forces, represented as a viscous damper in Fig. 5 (constant C is the viscous damping coefficient), are neglected due to the fact that impact occurs so fast than structural damping has no time to develop significantly.

(a)

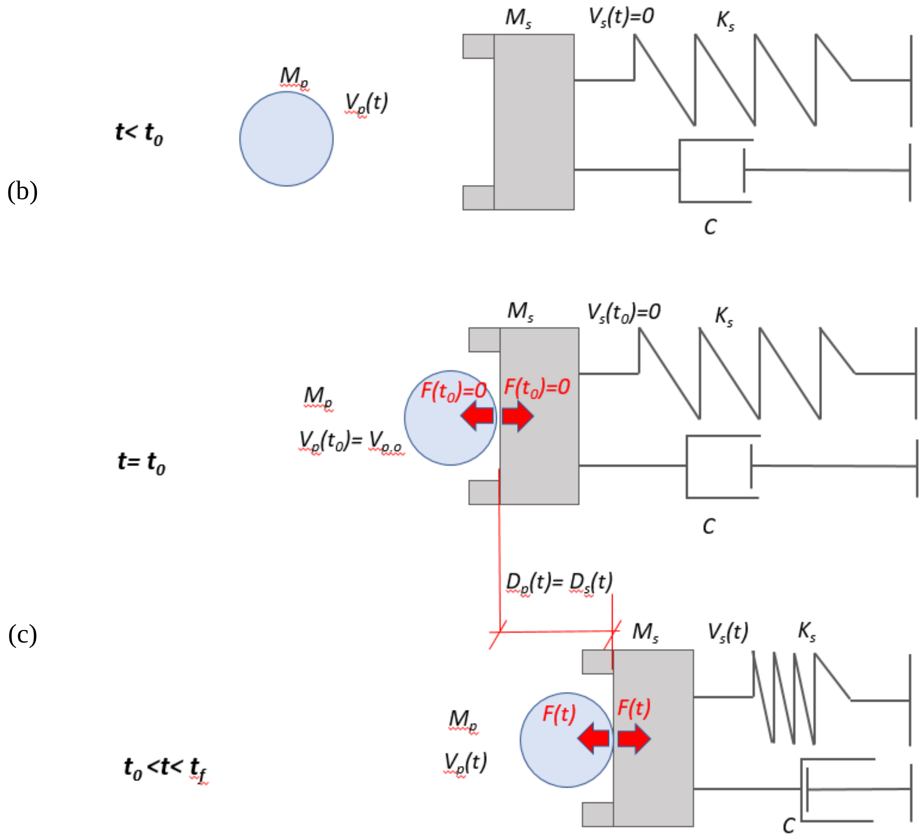

Fig. 5: Idealized impact sequence. **(a)** Situation before the impact between the structure and the projectile. **(b)** Instant of impact ($t = t_0$). **(c)** Situation between the beginning and the end of the impact ($t_0 < t < t_f$).

The structure is initially in idle state ($t < t_0$), Fig. 5(a). When the impact between the structure and the projectile begins ($t = t_0$), the projectile has a previous impact velocity $V_{p,0}$, the structure remains idle, and the contact force between the bodies *F(t)* has not been developed yet, and hence its value is null, Fig. 5(b). From this point on ($t > t_0$), as a consequence of the impact, the structure deforms at the same time as the projectile's velocity gets reduced. At any moment between the beginning and the end of the impact ($t_0 < t < t_f$), the displacement of the projectile $D_p(t)$ and the structure $D_s(t)$ are the same as a consequence of the initial hypotheses, Fig. 5(c). Equations (1) and (2) express, respectively, these displacements at time t.

$$D_p(t) = V_{p,0}t - \iint_0^t \frac{F(t)}{M_p} d\tau\, dt \tag{1}$$

$$D_s(t) = \iint_0^t \frac{F(t) - K_s(D_s(t))}{M_s} d\tau\, dt \tag{2}$$

By developing the following change of variable, $\frac{d^4y}{dt^4} = F(t)$, and operating in equations (1) and (2), as can be verified in [46], equation (3) is obtained.

$$\ddot{y} + W_{sp}^2 y = \frac{M_s M_p}{M_s+M_p}\frac{K_s\, V_{p,0}}{6M_s}t^3 + \frac{M_s\, M_p}{M_s+M_p}V_{p,0}t \quad (3)$$

Where W_{sp} is the vibration frequency during impact, which is related to the effective stiffness, K_s, equivalent mass, M_s, and projectile mass, M_p, as equation (4) expresses.

$$W_{sp}^2 = \frac{K_s}{M_s+M_p} \quad (4)$$

Equation (5) shows the natural vibration frequency of the structure W_s before the impact.

$$W_s^2 = \frac{K_s}{M_s} \quad (5)$$

Meaning that, based on equation (4), the frequency of vibration during the impact includes the projectile mass as part of the structure mass.

3.2.3. Problem resolution

The general solution of equation (3) is defined as the sum of the particular solution and the homogeneous solution (solution details in [46]), applying the corresponding boundary conditions.

The first boundary condition results from making the impact force null at $t = t_0$.

The second boundary condition results from attributing an initial velocity V_I, of value initially unknown, to the projectile (and structure) immediately after the impact, equation (6).

$$V_p(t = 0) = V_I \quad (6)$$

The displacement of the structure and projectile, equation (7), the contact force, equation (8), and, based on that, the equivalent static force, equation (9), can be calculated after applying the boundary conditions (the whole development can be consulted in [46]).

$$D_p(t) = D_s(t) = \frac{V_I\, \sin(W_{sp}\, t)}{W_{sp}};\ D_{s,max} = \frac{V_I}{W_{sp}} \quad (7)$$

$$F(t) = M_p W_{sp} V_I \sin(W_{sp}\, t) \quad (8)$$

$$F_{eq,st} = K_s \frac{V_I}{W_{sp}} \quad (9)$$

Both the displacement and the contact forces, as a function of the velocity just after the impact contact V_I, are still unknown.

The duration of the impact t_f is obtained estimating the time it takes to make the velocity of the projectile null, equation (10).

$$V_p(t = t_f) = V_I \cos(W_{sp}\, t_f) = 0 \rightarrow W_{sp} t_f = \frac{\Pi}{2} \quad \rightarrow \quad t_f = \frac{\Pi}{2\, W_{sp}} = \frac{T_{sp}}{4} \quad (10)$$

Where T_{sp} is the vibration period during impact associated with the frequency W_{sp}.

At $t > t_f$, the movement of the structure changes to the opposite direction and the recovery phase begins, returning to its initial position. As for the projectile, either the contact with the structure remains or it does not. If the former happens, that means the recovery movement of the structure is quick enough to maintain the contact, pushing the projectile in the opposite direction to the one the system had until t_f. The latter though, would occur if the recovery movement is not quick enough, or if the projectile falls laterally or breaks, losing contact with the structure. Equations (7) and (8) remain valid only during the impact $t > t_f$ in the scenario in which the projectile and the structure remain in contact during recovery time. Otherwise, the assumptions about the impact would be different.

In general, the time for impact analysis is restricted to $t \leq t_f$, as the maximum displacement and force on the structure are given within this time period.

3.2.4. Additional hypotheses

The resulting statement of the impact problem expressed in equations (7), (8) and (9) requires two additional assumptions to obtain a general solution.

H.1.: The initial velocity in compression phase. Given the displacement of the projectile and the structure are identical at time *t*, the velocity must also be so. Since prior to the impact the velocities of the projectile and the structure are different, this means there is a previous phase (which will be called *initial phase* from now on) in which the velocities of the bodies pass from being different to equal. After that initial phase, the expressions deduced previously, equations (7), (8) and (9), will then be valid. Assuming the initial phase occurs almost instantaneously, there is no time for the spring K_s to be displaced. If K_s is not displaced, then the force produced by the spring is null. This consideration enables the initial contact to be treated as a classic impact of two free stiff masses, Fig. 6.

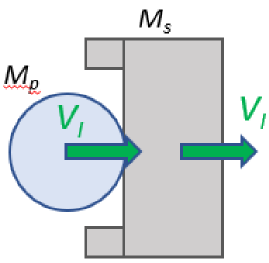

Fig. 6: Initial impact of the bodies without the influence of the spring

In classic impact of stiff masses, Fig. 4, the moment at which the velocities of the two masses become equal during impact is always when the compression phase (compression of the bodies) changes to the restitution phase (decompression of the bodies). At that particular instant in time, the velocities of both masses are the same and the value matches with the final velocity of a perfectly plastic impact. After that instant, the projectile keeps moving forward for as long its mass is greater than the equivalent mass of the structure ($\alpha > 1$). The energy expended in this initial phase is considered negligible (this assumption will be analysed later in the article).

The classic formulation enables the final impact velocities of two bodies 1 and 2 to be obtained, e being the so-called restitution coefficient, equation (11).

$$v_1 = \frac{(m_1 - m_{2e})u_1 + m_2(1+e)u_2}{m_1 + m_2};\ v_2 = \frac{(m_2 - m_{1e})u_2 + m_1(1+e)u_1}{m_1 + m_2} \qquad (11)$$

If equation (11) is applied for a perfectly plastic impact (restitution coefficient $e=0$), the initial impact velocity V_I for both masses can be calculated, considering $u_1=V_{p0}$; $u_2=0$; $m_1=M_p$; $m_2=M_s$, equation (12).

$$v_1 = v_2 = V_I = V_{po} C_m \tag{12}$$

Where C_m is the mass coefficient, $C_m = \frac{M_p}{M_p+M_s} = \frac{\alpha}{\alpha+1}$

Then, finally, bearing in mind equations (9) and (12), the equivalent static force, equation (13), and structural displacement, equation (14), can be written based on initial parameters.

$$F_{eq,st} = M_p V_{po} \sqrt{\frac{K_s}{M_p+M_s}} \tag{13}$$

$$D_s(t) = \frac{V_{po} \cdot \frac{M_p}{M_p+M_s} \, sin\left(\sqrt{\frac{K_s}{M_p+M_s}}\, t\right)}{\sqrt{\frac{K_s}{M_p+M_s}}} \tag{14}$$

H.2.: The stiffness of the fundamental vibration mode at the impact point K_s. The second hypothesis is made based on equation (4), that is, the mass of the projectile becomes part of the structure. Due to limitations of the spring-mass model which lead to the given expression, the mass of the projectile cannot change the spring stiffness. In a real structure, given the deflection of the mode would change when absorbing the mass of the projectile, the stiffness of the mode could be modified. From this statement, the hypothesis can be defined as follows; the spring stiffness must be calculated from the deflection of the structure with an added mass equal to the mass of the projectile at the point of impact. These two hypotheses, together with equations (13) and (14), represent the solution to the impact problem due to a large mass for any given structure.

4. RESULTS

From the values of equivalent stiffness and equivalent masses previously calculated, and through equations (13) and (14), the equivalent static force $F_{eq,st}$ and the displacement of the structure $D_s(t)$ are obtained for the seven impact cases studied, Table 5.

Case	Struct.	α	W_{sp} (rad/s)	V_I (m/s)	$V_I/V_{p,o}$ (%)	$D_s(t)_{max}$ (m)	$t_f = t(T_{sp}/4)$ (s)	$F_{eq,st}$ (kN)	Stress (MPa)	Error (%)
1	**1**	10	4.443	1.818	90.9%	0.4090	0.35400	34.4	515	1.3
2	**1**	5	6.015	1.667	83.4%	0.2770	0.26100	23.3	349	1.4
3	**1**	3	7.367	1.500	75.0%	0.2040	0.21300	17.1	257	6.9
4	**1**	2	8.507	1.333	66.7%	0.1570	0.18500	13.2	198	3.2
5	**1**	1	10.419	1.000	50.0%	0.0960	0.15100	8.1	120	21.7
6	**2**	10	478.141	2.727	90.9%	0.0057	0.00329	23.9	239	0.3
7	**3**	8	2.467	0.444	88.8%	0.1800	0.63700	127.8	36	3.7

* Relative error in equivalent static force has been calculated based on the ratio between maximum displacement as per FEM model and the proposed formulation.

Table 5: Main results per case.

In Fig. 7, the results given by the proposed methodology (red line) and those obtained from Midas NFX (blue line) for the different cases studied in structure 1 are depicted.

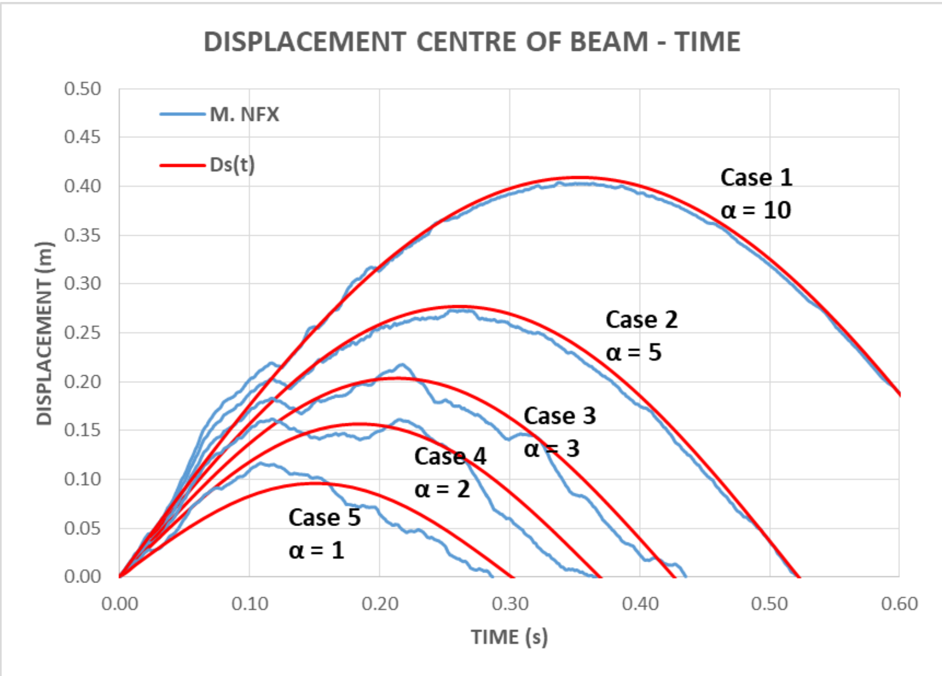

Fig. 7: Comparison of results between the proposed method (red) and Midas NFX (blue) for cases 1-5 (structure 1).

It can be observed that the hypothesis referring to the initial velocity of the impact V_I adapts perfectly in all cases regardless of the conditions. In all cases, the velocity of the projectile prior to contact is the same (2m/s), yet the initial velocity of the impact, dependent on the mass coefficient C_m, is different in each case, Table 5. This initial velocity of the impact is shown on Fig. 7 as the initial tangent (at $t = 0$) of the displacement, and it can be observed how well it approximates the initial behaviour of the structure. It must be pointed out that the formulation proposed in this paper only covers the fundamental mode, and thus, it does not capture all the small displacement variations added by the rest of the vibration modes.

It can also be observed that the match between the results obtained by means of the proposed formulation and FEM becomes better as the coefficient α becomes greater. For α values equal to or less than 2, the differences might be only slightly significant, whereas for α values greater than 5 the match proves to be quite good.

Fig. 8 represents the values obtained from Midas NFX (solid blue) and the results from the proposed formulation (solid red, coinciding with the blue line) for structure 2 (impact case 6).

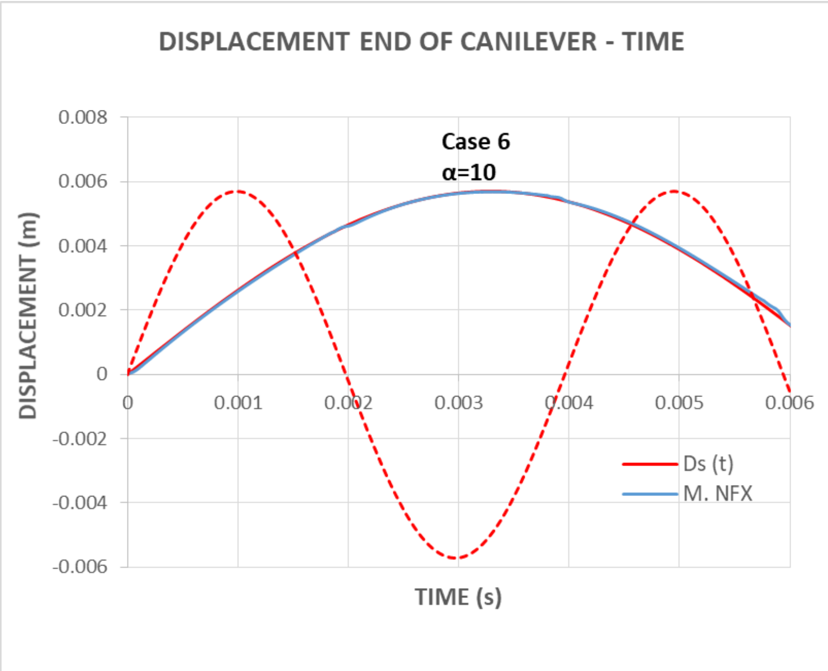

Fig. 8: Comparison of results between the proposed formulation (red) and Midas NFX (blue), and the proposed formulation with natural frequency (dashed line) for case 6 (structure 2).

In order to understand how correct the hypothesis of the projectile mass as a structural mass is (as equation (6) says), an additional dashed red curve is shown in Fig. 8. This curve is obtained from the proposed formulation without projectile mass as a structural mass. So, the latter curve is obtained from equation (7), changing $sin(W_{sp}\ t)$ for $sin(W_s\ t)$, and it is shown to identify the variation in vibration frequency produced by the projectile (dashed red).

The results show that the hypothesis referring to the vibration frequency during impact (changing as if the projectile was part of the structure itself, with the corresponding changes in mass and stiffness), approximates very well to the values provided by Midas NFX (almost coincident).

As for the validity of the formulas when $t > t_f$, more difference is observed between the curves from Midas NFX and the proposed formulation for that period than when $t < t_f$. This can be clearly seen in Fig. 9 where the displacement of the structure described by Midas NFX and the proposed formulation coincides before the maximum is reached at $t = 0.637$. From this moment, the contact between the projectile and the structure is not continuous, giving rise to additional vibrations and slightly changing the vibration frequency during the strain recovery process.

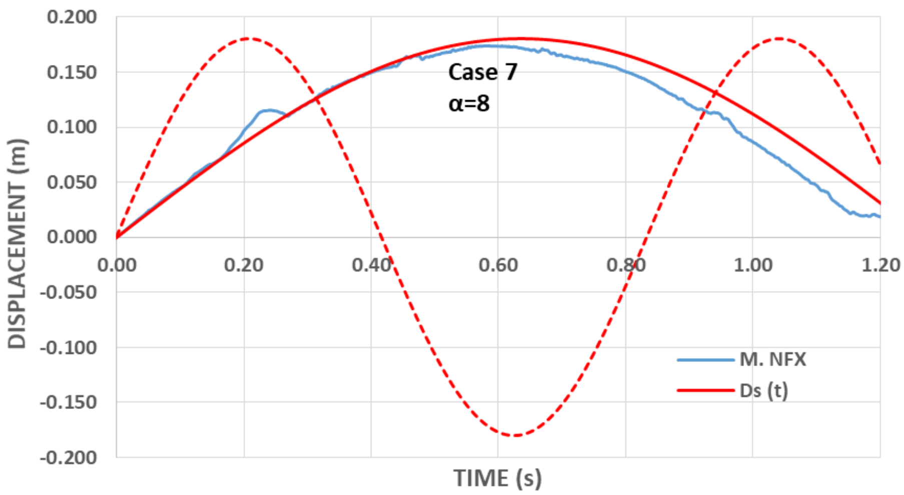

Fig. 9: Comparison of results between the proposed formulation (red) and Midas NFX (blue), and proposed formulation with natural frequency (dashed line) for case 7 (structure 3).

The dashed red curve is obtained in the same way as was explained for structure 2 in Fig. 9. Moreover, in such cases the projectile mass has a clear influence on the vibration frequency during impact, as can be seen in the difference between the solid red curve and dashed red one.

5. DISCUSSION

The match in the displacement of the structure at the point of contact between the proposed formulation and Midas NFX depends on the mass ratio (α). Regardless of the type of structure and energy absorption mechanism, when α is greater than 5, the proposed formulation returns very similar results in all cases. Even though the displacement is similar, when α is lower than 2, the vibration created by the other vibration modes is larger and introduces more error. Increasing vibrations from other modes of vibration when α is small can be explained by the initial hypothesis about the continuous contact between projectile and structure until $t = t_f$. When the participation of high vibration modes increases, this hypothesis is not fulfilled, and gives rise to multiple point contacts between the bodies. This is supported by the following energetic approach: at $t = t_f$, the structure, represented by its fundamental mode, reaches the maximum displacement and loses all the velocity exerted by the projectile leading to the initiation of the recovery process to its initial position. Therefore, at $t = t_f$, the maximum amount of energy absorbed is reached and translated into deformation energy ($E_{1^{st}\ mode}$), equation (15).

$$E_{1^{st}\ mode} = \frac{1}{2} K_e (D_{e,max})^2 = \frac{1}{2} K_e (\frac{V_I}{W_{ep}})^2 = E_o \frac{\alpha}{\alpha+1} \qquad (15)$$

Given that the projectile stops at a given instant $t = t_f$, and assuming it is non-deformable, all the initial kinetic energy from the impact has been conveyed to the structure. Nonetheless, based on equation (15), the fundamental mode only absorbs a portion of the total initial energy, E_0, depending on the mass coefficient, α, and hence, the rest is absorbed by the other modes of vibration. This means that if the mass of the projectile is infinite with respect to the structure ($\alpha = \infty$), all the energy will be absorbed by the fundamental mode. When $\alpha < 1$ more than 50% of the initial energy will be absorbed by other modes of vibration. In this way, the lower precision in the proposed formulation when α is small is explained, as this represents a small portion with respect to the total energy. The rest of the energy is held in other distinct modes of vibration, not the fundamental one (transferred to the structure in the so-called initial phase), making the contact between the projectile and the structure become discontinuous.

From the results presented in Table 6, the energy can be accounted for in the proposed formulation for the seven cases analysed.

Case	α	C_m	$E_{1^{st}mode}/E_0$ (%)	$E_{initial\ phase}$ (%)
1	10	0.909	90.9	9.1
2	5	0.833	83.3	16.7
3	3	0.750	75.0	25.0
4	2	0.667	66.7	33.3
5	1	0.500	50.0	50.0
6	10	0.909	90.9	9.1
7	8	0.889	88.9	11.1

Table 6: Energy ratio between the fundamental mode and the total impact.

As shown on Table 6, the energy consumed in the initial phase of the impact is the non-absorbed energy by the fundamental mode, and released by the projectile in a short period of time. In other words, the additional hypothesis where initial phase energy consumed in the impact is considered negligible will be fulfilled when the mass of the projectile is large compared to the equivalent mass of the structure ($\alpha >> 1$). In that case, the fundamental mode will absorb most of the energy of the impact, and hence, the contact force calculated in equation (8) will be representative. A lot of real situations like a truck crashing into a slender pier of a bridge, a big piece ejected towards a near wall from a mechanical machine due to a failure, or any large load dropped by error from a crane are good examples where normally $\alpha >> 1$. Thus, this kind of situations can be analysed with the proposed formulation in two scenarios: as a pre-design of a new structure (avoiding unnecessary iterations with FEM models), or to consider easily whether an accident may cause integrity or operational problems in an existing structure.

When it comes to the displacement of the structure at the point of contact, even at small values of α, the proposed formulation provides reasonably good results. In order to achieve greater accuracy in results at small values of α, the rest of the non-fundamental vibration modes should be considered. It must be considered that contact force is more influenced by the energy expended in the initial phase than structural displacement (and equivalent static force). The influence of the rest of the modes is currently under investigation, as well as the possible influence of other issues such as gravity, projectile stiffness, and membrane force.

After this energy analysis, it is clear that the errors shown in Table 5 are related essentially with the α parameter. When α is close to 1, the error increases due to influence of higher modes of vibration, equation (15), and if α increases, this error decreases and the curves are well matched. This fact confirms that the hypothesis of considering the influence of shear and local deformation negligible (which are considered in FEM results) is appropriate for the aim of the investigation.

Lastly, regarding the maximum stresses calculated from equivalent static force, Table 5, it shows that the first case of impact on structure 1 would produce plastic behaviour. The stress calculated based on elastic parameters exceeds the yield strength (515 MPa > 355 MPa), and a plastic analysis would be required to refine the results, if necessary. The rest of the cases are under the yield strength so results are totally correct. It must be pointed out that in case 7, impact on the concrete slab, the strees is quite high although lower than maximum compressive stress (36 MPa < 45 MPa), so it would require a prestressed solution to avoid tension. Otherwise the elastic parameters must be replaced by cracking parameters.

6. CONCLUSIONS

The proposed formulation (1 DoF) to solve impacts on any kind of structure, by means of a transformation into a spring-mass system in a simple way, is able to reproduce the fundamental mode of vibration, and provides results close to those offered by FEM. Note that the proposed formulation works without considering any local stiffness, which is one of the issues of 2 DoF models.

The proposed formulation's displacements are well-matched with the FEM ones; thus, the equivalent static force equation is justified. Results demonstrated that, during the impact, projectile mass behaves as a structural mass, the vibration frequency varying as per the new theory and proposed formulation.

The precision of the proposed formulation depends on the ratio between the mass of the projectile and the equivalent mass of the structure (α), as this parameter governs the amount of energy that the fundamental mode of vibration of the structure absorbs with respect to the total in the impact. This means that the energy criteria developed to establish when the formulation can be used is simple and clear in every structure (if $\alpha > 2$, the error remains under 5% in all cases. Only when α is around 1 does the error start to grow to about 20%, establishing the limit of the formulation).

Given the proposed formulation is only based on the fundamental mode of vibration, it does not manage to account for the vibrations from the rest of the modes. It is more important when the value of α is small. To ensure sufficient precision, the formulation is recommended for use when:

- $\alpha \geq 1$ to apply in bending moments due to equivalent static force and displacements.
- $\alpha > 3$ to apply in shear forces due to equivalent static force and contact force.

What is important for structural design is the equivalent static force, because this is the force that produces the same shear forces and bending moments in the structure as the impact load, but with static analysis. Contact force, in some cases, can be extremely high but in a very short time and it does not produce any internal force in the structure. Thus, the static equivalent force is the only important parameter from a structural designer's point of view.

It is possible to analyse impacts for any value of α with this formulation considering more than one mode of vibration. This issue is currently being studied and it will be addressed in future research.

The seven impact cases analysed in this paper contain different structures (simple supported beams, cantilever and slabs), impacts (transverse and longitudinal), materials (steel, concrete) etc. Moreover, displacement and impact time results vary from nearly 400 mm and 350 μs (case 1) to 5 mm and 3μs (case 6). Therefore, it can be concluded that the proposed formulation is suitable for any elastic impact, or plastic one if equivalent parameters are used. With minor differences, the proposed formulation demonstrates that there is a general pattern governing impacts regardless of the type of structure.

The investigation also shows that some considerations like shear and local deformations are not necessary to approach the general behaviour of the structure under impact load. The proposed formulation enables structural engineers to pre-design or perform FEM checks under impact loads without considering these effects, in the same way that engineers do in static cases.

The proposed formulation is also useful to establish the integration parameters in a FEM model, greatly reducing computational cost, in which it is not always easy to achieve convergence; because impact duration, equation (10), and maximum displacement, equation (14), are already well-known from the proposed formulation.

Finally, as a summary of the previous comments, since in some cases the FEM analysis needs hours to obtain the exact solution for the impact, by using the simple equation suggested, equation (13), it is possible to calculate the equivalent static force ($F_{eq,st}$) and perform a pre-design of the structure prior to performing the FEM analysis, and thus to be more time efficient.

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