Optimal Control of Semilinear Parabolic Equations with Non-smooth Pointwise-Integral Control Constraints in Time-Space

Eduardo Casas[1] · **Karl Kunisch**[2,3]

Abstract

This work concentrates on a class of optimal control problems for semilinear parabolic equations subject to control constraint of the form $\|u(t)\|_{L^1(\Omega)} \leq \gamma$ for $t \in (0, T)$. This limits the total control that can be applied to the system at any instant of time. The L^1-norm of the constraint leads to sparsity of the control in space, for the time instants when the constraint is active. Due to the non-smoothness of the constraint, the analysis of the control problem requires new techniques. Existence of a solution, first and second order optimality conditions, and regularity of the optimal control are proved. Further, stability of the optimal controls with respect to γ is investigated on the basis of different second order conditions.

Keywords Optimal control · Semilinear parabolic equations · Non-smooth control constraints · First and second order optimality conditions

Mathematics Subject Classification 35K58 · 49J20 · 49J52 · 49K20

✉ Eduardo Casas
eduardo.casas@unican.es

Karl Kunisch
karl.kunisch@uni-graz.at

[1] Departmento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain

[2] Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria

[3] Johann Radon Institute, Austrian Academy of Sciences, Linz, Austria

🖄 Springer

1 Introduction

We study the optimal control problem

$$(P) \quad \inf_{u \in U_{ad} \cap L^\infty(Q)} J(u) := \frac{1}{2} \int_Q (y_u(x, t) - y_d(x, t))^2 \, dx \, dt + \frac{\kappa}{2} \int_Q u(x, t)^2 \, dx \, dt,$$

where $\kappa > 0$,

$$U_{ad} = \{u \in L^\infty(0, T; L^1(\Omega)) : \|u(t)\|_{L^1(\Omega)} \le \gamma \text{ for a.a. } t \in (0, T)\}$$

with $0 < \gamma < +\infty$, and y_u is the solution of the semilinear parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + a(x, t, y) = u \text{ in } Q = \Omega \times (0, T), \\ y = 0 \text{ on } \Sigma = \Gamma \times (0, T), \ y(0) = y_0 \text{ in } \Omega. \end{cases} \tag{1.1}$$

with

$$Ay = -\sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)\partial_{x_i} y).$$

We assume that Ω is a bounded, connected, and open subset of $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz boundary Γ, and that $0 < T < \infty$ is fixed.

The precise conditions on the nonlinearity a will be given below. Suffice it to say at this moment that strong nonlinearities such as $\exp(y)$, $\sin(y)$, or polynomial non-linearities with positive leading term of odd degree will be admitted. A first difficulty that arises in treating (P) relates to the proof of existence of an optimal control. The reader could think of choosing $L^2(Q)$ as the convenient space to prove the existence of a solution because of the coercivity of J on this space and since the constraint defines a closed and convex subset of $L^2(Q)$. However, the selection of controls in $L^2(Q)$ is not appropriate to deal with the non-linearity in the sate equation. Indeed, even if we can prove the existence of a solution of the state equation, its regularity is not enough (it is not an element of $L^\infty(Q)$, in general) to get the differentiability of the relation control to state. Looking at the control constraint and the cost functional, a second possibility is to consider $L^\infty(0, T; L^2(\Omega))$ as control space. But this is not a reflexive Banach space and, consequently, the proof of existence of a solution to (P) cannot be done by standard techniques. Nevertheless, we can prove existence of solutions in the spaces $L^r(0, T; L^2(\Omega))$ for all $r > \frac{4}{4-n}$. Moreover, all these solutions belong to $L^\infty(Q)$. This leads us to formulate the control problem in $L^\infty(Q)$; see Remark (4.2). To deal with the non-linearity of the state equation in the proof of a solution to (P) in $L^\infty(Q)$, one approach consists in introducing artificial bound constraints on the control and prove that they are inactive as the artificial constraint parameter is large enough; see, for instance [7]. In our case, this would lead to two control constraints with two Lagrange multipliers in the dual of L^∞. This makes the proof of boundedness of the optimal control very difficult. In this work we avoid such a technique and rather

modify (truncate) the non-linear term of the state equation and prove that for a large truncation parameter the cut off is not active on the optimal state.

A second difficulty results from the non-differentiability of the constraint on the control in the definition of U_{ad}. This is a natural constraint since it models a volumetric restriction, which represents a limit to the total amount of control acting at any time t. This technological constraint is an alternative to pointwise or to energy constraints which have been considered previously in the literature. Moreover, the L^1-norm in space leads to a spatially sparsifying effect for the solutions. It is different from the type of sparsification which results when considering such terms in the cost. While for the former, sparsification takes place only after the control becomes active, for the latter it takes place regardless of the norm of the control. For problem (P) the sparsity effect is described by the level set characterized by the functional values of the adjoint state at the height of the supremum norm of the multiplier associated to the control constraint in (P); see Corollary 3.3. We point out that while the L^2 norm appearing in the cost influences the optimal solution, it does not eliminate the sparsifying effect of L^1-terms, regardless of whether they appear in the cost or as a constraint. The literature on problems with an L^1 or measure-valued norm in the cost is quite rich, so we can only give selected references which consider evolutionary problems [1–5, 7–9, 11, 14–16, 18]. In all these papers, either there are no control constraints or they are box constraints. In [13], the authors study a control problem for the evolutionary Navier–Stokes system under the smooth control constraint $\|u(t)\|_{L^2(\Omega)}^2 \leq 1$, which is smooth and not sparsifying. In [6], the control of the 2d evolutionary Navier–Stokes system is analyzed, where the controls are measured valued functions subject to the constraint $\|u(t)\|_{M(\Omega)} \leq \gamma$.

The structure of the paper is the following. The analysis of the state equation and its first and second derivatives with respect to the controls is carried out in Sect. 2. Here special attention is paid to the $L^\infty(Q)$ regularity of the state variable. In Sect. 3 first order optimality conditions are derived and the structural properties of the involved functions are analyzed. In particular, the regularity of the optimal control is proved, which is a crucial point for the numerical analysis of the control problem. The proof of existence of an optimal control is given in Sect. 4. Section 5 is devoted to necessary and sufficient second order optimality conditions. In the final section, as a consequence of the second order condition, Hölder and Lipschitz stability of local solutions with respect to the control bound γ is investigated.

2 Analysis of the State Equation

In this section we establish the well posedness of the state equation, the regularity of the solution, and the differentiable dependence of the solution with respect to the control. To this end we make the following assumptions.

We assume that $y_0 \in L^\infty(\Omega)$, $a_{ij} \in L^\infty(\Omega)$ for every $1 \leq i, j \leq n$, and

$$\Lambda_A |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n \text{ for a.a. } x \in \Omega \tag{2.1}$$

for some $\Lambda_A > 0$. We also assume that $a : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable satisfying the following properties:

$$\exists C_a \in \mathbb{R} : \frac{\partial a}{\partial y}(x, t, y) \geq C_a \ \forall y \in \mathbb{R}, \tag{2.2}$$

$$a(\cdot, \cdot, 0) \in L^{\hat{r}}(0, T; L^{\hat{p}}(\Omega)), \ \text{with } \hat{r}, \hat{p} \geq 2 \ \text{and} \ \frac{1}{\hat{r}} + \frac{n}{2\hat{p}} < 1, \tag{2.3}$$

$$\forall M > 0 \ \exists C_{a,M} > 0 : \left| \frac{\partial^j a}{\partial y^j}(x, t, y) \right| \leq C_{a,M} \ \forall |y| \leq M \ \text{and} \ j = 1, 2, \tag{2.4}$$

$\forall \rho > 0$ and $\forall M > 0 \ \exists \varepsilon > 0$ such that
$$\left| \frac{\partial^2 a}{\partial y^2}(x, t, y_1) - \frac{\partial^2 a}{\partial y^2}(x, t, y_2) \right| < \rho \ \ \forall |y_1|, |y_2| \leq M \ \text{with} \ |y_1 - y_2| < \varepsilon, \tag{2.5}$$

for almost all $(x, t) \in Q$.

As usual $W(0, T)$ denotes the Hilbert space

$$W(0, T) = \{ y \in L^2(0, T; H_0^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2(0, T; H^{-1}(\Omega)) \}.$$

We recall that $W(0, T)$ is continuously embedded in $C([0, T]; L^2(\Omega))$ and compactly embedded in $L^2(Q)$.

Theorem 2.1 *Under the previous assumptions, for every $u \in L^r(0, T; L^p(\Omega))$ with $\frac{1}{r} + \frac{n}{2p} < 1$ and $r, p \geq 2$ there exists a unique solution $y_u \in L^\infty(Q) \cap W(0, T)$ of (1.1). Moreover, the following estimates hold*

$$\|y_u\|_{L^\infty(Q)} \leq \eta \big(\|u\|_{L^r(0,T;L^p(\Omega))} + \|a(\cdot, \cdot, 0)\|_{L^{\hat{r}}(0,T;L^{\hat{p}}(\Omega))} + \|y_0\|_{L^\infty(\Omega)} \big), \tag{2.6}$$

$$\|y_u\|_{C([0,T];L^2(\Omega))} + \|y_u\|_{L^2(0,T;H_0^1(\Omega))}$$
$$\leq K \big(\|u\|_{L^2(Q)} + \|a(\cdot, \cdot, 0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \big), \tag{2.7}$$

for a monotone non-decreasing function $\eta : [0, \infty) \longrightarrow [0, \infty)$ and some constant K both independent of u.

Proof We decompose the state equation into two parts. First, we consider

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az = u \ \text{in} \ Q, \\ z = 0 \ \text{on} \ \Sigma, \quad z(0) = y_0 \ \text{in} \ \Omega. \end{cases} \tag{2.8}$$

It is well known that it has a unique solution $z \in W(0, T) \cap L^\infty(Q)$. Moreover, we have the estimates

$$\|z\|_{W(0,T)} \leq C_W (\|u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}), \tag{2.9}$$

$$\|z\|_{L^\infty(Q)} \leq C_\infty (\|u\|_{L^r(0,T;L^p(\Omega))} + \|y_0\|_{L^\infty(\Omega)}); \tag{2.10}$$

see, for instance [17,Chapter III]. Now, we define $b : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ as follows

$$b(x, t, s) = e^{-|C_a|t}[a(x, t, e^{|C_a|t}s + z(x, t)) - a(x, t, z(x, t))] + |C_a|s,$$

where C_a is as in (2.2). Then, $b(x, t, 0) = 0$ and according to (2.2)

$$\frac{\partial b}{\partial s}(x, t, s) = \frac{\partial a}{\partial s}(x, t, e^{|C_a|t}s + z(x, t)) + |C_a| \geq 0.$$

We consider the equation

$$\begin{cases} \dfrac{\partial w}{\partial t} + Aw + b(x, t, w) = -e^{-|C_a|t}a(x, t, z(x, t)) \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \ \ w(0) = 0 \text{ in } \Omega. \end{cases} \tag{2.11}$$

Due to the properties of b, the existence and uniqueness of a solution $w \in L^\infty(Q) \cap W(0, T)$ is well known; see [20,Theorem 5.5]. Moreover, the following estimates hold

$$\|w\|_{W(0,T)} \leq C_b(\|a(\cdot, \cdot, z)\|_{L^2(Q)} + \|b(\cdot, \cdot, w)\|_{L^2(Q)}), \tag{2.12}$$

$$\|w\|_{L^\infty(Q)} \leq C'_\infty \|a(\cdot, \cdot, z)\|_{L^{\hat{r}}(0,T;L^{\hat{p}}(\Omega))}. \tag{2.13}$$

Denoting $M = \|z\|_{L^\infty(Q)}$ and using (2.4) we infer with the mean value theorem

$$|a(x, t, z(x, t))| \leq |a(x, t, z(x, t)) - a(x, t, 0)| + |a(x, t, 0)|$$
$$= \left| \frac{\partial a}{\partial y}(x, t, \theta(x, t)z(x, t))z(x, t) \right| + |a(x, t, 0)| \leq C_{a,M}M + |a(x, t, 0)|.$$

Combining this with (2.10) and (2.13) we get

$$\|w\|_{L^\infty(Q)} \leq \sigma(\|u\|_{L^r(0,T;L^p(\Omega))} + \|a(\cdot, \cdot, 0)\|_{L^{\hat{r}}(0,T;L^{\hat{p}}(\Omega))} + \|y_0\|_{L^\infty(\Omega)}), \tag{2.14}$$

for a non-decreasing function $\sigma : [0, \infty) \longrightarrow [0, \infty)$.

If we set $w = e^{-|C_a|t}\psi$ and insert this in (2.11), we infer

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + A\psi + a(x, t, z(t) + \psi) = 0 \text{ in } Q, \\ \psi = 0 \text{ on } \Sigma, \ \ \psi(0) = 0 \text{ in } \Omega. \end{cases} \tag{2.15}$$

Adding (2.8) and (2.15) we deduce that $y_u = z + \psi$ solves (1.1). Moreover, any solution of (1.1) is the sum of the solutions of (2.8) and (2.15). Since these equations have a unique solution, the uniqueness of y_u follows. Furthermore, (2.10) and (2.14) imply (2.6).

To prove (2.7), we take $\phi = e^{-|C_a|t}y_u$ and introduce the function $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x, t, s) = e^{-|C_a|t}[a(x, t, e^{|C_a|t}s) - a(x, t, 0)] + |C_a|s.$$

Then, ϕ satisfies

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + A\phi + f(x, t, \phi) = e^{-|C_a|t}[u - a(x, t, 0)] \text{ in } Q, \\ \phi = 0 \text{ on } \Sigma, \quad \phi(0) = y_0 \text{ in } \Omega. \end{cases} \tag{2.16}$$

Since $f(x, t, 0) = 0$ and $\frac{\partial f}{\partial s}(x, t, s) \geq 0$, multiplying the above equation by ϕ, integrating in Ω, and using (2.1) we get

$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{L^2(\Omega)} + \Lambda_A \int_\Omega |\nabla \phi(t)|^2 \, dx$$

$$\leq \frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{L^2(\Omega)} + \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_{x_i}\phi(t)\partial_{x_j}\phi(t) \, dx + \int_\Omega f(x, t, \phi(t))\phi(t) \, dx$$

$$= \int_\Omega e^{-|C_a|t}(u - a(x, t, 0))\phi \, dx \leq \big(\|u(t)\|_{L^2(\Omega)} + \|a(\cdot, t, 0)\|_{L^2(\Omega)}\big)\|\phi(t)\|_{L^2(\Omega)}.$$

Estimate (2.7) follows from this inequality as usual.   □

We apply Theorem 2.1 with $p = 2$ and $r \in \left(\frac{4}{4-n}, \infty\right]$. Observe that $\frac{1}{r} + \frac{n}{4} < 1$ and $r > 2$. Then, the mapping $G : L^r(0, T; L^2(\Omega)) \longrightarrow L^\infty(Q) \cap W(0, T)$ given by $G(u) = y_u$ solution of (1.1) is well defined. We have the following differentiability properties of G.

Theorem 2.2 *The mapping G is of class C^2. For $u, v, v_1, v_2 \in L^r(0, T; L^2(\Omega))$ the derivatives $z_v = G'(u)v$ and $z_{v_1, v_2} = G''(u)(v_1, v_2)$ are the solutions of the equations*

$$\begin{cases} \dfrac{\partial z_v}{\partial t} + Az_v + \dfrac{\partial a}{\partial y}(x, t, y_u)z_v = v \text{ in } Q, \\ z_v = 0 \text{ on } \Sigma, \quad z_v(0) = 0 \text{ in } \Omega, \end{cases} \tag{2.17}$$

$$\begin{cases} \dfrac{\partial z_{v_1, v_2}}{\partial t} + Az_{v_1, v_2} + \dfrac{\partial a}{\partial y}(x, t, y_u)z_{v_1, v_2} + \dfrac{\partial^2 a}{\partial y^2}(x, t, y_u)z_{v_1}z_{v_2} = 0 \text{ in } Q, \\ z_{v_1, v_2} = 0 \text{ on } \Sigma, \quad z_{v_1, v_2}(0) = 0 \text{ in } \Omega. \end{cases} \tag{2.18}$$

Proof Let us consider the Banach space

$$\mathcal{Y} = \{y \in L^\infty(Q) \cap W(0, T) : \frac{\partial y}{\partial t} + Ay \in X\},$$

where $X = L^{\hat{r}}(0, T; L^{\hat{p}}(\Omega)) + L^r(0, T; L^2(\Omega))$, endowed with the norm

$$\|y\|_{\mathcal{Y}} = \|y\|_{L^\infty(Q)} + \|y\|_{W(0,T)} + \|\frac{\partial y}{\partial t} + Ay\|_X.$$

Now, we define the mapping

$$\mathcal{F} : \mathcal{Y} \times L^\infty(\Omega) \times L^r(0, T; L^2(\Omega)) \longrightarrow X \times L^\infty(\Omega)$$

$$\mathcal{F}(y, w, u) = \left(\frac{\partial y}{\partial t} + Ay + a(\cdot, \cdot, y) - u, \, y(0) - w \right).$$

We have that $\mathcal{F}$ is of class C^2, $\mathcal{F}(y_u, y_0, u) = (0, 0)$ for every $u \in L^r(0, T; L^2(\Omega))$, and

$$\frac{\partial \mathcal{F}}{\partial y}(y_u, y_0, u) : \mathcal{Y} \longrightarrow X \times L^\infty(\Omega)$$

$$\frac{\partial \mathcal{F}}{\partial y}(y_u, y_0, u)z = \left(\frac{\partial z}{\partial t} + Az + \frac{\partial a}{\partial y}(\cdot, \cdot, y_u)z, \, z(0) \right)$$

is an isomorphism. Hence, an easy application of the implicit function theorem proves the result. $\qquad \square$

As a consequence of the above theorem and the chain rule we infer the differentiability of the mapping $J : L^r(0, T; L^2(\Omega)) \longrightarrow \mathbb{R}$. From now on, we assume

$$y_d \in L^{\hat{r}}(0, T; L^{\hat{p}}(\Omega)), \tag{2.19}$$

where $\hat{r}$ and $\hat{p}$ are defined in (2.3).

Corollary 2.1 *If $r > \frac{4}{4-n}$, then J is of class C^2 and its derivatives are given by the expressions*

$$J'(u)v = \int_Q (\varphi + \kappa u)v \, dx \, dt, \tag{2.20}$$

$$J''(u)(v_1, v_2) = \int_Q \left[\left(1 - \frac{\partial^2 a}{\partial y^2}(x, t, y_u)\varphi \right) z_{v_1} z_{v_2} + \kappa v_1 v_2 \right] dx \, dt, \tag{2.21}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and $\varphi \in C(\bar{Q}) \cap H^1(Q)$ is the solution of the adjoint state equation

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial a}{\partial y}(x, t, y_u)\varphi = y_u - y_d \ \text{in } Q, \\ \varphi = 0 \ \text{on } \Sigma, \quad \varphi(T) = 0 \ \text{in } \Omega. \end{cases} \tag{2.22}$$

Above A^* denotes the adjoint operator of A

$$A^*\varphi = -\sum_{i,j=1}^n \partial_{x_j}(a_{ji}(x)\partial_{x_i}\varphi).$$

The regularity $\bar{\varphi} \in C(\bar{Q}) \cap H^1(Q)$ follows from Theorems III-6.1 and III-10.1 of [17]. Moreover, we observe that $J'(u)$ and $J''(u)$ can be extended to continuous linear and bilinear forms $J'(u) : L^2(Q) \longrightarrow \mathbb{R}$ and $J''(u) : L^2(Q) \times L^2(Q) \longrightarrow \mathbb{R}$ for every $u \in L^r(0, T; L^2(\Omega))$.

Remark 2.1 Hypotheses (2.1)–(2.5) are satisfied, for instance, for the nonlinearity $a(y) = \exp(y)$. They are also satisfied for $a(y) = (y - z_1)(y - z_2)(y - z_3)$ for constants z_i, with $i \in \{1, 2, 3\}$. This latter nonlinearity is known in neurology as Nagumo equation and in physical chemistry as Schlögl model. Formulating the optimal control problem with an $L^1(\Omega)$ constraint implies that one looks for the action of a controlling laser whose optimal support is small; see [12].

3 Existence of Optimal Controls and First Order Optimality Conditions

Since the control problem (P) is not convex, we need to distinguish between local and global minimizers. We call $\bar{u}$ a local minimizer for (P) in the $L^r(0, T; L^2(\Omega))$ sense with $r > \frac{4}{4-n}$ if $\bar{u} \in U_{ad} \cap L^\infty(Q)$ and there exists $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in B_\varepsilon \cap U_{ad}, \tag{3.1}$$

where

$$B_\varepsilon = \{u \in L^r(0, T; L^2(\Omega)) : \|u - \bar{u}\|_{L^r(0,T;L^2(\Omega))} \leq \varepsilon\}.$$

It is immediate to check that if $\bar{u}$ is a local minimizer in the $L^r(0, T; L^2(\Omega))$ sense, then it is also a local minimizer in the $L^{r'}(0, T; L^2(\Omega))$ sense for every $r < r' \leq \infty$.

Theorem 3.1 *There exists at least one solution of (P). Moreover, for every local minimizer $\bar{u}$ in the $L^r(0, T; L^2(\Omega))$ sense with $r > \frac{4}{4-n}$, there exist $\bar{y} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$, $\bar{\varphi} \in C(\bar{Q}) \cap H^1(Q)$, and $\bar{\mu} \in L^\infty(Q)$ such that*

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} + A\bar{y} + a(x, t, \bar{y}) = \bar{u} \ \text{ in } Q, \\ \bar{y} = 0 \ \text{on } \Sigma, \ \ \bar{y}(0) = y_0 \ \text{in } \Omega, \end{cases} \tag{3.2}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} + A^*\bar{\varphi} + \dfrac{\partial a}{\partial y}(x, t, \bar{y})\bar{\varphi} = \bar{y} - y_d \ \text{ in } Q, \\ \bar{\varphi} = 0 \ \text{on } \Sigma, \ \ \bar{\varphi}(T) = 0 \ \text{in } \Omega, \end{cases} \tag{3.3}$$

$$\int_Q \bar{\mu}(u - \bar{u}) \, dx \, dt \leq 0 \quad \forall u \in U_{ad}, \tag{3.4}$$

$$\bar{\varphi} + \kappa\bar{u} + \bar{\mu} = 0. \tag{3.5}$$

Proof The proof of existence of a solution for (P) is postponed to the next section, see Theorem 4.5. Given a local minimizer $\bar{u}$, we take $\bar{y}$ and $\bar{\varphi}$ as solutions of (3.2) and (3.3), respectively. Using the convexity of U_{ad} and (2.20) we get

$$0 \leq J'(\bar{u})(u - \bar{u}) = \int_Q (\bar{\varphi} + \kappa \bar{u})(u - \bar{u}) \, dx \, dt \quad \forall u \in U_{ad} \cap L^\infty(Q).$$

Now, given $u \in U_{ad}$ arbitrary, we set $u_k(x, t) = \text{Proj}_{[-k, +k]}(u(x, t))$ for $k \geq 1$, thus $\{u_k\}_{k=1}^\infty \subset L^\infty(Q) \cap U_{ad}$ and $u_k \to u$ in $L^1(Q)$. Then, we can pass to the limit in the inequality $J'(\bar{u})(u_k - \bar{u}) \geq 0$ and, hence, we obtain

$$\int_Q (\bar{\varphi} + \kappa \bar{u})(u - \bar{u}) \, dx \, dt \geq 0 \quad \forall u \in U_{ad}.$$

This inequality is equivalent to the fact $-(\bar{\varphi} + \kappa \bar{u}) \in \partial I_{U_{ad}}(\bar{u}) \subset L^\infty(Q)$. Here $\partial I_{U_{ad}}$ denotes the subdifferential of the indicator function $I_{U_{ad}} : L^1(Q) \longrightarrow [0, +\infty]$, which takes the value $I_{U_{ad}}(u) = 0$ if $u \in U_{ad}$ and $+\infty$ otherwise. Therefore, there exists $\bar{\mu} \in \partial I_{U_{ad}}$ such that (3.4) and (3.5) holds. $\qquad\square$

Let us denote by $\text{Proj}_{B_\gamma} : L^2(\Omega) \longrightarrow B_\gamma \cap L^2(\Omega)$ the $L^2(\Omega)$ projection, where $B_\gamma = \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \leq \gamma\}$. Then, we have the following consequence of the previous theorem.

Corollary 3.1 *Let $\bar{u}$, $\bar{\varphi}$, and $\bar{\mu}$ satisfy (3.2)–(3.5). Then, the following properties hold*

$$\int_\Omega \bar{\mu}(t)(v - \bar{u}(t)) \, dx \leq 0 \quad \forall v \in B_\gamma \text{ and for a.a. } t \in (0, T), \tag{3.6}$$

$$\bar{u}(t) = \text{Proj}_{B_\gamma} \left(-\frac{1}{\kappa} \bar{\varphi}(t) \right) \text{ for a.a. } t \in (0, T), \tag{3.7}$$

$$\begin{cases} \bar{u}(x, t)\bar{\mu}(x, t) = |\bar{u}(x, t)||\bar{\mu}(x, t)| \text{ for a.a. } (x, t) \in Q, \\ \text{if } \|\bar{u}(t)\|_{L^1(\Omega)} < \gamma \text{ then } \bar{\mu}(t) \equiv 0 \text{ in } \Omega \text{ a.e. in } (0, T), \\ \text{if } \|\bar{u}(t)\|_{L^1(\Omega)} = \gamma \text{ and } \bar{\mu}(t) \not\equiv 0 \text{ in } \Omega, \\ \qquad \text{then } \text{supp}(\bar{u}(t)) \subset \{x \in \Omega : |\bar{\mu}(x, t)| = \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\}. \end{cases} \tag{3.8}$$

Proof Let us show that (3.4) and (3.6) are equivalent. Using Fubini's theorem, it is obvious that (3.6) implies (3.4). Let us prove the contrary implication. Let $v \in B_\gamma$ be arbitrary and set

$$I_v = \left\{ t \in (0, T) : \int_\Omega \bar{\mu}(x, t)(v(x) - \bar{u}(x, t)) \, dx > 0 \right\}$$

and

$$u(x, t) = \begin{cases} v(x) & \text{if } t \in I_v, \\ \bar{u}(x, t) & \text{otherwise.} \end{cases}$$

Then, $u \in U_{ad}$ and (3.4) yields

$$0 \geq \int_Q \bar{\mu}(x,t)(u - \bar{u}) \, dx \, dt = \int_{I_v} \int_\Omega \bar{\mu}(x,t)(v(x) - \bar{u}(x,t)) \, dx.$$

This is only possible if $|I_v| = 0$. In order to prove (3.7) we use (3.5) and (3.6) to get

$$\int_\Omega \left(-\frac{1}{\kappa} \bar{\varphi}(t) - \bar{u}(t) \right)(v - \bar{u}(t)) \leq 0 \ \forall v \in B_\gamma \cap L^2(\Omega) \text{ and for a.a. } t \in (0, T).$$

Since $B_\gamma \cap L^2(\Omega)$ is a convex and closed subset of $L^2(\Omega)$, the above inequality is the well known characterization of (3.7).

Let us prove the first statement of (3.8). Take $u(x,t) = \text{sign}(\bar{\mu}(x,t))|\bar{u}(x,t)|$. Then, $u \in U_{ad}$ and with (3.4) we obtain

$$\int_Q |\bar{\mu}(x,t)||\bar{u}(x,t)| \, dx \, dt = \int_Q \bar{\mu}(x,t)u(x,t) \, dx \, dt \leq \int_Q \bar{\mu}(x,t)\bar{u}(x,t) \, dx \, dt,$$

which proves the desired identity. We prove the second statement of (3.8). For every $\varepsilon > 0$ we define

$$I_\varepsilon = \{t \in (0,T) : \|\bar{u}(t)\|_{L^1(\Omega)} \leq \gamma - \varepsilon\}.$$

Denote B_ε the closed ball of $L^1(\Omega)$ centered at 0 and radius ε. Take $v \in B_\varepsilon$ arbitrary. Then, we have that $v + \bar{u}(t) \in B_\gamma$ for $t \in I_\varepsilon$, and (3.6) yields

$$\int_\Omega \bar{\mu}(x,t)v(x) \, dx \leq 0 \ \forall v \in B_\varepsilon \text{ and } t \in I_\varepsilon,$$

which implies that $\bar{\mu}(t) \equiv 0$ in Ω for $t \in I_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we infer the second statement of (3.8). Let us prove the third statement. Under the assumption $\|\bar{u}(t)\|_{L^1(\Omega)} = \gamma$ and $\bar{\mu}(t) \not\equiv 0$ in Ω. For every $\varepsilon > 0$ and $t \in (0, T)$ we consider the sets

$$\Omega^\varepsilon(t) = \{x \in \Omega : |\bar{u}(x,t)| > \varepsilon \text{ and } |\bar{\mu}(x,t)| < \|\bar{\mu}(t)\|_{L^\infty(\Omega)} - \varepsilon\},$$
$$\tilde{\Omega}^\varepsilon(t) = \{x \in \Omega : |\bar{\mu}(x,t)| > \|\bar{\mu}(t)\|_{L^\infty(\Omega)} - \varepsilon\}.$$

We are going to prove that $|\Omega^\varepsilon(t)| = 0$ for almost all $t \in (0, T)$. Assume that $|\Omega^\varepsilon(t)| > 0$ for some $\varepsilon > 0$ and $t \in (0, T)$. Since $|\tilde{\Omega}^\varepsilon(t)| > 0$ by definition of the essential supremum, we can find two sets $E \subset \Omega^\varepsilon(t)$ and $F \subset \tilde{\Omega}^\varepsilon(t)$ such that $|E| = |F| > 0$. We define the control

$$v(x) = \begin{cases} \bar{u}(x,t) - \varepsilon \, \text{sign}(\bar{u}(x,t)) & \text{if } x \in E, \\ \bar{u}(x,t) + \varepsilon \, \text{sign}(\bar{u}(x,t)) & \text{if } x \in F, \\ \bar{u}(x,t) & \text{otherwise.} \end{cases}$$

Since $\|\bar{u}(t)\|_{L^1(\Omega)} = \gamma$, we get

$$\|v\|_{L^1(\Omega)} = \int_E |\bar{u}(x)|\,\mathrm{d}x - \varepsilon|E| + \int_F |\bar{u}(x)|\,\mathrm{d}x + \varepsilon|F| + \int_{\Omega \setminus (E \cup F)} |\bar{u}(x)|\,\mathrm{d}x = \gamma.$$

Moreover, we get with the first statement of (3.8)

$$\int_\Omega \bar{\mu}(x,t)(v(x) - \bar{u}(x,t))\,\mathrm{d}x = -\varepsilon \int_E |\bar{\mu}(x,t)|\,\mathrm{d}x + \varepsilon \int_F |\bar{\mu}(x,t)|\,\mathrm{d}x > 0,$$

which contradicts (3.6) unless it is satisfied for a set of points t of zero Lebesgue measure. Taking

$$\Omega(t) = \{x \in \Omega : |\bar{u}(x,t)| > 0 \text{ and } |\bar{\mu}(x,t)| < \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\},$$

since $\varepsilon > 0$ was arbitrary, we deduce that $|\Omega(t)| = 0$ for almost all $t \in (0, T)$. This implies that $\mathrm{supp}(\bar{u}(t)) \subset \{x \in \Omega : |\bar{\mu}(x,t)| = \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\}$. □

Remark 3.1 Let us observe that the first statement of (3.8) and (3.5) imply

$$|\bar{\varphi}(x,t)| = \kappa|\bar{u}(x,t)| + |\bar{\mu}(x,t)|.$$

This yields

$$\|\bar{\varphi}(t)\|_{L^1(\Omega)} = \kappa\|\bar{u}(t)\|_{L^1(\Omega)} + \|\bar{\mu}(t)\|_{L^1(\Omega)}.$$

From this identity and the second statement of (3.8) we infer that $\bar{\mu}(t) \not\equiv 0$ in Ω if and only if $\|\bar{\varphi}(t)\|_{L^1(\Omega)} > \kappa\gamma$.

Remark 3.2 From (3.8) we deduce that $\bar{\mu}(x,t) \in \|\bar{\mu}(t)\|_{L^\infty(\Omega)} \partial|\cdot|(\bar{u}(x,t))$ for almost every point $(x,t) \in Q$.

Corollary 3.2 *Let $\bar{u} \in U_{ad} \cap L^\infty(\Omega)$ satisfy (3.5) and (3.8). Then, the following identities are satisfied*

$$\begin{aligned}
\bar{u}(x,t) &= -\frac{1}{\kappa}\,\mathrm{sign}(\bar{\varphi}(x,t))\big(|\bar{\varphi}(x,t)| - \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\big)^+ \\
&= -\frac{1}{\kappa}\left\{\Big[\bar{\varphi}(x,t) + \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\Big]^- + \Big[\bar{\varphi}(x,t) - \|\bar{\mu}(t)\|_{L^\infty(\Omega)}\Big]^+\right\}. \quad (3.9)
\end{aligned}$$

Moreover, the regularity $\bar{u} \in H^1(Q)$ and $\bar{\mu} \in H^1(Q)$ hold.

Proof If $\|\bar{\mu}(t)\|_{L^\infty(\Omega)} = 0$, then $\bar{u}(x,t) = -\frac{1}{\kappa}\bar{\varphi}(x,t)$ follows from (3.5), which coincides with the identity (3.9). Now, we assume that $\|\bar{\mu}(t)\|_{L^\infty(\Omega)} > 0$. Using (3.8) we obtain that $\|\bar{u}(t)\|_{L^1(\Omega)} = \gamma$. Then, the third statement of (3.8) implies that $|\bar{\mu}(x,t)| = \|\bar{\mu}(t)\|_{L^\infty(\Omega)}$ if $|\bar{u}(x,t)| > 0$. We distinguish three cases.

(i) If $\bar{u}(x, t) > 0$, (3.5) and the first statement of (3.8) leads to $\bar{u}(x, t) = -\frac{1}{\kappa}(\bar{\varphi}(x, t) + \|\bar{\mu}(t)\|_{L^\infty(\Omega)})$, which coincides with the expression (3.9).

(ii) If $\bar{u}(x, t) = 0$, using again (3.5) we get $|\bar{\varphi}(x, t)| = |\bar{\mu}(x, t)| \leq \|\bar{\mu}(t)\|_{L^\infty(\Omega)}$. Then, the identity (3.9) holds.

(iii) If $\bar{u}(x, t) < 0$, from the first statement of (3.8) and (3.5) we infer that $\bar{u}(x, t) = -\frac{1}{\kappa}(\bar{\varphi}(x, t) - \|\bar{\mu}(t)\|_{L^\infty(\Omega)})$. Then, (3.9) holds too.

The spatial regularity $\bar{u} \in L^2(0, T; H_0^1(\Omega))$ is an immediate consequence of (3.9) and the fact that $\bar{\varphi} \in H^1(Q)$. For the temporal regularity of $\bar{u}$, we first observe

$$\|\bar{u}(t) - \bar{u}(t')\|_{L^2(\Omega)}$$
$$= \|\operatorname{Proj}_{B_\gamma}(-\frac{1}{\kappa}\bar{\varphi}(t)) - \operatorname{Proj}_{B_\gamma}(-\frac{1}{\kappa}\bar{\varphi}(t'))\|_{L^2(\Omega)} \leq \frac{1}{\kappa}\|\bar{\varphi}(t) - \bar{\varphi}(t')\|_{L^2(\Omega)}.$$

Since $\bar{\varphi} : [0, T] \longrightarrow L^2(\Omega)$ is absolutely continuous, using the above inequality we infer that $\bar{u} : [0, T] \longrightarrow L^2(\Omega)$ is also absolutely continuous. Moreover, the same inequality yields $\|\bar{u}'(t)\|_{L^2(\Omega)} \leq \frac{1}{\kappa}\|\bar{\varphi}'(t)\|_{L^2(\Omega)}$ and $\bar{u} \in W^{1,2}(0, T; L^2(\Omega))$. All together, this implies that $\bar{u} \in H^1(Q)$. The regularity of $\bar{\mu}$ follows from (3.5). □

Corollary 3.3 *Let $\bar{u}$ be as in Corollary 3.2. Then, we have the following property*

$$\bar{u}(x, t) = 0 \text{ if and only if } |\bar{\varphi}(x, t)| \leq \|\bar{\mu}(t)\|_{L^\infty(\Omega)}. \tag{3.10}$$

This corollary is a straightforward consequence of (3.9).

Theorem 3.2 *There exists a constant $K_\infty > 0$ independent of γ such that $\|\bar{u}\|_{L^\infty(Q)} \leq K_\infty$ for every global minimizer $\bar{u}$ of (P). In addition, if we set $\gamma_0 = K_\infty|\Omega|$, then for every $\gamma > \gamma_0$ and every solution $\bar{u}$ of (P) we have $\|\bar{u}(t)\|_{L^1(\Omega)} < \gamma$ for almost every t and $\bar{u} = -\frac{1}{\kappa}\bar{\varphi}$.*

To prove this theorem, we can argue as in the proof of Theorem 4.4 below to deduce the existence of $K_\infty > 0$ independent of γ such that $\|\bar{u}\|_{L^\infty(Q)} \leq K_\infty$. The last statement is a straightforward consequence of this estimate and the definition of γ_0.

4 Proof of Existence of a Solution for (P)

The proof of existence of a solution of (P) can not be performed by the classical method of calculus of variations due to the lack of boundedness of U_{ad} in $L^\infty(\Omega)$ and the non coercivity of J on this space. One can try to prove the existence of a solution $\bar{u}$ of (P) in $L^2(Q)$ and then to deduce that $\bar{u} \in L^\infty(Q)$ from the optimality conditions. However, the differentiability of J in $L^2(Q)$ can fail due to the nonlinearity of the state equation. To overcome this difficulty we are going to truncate the nonlinear term

$a(x, t, y)$ as follows. For every $M > 0$ we define the function $f_M : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
f_M(s) = \begin{cases}
M + 1 & \text{if } s > M + 1, \\
s + (M - s)^2 + (M - s)^3 & \text{if } M \leq s \leq M + 1, \\
s & \text{if } -M < s < +M, \\
s - (M + s)^2 - (M + s)^3 & \text{if } -M - 1 \leq s \leq -M, \\
-M - 1 & \text{if } s < -M - 1.
\end{cases}
$$

It can be easily checked that $f_M \in C^1(\mathbb{R})$ and $0 \leq f_M'(s) \leq 1$ for every $s \in \mathbb{R}$. Now, we set $a_M(x, t, s) = a(x, t, f_M(s))$. It is obvious that a_M is of class C^1 with respect to the last variable and (2.2)–(2.4) imply

$$
\frac{\partial a_M}{\partial y}(x, t, y) = \frac{\partial a}{\partial y}(x, t, f_M(y)) f_M'(y) \geq \min(0, C_a) \ \forall y \in \mathbb{R}, \tag{4.1}
$$

$$
a_M(\cdot, \cdot, 0) = a(\cdot, \cdot, 0) \in L^{\hat{r}}(0, T; L^{\hat{p}}(\Omega)), \tag{4.2}
$$

$$
\left| \frac{\partial a_M}{\partial y}(x, t, y) \right| \leq C_{a, M+1} \ \forall y \in \mathbb{R}, \tag{4.3}
$$

for almost all $(x, t) \in Q$.

Theorem 4.1 *For any $M > 0$ and all $u \in L^2(Q)$ the equation*

$$
\begin{cases}
\dfrac{\partial y}{\partial t} + Ay + a_M(x, t, y) = u \ \text{in } Q, \\
y = 0 \ \text{on } \Sigma, \quad y(0) = y_0 \ \text{in } \Omega,
\end{cases} \tag{4.4}
$$

has a unique solution $y_u^M \in W(0, T)$. Moreover, y_u^M satisfies the inequalities

$$
\|y_u^M\|_{C(0,T;L^2(\Omega))} + \|y_u^M\|_{L^2(0,T;H_0^1(\Omega))}
$$
$$
\leq K \big(\|u\|_{L^2(Q)} + \|a(\cdot, \cdot, 0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \big), \tag{4.5}
$$

$$
\|y_u^M\|_{W(0,T)}
$$
$$
\leq K' \big(\|u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} + \|a(\cdot, \cdot, 0)\|_{L^2(Q)} + C_{a, M+1}(M + 1)|Q|^{\frac{1}{2}} \big), \tag{4.6}
$$

where K is the same constant as in (2.7) and K' is independent of M and u.

Proof From (4.3) and the mean value theorem we infer that $|a_M(\cdot, \cdot, s) - a_M(\cdot, \cdot, 0)| \leq C_{a, M+1}(M + 1)$ for all $s \in \mathbb{R}$. Consequently, the estimate

$$
\|a_M(\cdot, \cdot, y) - a_M(\cdot, \cdot, 0)\|_{L^2(Q)} \leq C_{a, M+1}(M + 1)|Q|^{\frac{1}{2}}
$$

holds. Hence, an easy application of fixed point Schauder's theorem yields the existence of a solution y_u^M in $W(0, T)$. The uniqueness follows in the standard way noting that

$$
\int_\Omega [a_M(x, t, y_2) - a_M(x, t, y_1)](y_2 - y_1) \, dx \geq \min\{0, C_a\} \|y_2 - y_1\|_{L^2(\Omega)}^2.
$$

The proof of the estimate (4.5) is the same as the one of (2.7). Inequality (4.6) follows from (4.5) and the fact that

$$\|a_M(\cdot,\cdot,y)\|_{L^2(Q)} \le \|a_M(\cdot,\cdot,0)\|_{L^2(Q)} + C_{a,M+1}(M+1)|Q|^{\frac{1}{2}}.$$

□

Let us define the mapping $G_M : L^2(Q) \longrightarrow W(0,T)$ associating to every u the corresponding solution y_u^M of (4.4).

Theorem 4.2 *The mapping G_M is of class C^1. For all $u, v \in L^2(Q)$ the derivative $z_v = G'_M(u)v$ is the solution of the linearized equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \dfrac{\partial a_M}{\partial y}(x,t,y_u^M)z = v \ in \ Q, \\ z = 0 \ on \ \Sigma, \quad z(0) = 0 \ in \ \Omega, \end{cases} \tag{4.7}$$

where $y_u^M = G_M(u)$.

Proof Let us introduce the space

$$Y = \left\{ y \in W(0,T) : \frac{\partial y}{\partial t} + Ay \in L^2(Q) \right\}.$$

This is a Banach space when it is endowed with the graph norm

$$\|y\|_Y = \|y\|_{W(0,T)} + \|\frac{\partial y}{\partial t} + Ay\|_{L^2(Q)}.$$

Now, we define the mapping

$$\mathcal{F}_M : Y \times L^2(\Omega) \times L^2(Q) \longrightarrow L^2(Q) \times L^2(\Omega)$$

$$\mathcal{F}_M(y,w,u) = \left(\frac{\partial y}{\partial t} + Ay + a_M(\cdot,\cdot,y) - u, y(0) - w \right).$$

Let us prove that the mapping

$$F_M : W(0,T) \longrightarrow L^2(Q), \quad F_M(y) = a_M(\cdot,\cdot,y)$$

is of class C^1 with

$$DF_M : W(0,T) \longrightarrow \mathcal{L}(W(0,T), L^2(Q)), \quad DF_M(y)z = \frac{\partial a_M}{\partial s}(\cdot,\cdot,y)z.$$

First, we observe that a standard application of a Gagliardo–Nirenberg inequality leads to

$$\|z\|_{L^{\frac{8}{3}}(0,T;L^4(\Omega))} \le C\|z\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{4}} \|z\|_{L^2(0,T;H_0^1(\Omega))}^{\frac{3}{4}} \le C'\|z\|_{W(0,T)}$$

for every $z \in W(0, T)$. Using this inequality, (4.3), and the mean value theorem we infer

$$
\begin{aligned}
&\|F_M(y + z) - F_M(y) - DF_M(y)z\|^2_{L^2(Q)} \\
&= \int_Q \left| a_M(x, t, y(x, t) + z(x, t)) - a_M(x, t, y(x, t)) - \frac{\partial a_M}{\partial s}(x, t, y(x, t))z(x, t) \right|^2 dx\, dt \\
&= \int_Q \left| \frac{\partial a_M}{\partial s}(x, t, y(x, t) + \theta(x, t)z(x, t)) - \frac{\partial a_M}{\partial s}(x, t, y(x, t)) \right|^2 z^2(x, t)\, dx\, dt \\
&\le \int_0^T \left\| \frac{\partial a_M}{\partial s}(\cdot, t, y(t) + \theta(t)z(t)) - \frac{\partial a_M}{\partial s}(\cdot, t, y(t)) \right\|^2_{L^4(\Omega)} \|z(t)\|^2_{L^4(\Omega)}\, dt \\
&\le \left\| \frac{\partial a_M}{\partial s}(\cdot, \cdot, y + \theta z) - \frac{\partial a_M}{\partial s}(\cdot, \cdot, y) \right\|^2_{L^8(0,T;L^4(\Omega))} \|z\|^2_{L^{\frac{8}{3}}(0,T;L^4(\Omega))}.
\end{aligned}
$$

From here we deduce

$$
\lim_{\|z\|_{W(0,T)} \to 0} \frac{\|F_M(y + z) - F_M(y) - DF_M(y)z\|_{L^2(Q)}}{\|z\|_{W(0,T)}} = 0.
$$

Hence, F_M is Fréchet differentiable. The continuity of DF_M is immediate and, consequently, F_M is of class C^1. Using this and the continuity of the embedding $Y \subset W(0, T) \subset C([0, T]; L^2(\Omega))$, we conclude that $\mathcal{F}_M$ is of class C^1. Moreover, we have $\mathcal{F}_M(y_u^M, y_0, u) = (0, 0)$. An easy application of the implicit function theorem proves Theorem 4.2. □

For every $M > 0$ we consider the control problems

$$
(\mathrm{P}_M) \quad \inf_{u \in U_{ad} \cap L^2(Q)} J_M(u) := \frac{1}{2} \int_Q (y_u^M(x, t) - y_d(x, t))^2\, dx\, dt \\
+ \frac{\kappa}{2} \int_Q u(x, t)^2\, dx\, dt,
$$

where y_u^M denotes the solution of (4.4). Problem (P_M) has at least a solution u_M. This is consequence of the coercivity of J_M on $L^2(Q)$, the fact that $U_{ad} \cap L^2(Q)$ is closed and convex in $L^2(Q)$, and the lower semicontinuity of J_M with respect to the weak topology of $L^2(Q)$. The last statement follows easily from the estimate (4.6) and the compactness of the embedding $W(0, T) \subset L^2(Q)$.

From the chain rule and Theorem 4.2 we infer that $J_M : L^2(Q) \longrightarrow \mathbb{R}$ is of class C^1 and its derivative is given by the expression

$$
J_M'(u)v = \int_Q (\varphi + \kappa u)v\, dx\, dt, \tag{4.8}
$$

where $\varphi \in W(0, T)$ is the solution of the adjoint state equation

$$
\begin{cases}
-\dfrac{\partial \varphi}{\partial t} + A^* \varphi + \dfrac{\partial a_M}{\partial y}(x, t, y_u^M)\varphi = y_u^M - y_d \ \text{ in } Q, \\
\varphi = 0 \text{ on } \Sigma, \ \ \varphi(T) = 0 \text{ in } \Omega.
\end{cases} \tag{4.9}
$$

Theorem 4.3 *Let u_M be a solution of (P_M). Then, there exist functions $y_M, \varphi_M \in W(0, T)$ and $\mu_M \in L^2(Q)$ such that*

$$
\begin{cases}
\dfrac{\partial y_M}{\partial t} + A y_M + a_M(x, t, y_M) = u_M \ \text{ in } Q, \\
y_M = 0 \text{ on } \Sigma, \ \ y_M(0) = y_0 \text{ in } \Omega,
\end{cases} \tag{4.10}
$$

$$
\begin{cases}
-\dfrac{\partial \varphi_M}{\partial t} + A^* \varphi_M + \dfrac{\partial a_M}{\partial y}(x, t, y_M)\varphi_M = y_M - y_d \ \text{ in } Q, \\
\varphi_M = 0 \text{ on } \Sigma, \ \ \varphi_M(T) = 0 \text{ in } \Omega,
\end{cases} \tag{4.11}
$$

$$
\int_Q \mu_M(x, t)(u(x, t) - u_M(x, t)) \, dx \, dt \le 0 \ \ \forall u \in U_{ad} \cap L^2(Q), \tag{4.12}
$$

$$
\varphi_M + \kappa u_M + \mu_M = 0. \tag{4.13}
$$

The proof of this theorem is the same as the one of Theorem 3.1.

Theorem 4.4 *Let $(u_M, y_M, \varphi_M, \mu_M)$ be as in Theorem 4.3. Then, there exists a constant $K_\infty > 0$ such that*

$$
\|(u_M, y_M, \varphi_M, \mu_M)\|_{L^\infty(Q)^4} \le K_\infty \ \ \forall M > 0. \tag{4.14}
$$

Proof As in the proof for the first statement of (3.8), we have that (4.12) and (4.13) yield $|\mu_M(x, t)||u_M(x, t)| = \mu_M(x, t)u_M(x, t)$ for almost all $(x, t) \in Q$.

We denote by y_M^0 the solution of (4.4) associated with the control identically zero. Then, according to Theorem 4.1, inequality (4.5) implies that

$$
\|y_M^0\|_{C(0, T; L^2(\Omega))} \le K \big(\|a(\cdot, \cdot, 0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \big) \ \ \forall M > 0.
$$

From this inequality we infer

$$
\|y_M^0\|_{L^2(Q)} \le C_1 = \sqrt{T} K \big(\|a(\cdot, \cdot, 0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \big) \ \ \forall M > 0.
$$

Since u_M is solution of (P_M) and $u \equiv 0$ is an admissible control for (P_M) we get

$$
\frac{\kappa}{2}\|u_M\|_{L^2(Q)}^2 \le J_M(u_M) \le J_M(0) = \frac{1}{2}\|y_M^0 - y_d\|_{L^2(Q)}^2.
$$

This leads to

$$
\|u_M\|_{L^2(Q)} \le \frac{1}{\sqrt{\kappa}}\|y_M^0 - y_d\|_{L^2(Q)} \le C_2 = \frac{1}{\sqrt{\kappa}}(C_1 + \|y_d\|_{L^2(Q)}) \ \ \forall M > 0.
$$

Using again (4.5) and this estimate we deduce

$$\|y_M\|_{L^\infty(0,T;L^2(\Omega))} \le C_3 = K_2\big(C_2 + \|a(\cdot,\cdot,0)\|_{L^2(Q)} \\ + \|b(\cdot,\cdot,0)\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\big) \quad \forall M > 0.$$

Using this estimate we can infer the boundedness of φ_M by a constant independent of M. The idea of the proof is to make the substitution $\varphi_M(x,t) = e^{-|C_a|t}\psi_M(x,t)$, where C_a is given in (2.2). Then, ψ satisfies the equation

$$\begin{cases} -\dfrac{\partial \psi_M}{\partial t} + A^*\psi_M + \Big(\dfrac{\partial a_M}{\partial y}(x,t,y_M) + |C_a|\Big)\psi_M = e^{|C_a|t}(y_M - y_d) & \text{in } Q, \\ \psi_M = 0 \text{ on } \Sigma, \quad \psi_M(T) = 0 \text{ in } \Omega. \end{cases}$$

Since (4.1) implies that $\frac{\partial a_M}{\partial y}(x,t,y_u^M) + |C_a| \ge 0$, we apply [17,Theorem III-7.1] to deduce the existence of a constant $C > 0$ independent of M such that

$$\|\psi_M\|_{L^\infty(Q)} \le C\big(e^{|C_a|T}\big[\|y_M\|_{L^\infty(0,T;L^2(\Omega))} + \|y_d\|_{L^{\hat r}(0,T;L^{\hat p}(\Omega))}\big]\big) \\ \le C_4 = C\big(e^{|C_a|T}\big[C_3 + \|y_d\|_{L^{\hat r}(0,T;L^{\hat p}(\Omega))}\big]\big) \quad \forall M > 0.$$

From here we infer the estimate $\|\varphi_M\|_{L^\infty(Q)} \le \|\psi_M\|_{L^\infty(Q)} \le C_4$ for every $M > 0$. Now, using that u_M and μ_M have the same sign almost everywhere in Q, we deduce from (4.13)

$$\kappa|u_M(x,t)| \le |\kappa u_M(x,t) + \mu_M(x,t)| = |\varphi_M(x,t)| \le C_4,$$

which proves that $\|u_M\|_{L^\infty(Q)} \le \frac{C_4}{\kappa}$ for every $M > 0$. Moreover, the bounds from u_M and φ_M along with (4.13) imply that $\|\mu_M\|_{L^\infty(Q)} \le 2C_4$. Finally, the estimate of y_M in $L^\infty(Q)$ independently of M follows from (4.10), Theorem 2.1, and the estimate for u_M. □

Remark 4.1 The assumption $\kappa > 0$ was used in an essential manner in the above proof.

Theorem 4.5 *Let $M \ge K_\infty$ be arbitrary, where K_∞ satisfies (4.14). Let u_M be a solution of (P_M). Then, u_M is a solution of (P).*

Proof First we observe that $\|y_M\|_{L^\infty(Q)} \le M$ and hence $a_M(x,t,y_M) = a(x,t,y_M)$. Therefore, y_M is the solution of (1.1) corresponding to u_M and, consequently, $J_M(u_M) = J(u_M)$.

Given $u \in U_{ad} \cap L^\infty(Q)$ arbitrary, let y_u be the associated solution of (1.1) and set $M_0 = \|y_u\|_{L^\infty(Q)}$. If $M_0 \le M$, then it is obvious that $a_M(x,t,y_u) = a(x,t,y_u)$ and, hence, $J_M(u) = J(u)$. Therefore, the optimality of u_M implies $J(u_M) = J_M(u_M) \le J_M(u) = J(u)$.

If $M_0 > M$, we take a solution u_{M_0} of (P_{M_0}). Then, Theorem 4.4 implies that the solution y_{M_0} of (4.10) with M replaced by M_0 satisfies $\|y_{M_0}\|_{L^\infty(Q)} \le M$

and, consequently, $a_{M_0}(x, t, y_{M_0}) = a_M(x, t, y_{M_0}) = a(x, t, y_{M_0})$ and $J_{M_0}(u_{M_0}) = J_M(u_{M_0}) = J(u_{M_0})$. These facts along with the optimality of u_M and u_{M_0} lead to

$$J(u_M) = J_M(u_M) \le J_M(u_{M_0}) = J_{M_0}(u_{M_0}) \le J_{M_0}(u) = J(u),$$

which proves that u_M is a solution of (P). □

Remark 4.2 Let us compare problem (P) with the control problems

$$(P_r) \quad \inf_{u \in U_{ad} \cap L^r(0,T;L^2(\Omega))} J(u) := \frac{1}{2} \int_Q (y_u(x, t) - y_d(x))^2 \, dx \, dt$$
$$+ \frac{\kappa}{2} \int_Q u(x, t)^2 \, dx \, dt,$$

where $r \in (\frac{4}{4-n}, \infty)$. We observe that Theorems 2.1 and 2.2 , and Corollary 2.1 are applicable to deduce that any solution of (P_r) satisfies the optimality conditions (3.2)–(3.5). Then, the arguments of Theorem 4.4 apply to deduce that any solution of (P_r) belongs to $L^\infty(Q)$. Let us check that problems (P) and (P_r) are equivalent in the sense that both have the same solutions. Indeed, since $U_{ad} \cap L^r(0, T; L^2(\Omega)) \supset U_{ad} \cap L^\infty(Q)$, it is obvious that every solution of (P_r) is a solution of (P). Conversely, let $\bar{u}$ be a solution of (P) and take $u \in U_{ad} \cap L^r(0, T; L^2(\Omega))$ arbitrarily. For every integer $k \ge 1$ we set $u_k = \text{Proj}_{[-k,+k]}(u)$. Then, it is obvious that $u_k \in U_{ad} \cap L^\infty(Q)$ and $u_k \to u$ in $L^r(0, T; L^2(\Omega))$. Using the optimality of $\bar{u}$ we have $J(\bar{u}) \le J(u_k)$ for all k, and passing to the limit we infer that $J(\bar{u}) \le J(u)$. Since u was arbitrary, this implies that $\bar{u}$ is a solution of (P_r).

5 Second Order Optimality Conditions

We consider the Lipschitz and convex mapping $j : L^1(\Omega) \longrightarrow \mathbb{R}$ defined by $j(v) = \|v\|_{L^1(\Omega)}$. Its directional derivative is given by the expression

$$j'(u; v) = \int_{\Omega_u^+} v(x) \, dx - \int_{\Omega_u^-} v(x) \, dx + \int_{\Omega_u^0} |v(x)| \, dx \quad \forall u, v \in L^1(\Omega), \tag{5.1}$$

where

$$\Omega_u^+ = \{x \in \Omega : u(x) > 0\}, \ \Omega_u^- = \{x \in \Omega : u(x) < 0\} \text{ and } \Omega_u^0 = \Omega \setminus (\Omega_u^+ \cup \Omega_u^-).$$

In order to derive the second order optimality conditions for (P), we define the cone of critical directions. For a control $\bar{u} \in U_{ad} \cap L^\infty(Q)$ satisfying the first order optimality conditions (3.2)–(3.5) we set

$$C_{\bar{u}} = \left\{ v \in L^2(Q) : J'(\bar{u})v = 0 \text{ and } j'(\bar{u}(t); v(t)) \begin{cases} = 0 \text{ if } t \in I_\gamma^+, \\ \le 0 \text{ if } t \in I_\gamma \setminus I_\gamma^+, \end{cases} \right\},$$

where

$$I_\gamma = \{t \in (0, T) : j(\bar{u}(t)) = \gamma\} \text{ and } I_\gamma^+ = \{t \in I_\gamma : \bar{\mu}(t) \not\equiv 0 \text{ in } \Omega\}.$$

We first prove the second order necessary conditions. Given an element $v \in C_{\bar{u}}$, the classical approach to prove these second order conditions consists of taking a sequence $\{v_k\}_{k=1}^\infty$ converging to v such that $\bar{u} + \rho v_k$ is a feasible control for (P) for every $\rho > 0$ small enough. The way of taking this sequence is different from the case where box control constraints are considered. The main reason for this difference is that the functional j, defining the constraint, is not differentiable and that it is non-local in space. Even the approach followed in the case where j is involved in the cost functional cannot be used in our framework; see [3]. The proof makes an essential use of the following lemma.

Lemma 5.1 *Let $v \in L^2(Q)$ satisfy $j'(\bar{u}(t); v(t)) = 0$ for almost all $t \in I_\gamma^+$. Then, $J'(\bar{u})v = 0$ holds if and only if*

$$\|\bar{\mu}(t)\|_{L^\infty(\Omega)}|v(x, t)| = \bar{\mu}(x, t)v(x, t) \text{ for a.a. } (x, t) \in \Omega_{\bar{u}(t)}^0 \times I_\gamma^+. \quad (5.2)$$

As a consequence, every element v of $C_{\bar{u}}$ satisfies (5.2).

Proof From (2.20), (3.5), and (3.8) we infer

$$J'(\bar{u})v = \int_Q (\bar{\varphi} + \kappa\bar{u})v \, dx \, dt = -\int_Q \bar{\mu}v \, dx \, dt = -\int_{I_\gamma^+} \int_\Omega \bar{\mu}v \, dx \, dt$$

$$= -\int_{I_\gamma^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} \left\{ \int_{\Omega_{\bar{u}(t)}^+} v \, dx - \int_{\Omega_{\bar{u}(t)}^-} v \, dx \right\} - \int_{I_\gamma^+} \int_{\Omega_{\bar{u}(t)}^0} \bar{\mu}v \, dx \, dt.$$

Using that $j'(\bar{u}(t); v(t)) = 0$ for almost all $t \in I_\gamma^+$ and (5.1) we get

$$\int_{\Omega_{\bar{u}(t)}^+} v \, dx - \int_{\Omega_{\bar{u}(t)}^-} v \, dx = -\int_{\Omega_{\bar{u}(t)}^0} |v| dx.$$

Inserting this in the previous identity we obtain

$$J'(\bar{u})v = \int_{I_\gamma^+} \int_{\Omega_{\bar{u}(t)}^0} [\|\mu(t)\|_{L^\infty(\Omega)}|v| - \bar{\mu}v] \, dx \, dt.$$

Since $\bar{\mu}v \leq \|\mu(t)\|_{L^\infty(\Omega)}|v|$, we deduce from the above equality that $J'(\bar{u})v = 0$ if and only if (5.2) holds. $\qquad\square$

Theorem 5.1 *Let $\bar{u}$ be a local solution of (P) in the $L^r(0, T; L^2(\Omega))$ sense with $r > \frac{4}{4-n}$. Then, the inequality $J''(\bar{u})v^2 \geq 0$ holds for all $v \in C_{\bar{u}}$.*

Proof Let v be an element of $C_{\bar{u}} \cap L^\infty(0, T; L^2(\Omega))$. We will prove that $J''(\bar{u})v^2 \geq 0$. Later, we will remove the assumption $v \in L^\infty(0, T; L^2(\Omega))$. Set

$$g(x, t) = \begin{cases} \dfrac{v(x, t)}{|\bar{u}(x, t)|} & \text{if } x \notin \Omega^0_{\bar{u}(t)}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a(t) = \int_\Omega g(x, t)\bar{u}(x, t)\, dx.$$

From (5.1) we infer

$$j'(\bar{u}(t); v(t)) = a(t) + \int_{\Omega^0_{\bar{u}(t)}} |v(x, t)|\, dx.$$

For every integer $k \geq 1$ we put

$$a_k(t) = \int_\Omega \mathrm{Proj}_{[-k,+k]}(g(x, t))\bar{u}(x, t)\, dx,$$

$$g_k(x, t) = \mathrm{Proj}_{[-k,+k]}(g(x, t))|\bar{u}(x, t)| + \frac{a(t) - a_k(t)}{\gamma}\bar{u}(x, t),$$

$$v_k(x, t) = \begin{cases} 0 & \text{if } \gamma - \frac{1}{k} < \|\bar{u}(t)\|_{L^1(\Omega)} < \gamma, \\ g_k(x, t) + v(x, t)\chi_{\Omega^0_{\bar{u}(t)}}(x) & \text{if } \|\bar{u}(t)\|_{L^1(\Omega)} = \gamma, \\ v(x, t) & \text{otherwise,} \end{cases}$$

where $\chi_{\Omega^0_{\bar{u}(t)}}(x)$ takes the value 1 if $x \in \Omega^0_{\bar{u}(t)}$ and 0 otherwise.

Using that $|\mathrm{Proj}_{[-k,+k]}(g(x, t))\bar{u}(x, t)| \leq |v(x, t)|$ and the pointwise convergence $\mathrm{Proj}_{[-k,+k]}(g(x, t))\bar{u}(x, t) \to g(x, t)\bar{u}(x, t)$ almost everywhere in Q, we deduce with Lebesgue's Theorem that $\lim_{k\to\infty} a_k(t) = a(t)$ for almost all $t \in (0, T)$. Therefore, we have that $v_k(x, t) \to v(x, t)$ for almost all $(x, t) \in Q$. Moreover, we have

$$|g_k(x, t)| \leq |v(x, t)| + \frac{2}{\gamma}\|v\|_{L^\infty(0,T;L^1(\Omega))}\|\bar{u}\|_{L^\infty(Q)}$$

and

$$|v_k(x, t)| \leq |v(x, t)| + \frac{2}{\gamma}\|v\|_{L^\infty(0,T;L^1(\Omega))}\|\bar{u}\|_{L^\infty(Q)} \text{ for a.a. } (x, t) \in Q.$$

Once again, with Lebesgue's Theorem we get $v_k \to v$ in $L^r(0, T; L^2(\Omega))$ for every $r < \infty$.

Let us prove that $J'(\bar{u})v_k = 0$. To this end, we apply Lemma 5.1. Actually, we are going to prove that $v_k \in C_{\bar{u}}$. Given $t \in I_\gamma$, taking into account (5.1) and the fact that

$j(\bar{u}(t)) = \|\bar{u}(t)\|_{L^1(\Omega)} = \gamma$ we get

$$j'(\bar{u}(t); v_k(t))$$

$$= \int_{\Omega_{\bar{u}(t)}^+} \mathrm{Proj}_{[-k,+k]}(g(x,t))|\bar{u}(x,t)|\,\mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-} \mathrm{Proj}_{[-k,+k]}(g(x,t))|\bar{u}(x,t)|\,\mathrm{d}x$$

$$+ \frac{a(t) - a_k(t)}{\gamma}\Big[\int_{\Omega_{\bar{u}(t)}^+} \bar{u}(x,t)\,\mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-} \bar{u}(x,t)\,\mathrm{d}x\Big] + \int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\,\mathrm{d}x$$

$$= \int_\Omega \mathrm{Proj}_{[-k,+k]}(g(x,t))\bar{u}(x,t)\,\mathrm{d}x + \frac{a(t) - a_k(t)}{\gamma} j(\bar{u}(t)) + \int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\,\mathrm{d}x$$

$$= a(t) + \int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\,\mathrm{d}x = j'(\bar{u}(t), v(t)) \begin{cases} = 0 \text{ if } t \in I_\gamma^+, \\ \leq 0 \text{ if } t \in I_\gamma \setminus I_\gamma^+, \end{cases}$$

where we used that $v \in C_{\bar{u}}$ in the last step.

In the case where $\|\bar{u}(t)\|_{L^1(\Omega)} < \gamma$, according to the definition of v_k, we have that $v_k(x,t)$ is equal to 0 or to $v(x,t)$. Since v satisfies (5.2) due to the fact that $v \in C_{\bar{u}}$, we deduce that v_k also satisfies (5.2). Then, Lemma 5.1 implies that $J'(\bar{u})v_k = 0$. Therefore, $v_k \in C_{\bar{u}}$ holds.

Take $\rho_k > 0$ such that

$$\rho_k\Big(k + \frac{2}{\gamma}\|v\|_{L^\infty(0,T;L^1(\Omega))}\Big) < \frac{1}{k\max\{1,\gamma\}}.$$

Then, we have for each fixed k and $\forall \rho \in (0, \rho_k)$

$$\rho\Big(|\mathrm{Proj}_{[-k,+k]}(g(x,t))| + \Big|\frac{a(t) - a_k(t)}{\gamma}\Big|\Big) \leq \rho\Big(k + \frac{2}{\gamma}\|v\|_{L^\infty(0,T;L^1(\Omega))}\Big) < \frac{1}{k}.$$

Using this estimate we have that $\|\bar{u}(t) + \rho v_k(t)\| \leq \gamma$ if $j(\bar{u}(t)) = \gamma$ and $0 < \rho < \rho_k$:

$$\|\bar{u}(t) + \rho v_k(t)\|_{L^1(\Omega)}$$

$$= \int_{\Omega\setminus\Omega_{\bar{u}(t)}^0} \Big|\bar{u}(t)[1 + \rho[\,\mathrm{Proj}_{[-k,+k]}(g(x,t))\,\mathrm{sign}(\bar{u}(x,t)) + \frac{a(t) - a_k(t)}{\gamma}]\Big|\,\mathrm{d}x$$

$$+ \rho\int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\,\mathrm{d}x$$

$$= \int_{\Omega\setminus\Omega_{\bar{u}(t)}^0} |\bar{u}(t)|[1 + \rho[\,\mathrm{Proj}_{[-k,+k]}(g(x,t))\,\mathrm{sign}(\bar{u}(x,t)) + \frac{a(t) - a_k(t)}{\gamma}]\,\mathrm{d}x$$

$$+ \rho\int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\,\mathrm{d}x$$

$$= \int_\Omega |\bar{u}(t)|\,\mathrm{d}x + \rho\Big\{\int_\Omega \big[\,\mathrm{Proj}_{[-k,+k]}(g(x,t))\bar{u}(x,t) + \frac{a(t) - a_k(t)}{\gamma}|\bar{u}(x,t)|\big]\,\mathrm{d}x$$

$$+ \int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\, dx \Bigg\}$$

$$= \|\bar{u}(t)\|_{L^1(\Omega)} + \rho \left\{ a(t) + \int_{\Omega_{\bar{u}(t)}^0} |v(x,t)|\, dx \right\} = \gamma + \rho\, j'(\bar{u}(t); v(t)) \le \gamma.$$

In the case $\gamma - \frac{1}{k} < \|\bar{u}(t)\|_{L^1(\Omega)} < \gamma$, we have that $v_k(t) = 0$ and, consequently

$$\|\bar{u}(t) + \rho v_k(t)\|_{L^1(\Omega)} = \|\bar{u}(t)\|_{L^1(\Omega)} < \gamma.$$

If $\|\bar{u}(t)\|_{L^1(\Omega)} < \gamma - \frac{1}{k}$, then we get

$$\|\bar{u}(t) + \rho v_k(t)\|_{L^1(\Omega)} \le \gamma - \frac{1}{k} + \rho \|v\|_{L^\infty(0,T;L^1(\Omega))} < \gamma.$$

Using the local optimality of $\bar{u}$, the fact that $\bar{u} + \rho v_k \in U_{ad}$, $J'(\bar{u})v_k = 0$, and making a Taylor expansion we get for every $\rho < \rho_k$ small enough

$$0 \le J(\bar{u} + \rho v_k) - J(\bar{u}) = \rho J'(\bar{u})v_k + \frac{\rho^2}{2} J''(\bar{u} + \theta \rho v_k) v_k^2 = \frac{\rho^2}{2} J''(\bar{u} + \theta \rho v_k) v_k^2.$$

Dividing the above inequality by $\rho^2/2$ and making $\rho \to 0$ we obtain with Corollary 2.1 that $J''(\bar{u})v_k^2 \ge 0$. Since $v_k \to v$ in $L^2(Q)$, we pass to the limit when $k \to \infty$ and conclude that $J''(\bar{u})v^2 \ge 0$.

Finally, we take $v \in C_{\bar{u}}$ arbitrary and for every $k \ge 1$ set

$$v_k(x,t) = \frac{v(x,t)}{1 + \frac{1}{k}\|v(t)\|_{L^1(\Omega)}}.$$

Then, we have

$$J'(\bar{u})v_k = \frac{1}{1 + \frac{1}{k}\|v(t)\|_{L^1(\Omega)}} J'(\bar{u})v = 0 \ \text{ and}$$

$$j'(\bar{u}(t); v_k(t)) = \frac{1}{1 + \frac{1}{k}\|v(t)\|_{L^1(\Omega)}} j'(\bar{u}(t); v(t)) \begin{cases} = 0 \text{ if } t \in I_\gamma^+, \\ \le 0 \text{ if } t \in I_\gamma \setminus I_\gamma^+. \end{cases}$$

Therefore, $v_k \in C_{\bar{u}} \cap L^\infty(0,T;L^1(\Omega))$ and $v_k \to v$ in $L^2(Q)$ is satisfied. Hence, we get $J''(\bar{u})v^2 = \lim_{k \to \infty} J''(\bar{u})v_k^2 \ge 0$, which concludes the proof.   □

Theorem 5.2 *Let $\bar{u} \in U_{ad} \cap L^\infty(Q)$ satisfy the first order optimality conditions* (3.2)–(3.5). *If $J''(\bar{u})v^2 > 0 \ \forall v \in C_{\bar{u}} \setminus \{0\}$ holds, then for each $r \in \left(\frac{4}{4-n}, \infty\right]$ there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|_{L^2(Q)}^2 \le J(u) \quad \forall u \in U_{ad} \cap B_\varepsilon(\bar{u}), \tag{5.3}$$

where $B_\varepsilon(\bar{u}) = \{u \in L^r(0, T; L^2(\Omega)) : \|u - \bar{u}\|_{L^r(0,T;L^2(\Omega))} \le \varepsilon\}$.

Proof We proceed by contradiction. If (5.3) is false for every $\delta > 0$ and $\varepsilon > 0$, then for every integer $k \ge 1$ there exists an element $u_k \in U_{ad}$ such that

$$\|u_k - \bar{u}\|_{L^r(0,T;L^2(\Omega))} < \frac{1}{k} \quad \text{and} \quad J(u_k) < J(\bar{u}) + \frac{1}{2k}\|u_k - \bar{u}\|^2_{L^2(Q)}. \quad (5.4)$$

Let us set $\rho_k = \|u_k - \bar{u}\|_{L^2(Q)}$ and $v_k = (u_k - \bar{u})/\rho_k$. Then, we have $\|v_k\|_{L^2(Q)} = 1$ and, taking a subsequence that we denote in the same way, we have $v_k \rightharpoonup v$ in $L^2(Q)$. We divide the proof in several steps.

Step I - $J'(\bar{u})v = 0$. From (3.4) and (3.5) we infer that $J'(\bar{u})(u_k - \bar{u}) \ge 0$ for every $k \ge 1$. Therefore, $J'(\bar{u})v_k \ge 0$ and passing to the limit we obtain $J'(\bar{u})v \ge 0$. Now, using (5.4) along with the mean value theorem we get for some $\theta_k \in (0, 1)$

$$J(u_k) - J(\bar{u}) = J'(\bar{u} + \theta_k(u_k - \bar{u}))(u_k - \bar{u}) < \frac{1}{2k}\|u_k - \bar{u}\|^2_{L^2(Q)}.$$

Dividing this inequality by ρ_k we obtain

$$J'(\bar{u} + \theta_k(u_k - \bar{u}))v_k < \frac{1}{2k}\|u_k - \bar{u}\|_{L^2(Q)}.$$

Then, passing to the limit when $k \to \infty$ it follows $J'(\bar{u})v \le 0$.

Step II - $v \in C_{\bar{u}}$. Since $\bar{u}(t) + \lambda v_k(t) = \bar{u}(t) + \frac{\lambda}{\rho_k}(u_k(t) - \bar{u}(t)) \in U_{ad}$ for every $0 < \lambda < \rho_k$, we get for almost every $t \in I_\gamma$

$$\begin{aligned} j'(\bar{u}(t); v_k(t)) &= \lim_{\lambda \searrow 0} \frac{\|\bar{u}(t) + \lambda v_k(t)\|_{L^1(\Omega)} - \|\bar{u}(t)\|_{L^1(\Omega)}}{\lambda} \\ &= \lim_{\lambda \searrow 0} \frac{\|\bar{u}(t) + \lambda v_k(t)\|_{L^1(\Omega)} - \gamma}{\lambda} \le 0. \end{aligned}$$

Take a measurable subset $J \subset I_\gamma$. Since the functional

$$u \in L^2(Q) \longrightarrow \int_J j'(\bar{u}(t); u(t))\, dt \in \mathbb{R}$$

is continuous and convex, recall (5.1), the weak convergence $v_k \rightharpoonup v$ in $L^2(Q)$ implies

$$\int_J j'(\bar{u}(t); v(t))\, dt \le \liminf_{k \to \infty} \int_J j'(\bar{u}(t); v_k(t))\, dt \le 0.$$

Since $J \subset I_\gamma$ is an arbitrary measurable set, we infer for almost all $t \in I_\gamma$

$$\int_{\Omega^+_{\bar{u}(t)}} v(t)\, dx - \int_{\Omega^-_{\bar{u}(t)}} v(t)\, dx + \int_{\Omega^0_{\bar{u}(t)}} |v(t)|\, dx = j'(\bar{u}(t); v(t)) \le 0. \quad (5.5)$$

Identities (3.5) and $J'(\bar{u})v = 0$, and (3.8) imply

$$
0 = \int_Q \bar{\mu}(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{I_\gamma^+}\int_\Omega \bar{\mu}(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t
$$

$$
= \int_{I_\gamma^+}\left\{\|\bar{\mu}(t)\|_{L^\infty(\Omega)}\Big[\int_{\Omega_{\bar{u}(t)}^+}v(t)\,\mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-}v(t)\,\mathrm{d}x\Big] + \int_{\Omega_{\bar{u}(t)}^0}\bar{\mu}(t)v(t)\,\mathrm{d}x\right\}\mathrm{d}t.
$$

(5.6)

From (5.5) we deduce

$$
\int_{I_\gamma^+}\left\{\|\bar{\mu}(t)\|_{L^\infty(\Omega)}\Big[\int_{\Omega_{\bar{u}(t)}^+}v(t)\,\mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-}v(t)\,\mathrm{d}x + \int_{\Omega_{\bar{u}(t)}^0}|v(t)|\,\mathrm{d}x\Big]\right\}\mathrm{d}t \le 0.
$$

The last two relations lead to

$$
\int_{I_\gamma^+}\left\{\int_{\Omega_{\bar{u}(t)}^0}\Big[\|\bar{\mu}(t)\|_{L^\infty(\Omega)}|v(t)| - \bar{\mu}(t)v(t)\Big]\mathrm{d}x\right\}\mathrm{d}t \le 0.
$$

This is possible if and only if $\|\bar{\mu}(t)\|_{L^\infty(\Omega)}|v(x,t)| = \bar{\mu}(x,t)v(x,t)$ for almost all $t \in I_\gamma^+$ and $x \in \Omega_{\bar{u}(t)}^0$. Inserting this identity in (5.6) we get

$$
0 = \int_Q \bar{\mu}(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{I_\gamma^+}\|\bar{\mu}(t)\|_{L^\infty(\Omega)}j'(\bar{u}(t); v(t))\,\mathrm{d}t.
$$

Finally, this identity and (5.5) yield $j'(\bar{u}(t); v(t)) = 0$ for almost all $t \in I_\gamma^+$. Therefore, we conclude with *Step I* that $v \in C_{\bar{u}}$.

Step III - $J''(\bar{u})v^2 \le 0$. From (5.4) and a Taylor expansion we infer

$$
\rho_k J'(\bar{u})v_k + \frac{\rho_k^2}{2}J''(\bar{u} + \theta_k\rho_k v_k)v_k^2 = J(u_k) - J(\bar{u}) < \frac{1}{2k}\|u_k - \bar{u}\|_{L^2(Q)}^2.
$$

Since $J'(\bar{u})v_k = \frac{1}{\rho_k}J'(\bar{u})(u_k - \bar{u}) \ge 0$, we deduce from the above inequality

$$
J''(\bar{u} + \theta_k(u_k - \bar{u}))v_k^2 = J''(\bar{u} + \theta_k\rho_k v_k)v_k^2 < \frac{1}{k}.
$$

(5.7)

The strong convergence $\bar{u} + \theta_k(u_k - \bar{u}) \to \bar{u}$ in $L^r(0, T; L^2(\Omega))$ yields the uniform convergences $y_{\theta_k} \to \bar{y}$ and $\varphi_{\theta_k} \to \bar{\varphi}$ in $L^\infty(Q)$, where y_{θ_k} and φ_{θ_k} are the state and adjoint state associated with $\bar{u}+\theta_k(u_k-\bar{u})$. This also implies that $z_{\theta_k,v_k} \to z_v$ strongly in $L^2(Q)$, where z_v is the solution of (2.20) for $y_u = \bar{y}$ and z_{θ_k,v_k}^2 is the solution of (2.20) with $v = v_k$ and $y_u = y_{\theta_k}$. Then, we can pass to the limit in (5.7) when $k \to \infty$ and deduce that $J''(\bar{u})v^2 \le 0$.

Step IV - Final contradiction. Since $v \in C_{\bar{u}}$ and $J''(\bar{u})v^2 \leq 0$, according to the assumptions of the theorem, this is only possible if $v = 0$. Therefore, we have that $v_k \rightharpoonup 0$ in $L^2(Q)$ and, consequently, $z^2_{\theta_k,v_k} \to 0$ strongly in $L^2(Q)$. Now, using that $\|v_k\|_{L^2(Q)} = 1$ and (2.21), we infer from (5.7)

$$0 \geq \liminf_{k \to \infty} J''(\bar{u} + \theta_k(u_k - \bar{u}))v_k^2$$

$$= \liminf_{k \to \infty} \int_Q \left[\left(1 - \frac{\partial^2 a}{\partial y^2}(x,t,y_{\theta_k})\varphi_{\theta_k}\right)z^2_{\theta_k,v_k} + \kappa v_k^2 \right] dx\, dt$$

$$\lim_{k \to \infty} \int_Q \left(1 - \frac{\partial^2 a}{\partial y^2}(x,t,y_{\theta_k})\varphi_{\theta_k}\right)z^2_{\theta_k,v_k}\, dx\, dt + \kappa = \kappa,$$

which contradicts our assumption $\kappa > 0$.   □

The next theorem establishes that the sufficient condition for local optimality, $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$, provides a useful tool for the numerical analysis of the control problem. Given $\tau > 0$ we define the extended cone

$$C_{\bar{u}}^\tau = \left\{ v \in L^2(Q) : |J'(\bar{u})v| \leq \tau \|v\|_{L^2(Q)} \text{ and} \right.$$

$$\left. \begin{cases} |j'(\bar{u}(t); v(t))| \leq \tau \|v\|_{L^2(Q)} & \text{if } t \in I_\gamma^+, \\ j'(\bar{u}(t); v(t)) \leq \tau \|v\|_{L^2(Q)} & \text{if } t \in I_\gamma \setminus I_\gamma^+, \end{cases} \right\}.$$

Theorem 5.3 *Let $\bar{u} \in U_{ad}$ satisfy the first order optimality conditions* (3.2)–(3.5) *and the second order condition $J''(\bar{u})v^2 > 0 \ \forall v \in C_{\bar{u}} \setminus \{0\}$. Then, for every $r \in \left(\frac{4}{4-n}, \infty\right]$ there exist strictly positive numbers ε, τ, ν such that*

$$J''(u)v^2 \geq \nu \|v\|_{L^2(Q)}^2 \quad \forall v \in C_{\bar{u}}^\tau \ \text{ and } \ \forall u \in B_\varepsilon(\bar{u}), \tag{5.8}$$

where $B_\varepsilon(\bar{u})$ denotes the $L^r(0,T;L^2(\Omega))$ closed ball.

Proof First we prove the existence of $\tau > 0$ and $\nu > 0$ such that

$$J''(\bar{u})v^2 \geq 2\nu \|v\|_{L^2(Q)}^2 \quad \forall v \in C_{\bar{u}}^\tau. \tag{5.9}$$

We proceed by contradiction. If (5.9) fails for all strictly positive numbers τ, ν, then for every integer $k \geq 1$ there exists a function $v_k \in C_{\bar{u}}^{\frac{1}{k}}$ such that $J''(\bar{u})v_k^2 < \frac{1}{k}\|v_k\|_{L^2(Q)}^2$. Dividing v_k by its $L^2(Q)$ norm and taking a subsequence we get

$$\|v_k\|_{L^2(Q)} = 1, \quad v_k \rightharpoonup v \text{ in } L^2(Q), \quad J''(\bar{u})v_k^2 < \frac{1}{k}, \tag{5.10}$$

$$|J'(\bar{u})v_k| \leq \frac{1}{k}, \quad \begin{cases} |j'(\bar{u}(t); v_k(t))| \leq \frac{1}{k} \text{ if } t \in I_\gamma^+, \\ j'(\bar{u}(t); v_k(t)) \leq \frac{1}{k} \text{ if } t \in I_\gamma \setminus I_\gamma^+. \end{cases} \tag{5.11}$$

We prove that $v \in C_{\bar{u}}$. First, from (5.10) and (5.11) we get

$$|J'(\bar{u})v| \leq \liminf_{k \to \infty} |J'(\bar{u})v_k| \leq 0.$$

Thus, we have $J'(\bar{u})v = 0$. Let us set

$$I = \{t \in I_\gamma : j'(\bar{u}(t); v(t)) > 0\}.$$

Then, we obtain with (5.10) and (5.11)

$$\int_I j'(\bar{u}(t); v(t)) \, \mathrm{d}t \leq \liminf_{k \to \infty} \int_I j'(\bar{u}(t); v_k(t)) \, \mathrm{d}t \leq 0.$$

This is not possible unless $|I| = 0$. Hence, we have that $j'(\bar{u}(t); v(t)) \leq 0$ for almost all $t \in I_\gamma$. Now, from the identity $J'(\bar{u})v = 0$, (5.1), and (3.8) it follows

$$
\begin{aligned}
0 &= \int_Q (\bar{\varphi} + \kappa \bar{u}) v \, \mathrm{d}x \, \mathrm{d}t = - \int_Q \bar{\mu} v \, \mathrm{d}x \, \mathrm{d}t \\
&= - \int_{I_\gamma^+} \left[\int_{\Omega_{\bar{u}(t)}^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \, \mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \, \mathrm{d}x + \int_{\Omega_{\bar{u}(t)}^0} \mu v \, \mathrm{d}x \right] \mathrm{d}t.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\int_{I_\gamma^+} &\left[\int_{\Omega_{\bar{u}(t)}^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \, \mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} v \, \mathrm{d}x \right] \mathrm{d}t \\
&= - \int_{I_\gamma^+} \int_{\Omega_{\bar{u}(t)}^0} \mu v \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{5.12}
$$

Now we have

$$
\begin{aligned}
\int_{I_\gamma^+} &\|\bar{\mu}(t)\|_{L^\infty(\Omega)} j'(\bar{u}(t); v(t)) \, \mathrm{d}t \\
&= \int_{I_\gamma^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} \left[\int_{\Omega_{\bar{u}(t)}^+} v \, \mathrm{d}x - \int_{\Omega_{\bar{u}(t)}^-} v \, \mathrm{d}x + \int_{\Omega_{\bar{u}(t)}^0} |v| \, \mathrm{d}x \right] \mathrm{d}t.
\end{aligned}
$$

From this identity and (5.12) we infer

$$\int_{I_\gamma^+} \|\bar{\mu}(t)\|_{L^\infty(\Omega)} j'(\bar{u}(t); v(t)) \, \mathrm{d}t = \int_{I_\gamma^+} \int_{\Omega_{\bar{u}(t)}^0} \left[\|\bar{\mu}(t)\|_{L^\infty(\Omega)} |v| - \bar{\mu}(t) v \right] \mathrm{d}x \, \mathrm{d}t \geq 0.$$

This inequality along with $j'(\bar{u}(t); v(t)) \leq 0$ for $t \in I_\gamma$ implies that $j'(\bar{u}(t); v(t)) = 0$ for almost all $t \in I_\gamma^+$. We have proved that $v \in C_{\bar{u}}$. From (5.10) we infer

$$J''(\bar{u})v^2 \leq \liminf_{k \to \infty} J''(\bar{u})v_k^2 \leq 0.$$

Since $\bar{u}$ satisfies the second order condition, the above inequality is only possible if $v = 0$. Therefore, we have that $v_k \rightharpoonup 0$ in $L^2(Q)$. Using (2.21) and the fact that $z_{v_k} \to 0$ strongly in $L^2(Q)$ this yields

$$\kappa = \liminf_{k \to \infty} \kappa \|v_k\|_{L^2(Q)}^2 = \liminf_{k \to \infty} J''(\bar{u})v_k^2 \leq 0,$$

which is a contradiction. Therefore, (5.9) holds.

Let us conclude the proof showing that (5.9) implies (5.8). Given $\rho > 0$ arbitrarily small, from Theorem 2.2 we deduce the existence of $\varepsilon > 0$ such that

$$\|y_u - \bar{y}\|_{L^\infty(Q)} = \|G(u) - G(\bar{u})\|_{L^\infty(Q)} < \rho \quad \forall u \in B_\varepsilon(\bar{u}).$$

Using this estimate, we get from (2.17) and (2.22), and taking a smaller ε if necessary

$$\|\varphi_u - \bar{\varphi}\|_{L^\infty(Q)} + \|z_{u,v} - z_v\|_{L^2(Q)} < \rho \quad \forall u \in B_\varepsilon(\bar{u}) \text{ and } \forall v \in L^2(Q),$$

where $z_{u,v} = G'(u)v, z_v = G'(\bar{u})v$, and φ_u and $\bar{\varphi}$ are the adjoint states corresponding to u and $\bar{u}$, respectively. Therefore, selecting ρ small enough we obtain with (2.21) for some $\varepsilon > 0$

$$|[J''(u) - J''(\bar{u})]v^2| \leq \nu \|v\|_{L^2(Q)}^2 \quad \forall u \in B_\varepsilon(\bar{u}) \text{ and } \forall v \in L^2(Q).$$

Combining this with (5.9) we infer (5.8). □

6 Stability of the Optimal Controls with Respect to γ

The aim of this section is to prove some stability of the local or global solutions of (P) with respect to γ. For every $\gamma > 0$ we consider the control problems

$$(\mathrm{P}_\gamma) \quad \inf_{u \in U_\gamma \cap L^\infty(Q)} J(u),$$

where

$$U_\gamma = \{u \in L^\infty(0, T; L^1(\Omega)) : \|u(t)\|_{L^1(\Omega)} \leq \gamma \text{ for a.a. } t \in (0, T)\}.$$

First, we prove some continuity of the solutions of (P_γ) with respect to γ.

Theorem 6.1 *Let $\{\gamma_k\}_{k=1}^{\infty} \subset (0, \infty)$ be a sequence converging to some $\gamma > 0$. For every k let u_{γ_k} be a global minimizer of the problem (P_{γ_k}). Then, the sequence $\{u_{\gamma_k}\}_{k=1}^{\infty}$ is bounded in $L^{\infty}(Q)$. Moreover, if u_{γ} is a weak* limit in $L^{\infty}(Q)$ of a subsequence of $\{u_{\gamma_k}\}_{k=1}^{\infty}$, then u_{γ} is a global minimizer of (P_{γ}) and the convergence is strong in $L^p(Q)$ for every $p < \infty$. Reciprocally, for every strict local minimizer u_{γ} of (P_{γ}) in the $L^r(0, T; L^2(\Omega))$ sense with $\frac{4}{4-n} < r < \infty$, there exists a sequence $\{u_{\gamma_k}\}_{k=1}^{\infty}$ such that u_{γ_k} is a $L^r(0, T; L^2(\Omega))$ local minimizer of (P_{γ_k}) and $u_{\gamma_k} \to u_{\gamma}$ strongly in $L^p(Q)$ for every $p < \infty$.*

Proof The boundedness of $\{u_{\gamma_k}\}_{k=1}^{\infty}$ in $L^{\infty}(Q)$ follows from Theorem 3.2. Therefore, we can take subsequences converging weakly* in $L^{\infty}(Q)$. Let us take one of these subsequences, that we denote in the same form, such that $u_{\gamma_k} \overset{*}{\rightharpoonup} \hat{u}$ in $L^{\infty}(Q)$. Let u_{γ} be a solution of (P_{γ}). For every k we define

$$u_k = \begin{cases} u_{\gamma} & \text{if } \gamma \leq \gamma_k, \\ \frac{\gamma_k}{\gamma} u_{\gamma} & \text{if } \gamma > \gamma_k, \end{cases} \quad \text{and} \quad \hat{u}_k = \begin{cases} u_{\gamma_k} & \text{if } \gamma_k \leq \gamma, \\ \frac{\gamma}{\gamma_k} u_{\gamma_k} & \text{if } \gamma_k > \gamma. \end{cases} \tag{6.1}$$

Then, it is immediate that $u_k \to u_{\gamma}$ and $\hat{u}_k \overset{*}{\rightharpoonup} \hat{u}$ in $L^{\infty}(Q)$, $\{\hat{u}_k\}_{k=1}^{\infty} \subset U_{\gamma}$ and $u_k \in U_{\gamma_k} \cap U_{\gamma}$ for every k. Since $U_{\gamma} \cap L^2(Q)$ is a closed and convex subset of $L^2(Q)$ and $\hat{u}_k \rightharpoonup \hat{u}$ in $L^2(Q)$, we deduce that $\hat{u} \in U_{\gamma}$. With the compactness of the embedding $W(0, T) \subset L^2(Q)$ we can easily prove that $y_{\hat{u}_k} \to y_{\hat{u}}$ in $L^2(Q)$. Using these convergences and the optimality of u_{γ_k} and u_{γ} we get

$$J(u_{\gamma}) \leq J(\hat{u}) \leq \liminf_{k\to\infty} J(u_{\gamma_k}) \leq \limsup_{k\to\infty} J(u_{\gamma_k}) \leq \limsup_{k\to\infty} J(u_k) = J(u_{\gamma}).$$

This implies that $J(u_{\gamma}) = J(\hat{u}) = \lim_{k\to\infty} J(u_{\gamma_k})$. This identity proves that $\hat{u}$ is a solution of (P_{γ}). Moreover, the convergence $y_{u_{\gamma_k}} \to y_{u_{\gamma}}$ in $L^2(Q)$ leads to $\lim_{k\to\infty} \|u_{\gamma_k}\|_{L^2(Q)} = \|\hat{u}\|_{L^2(Q)}$. From this fact and the weak convergence $u_{\gamma_k} \rightharpoonup u_{\gamma}$ in $L^2(Q)$, we obtain that $u_{\gamma_k} \to \hat{u}$ in $L^2(Q)$. This along with the boundedness of $\{u_{\gamma_k}\}_{k=1}^{\infty}$ in $L^{\infty}(Q)$ implies the strong convergence in $L^p(Q)$ for every $p < \infty$.

Let us prove the second part of the theorem. Let u_{γ} be an $L^r(0, T; L^2(\Omega))$ strict local minimizer to (P_{γ}). This means that there exists $\varepsilon > 0$ such that

$$J(u_{\gamma}) < J(u) \quad \forall u \in U_{\gamma} \cap B_{\varepsilon}(u_{\gamma}) \text{ with } u \neq u_{\gamma},$$

where $B_{\varepsilon}(u_{\gamma})$ is the closed ball in $L^r(0, T; L^2(\Omega))$ of radius ε and center u_{γ}. Now, we consider the problems

$$(PB_{\gamma}) \quad \min_{u\in U_{\gamma}\cap B_{\varepsilon}(u_{\gamma})} J(u) \quad \text{and} \quad (PB_{\gamma_k}) \quad \min_{u\in U_{\gamma_k}\cap B_{\varepsilon}(u_{\gamma})} J(u)$$

It is immediate that u_{γ} is the unique solution of (PB_{γ}). Observe that the controls u_k defined in (6.1) are elements of $U_{\gamma_k} \cap B_{\varepsilon}(u_{\gamma})$ for all k large enough. Hence, $U_{\gamma_k} \cap B_{\varepsilon}(u_{\gamma})$ is non-empty, closed, convex, and bounded in $L^r(0, T; L^2(\Omega))$. Therefore,

problem (PB_{γ_k}) has at least one solution u_{γ_k}. Let us prove that $u_{\gamma_k} \to u_\gamma$ in $L^p(Q)$ for every $p < \infty$. Denote y_{γ_k} and φ_{γ_k} the state and adjoint state associated with u_{γ_k}. Since $\{u_{\gamma_k}\}_{k=1}^\infty$ is bounded in $L^r(0, T; L^2(\Omega))$ we infer from Theorem 2.1 the boundedness of $\{y_{\gamma_k}\}_{k=1}^\infty$ in $L^\infty(Q)$. Hence, from the adjoint state equation and the classical estimates for linear equations we deduce that $\{\varphi_{\gamma_k}\}_{k=1}^\infty$ is also bounded in $L^\infty(Q)$. Due to the optimality of u_{γ_k} for (PB_{γ_k}) we obtain

$$\int_Q (\varphi_{\gamma_k} + \kappa u_{\gamma_k})(u - u_{\gamma_k})\, dx\, dt = J'(u_{\gamma_k})(u - u_{\gamma_k}) \geq 0 \ \ \forall u \in U_{\gamma_k} \cap B_\varepsilon(u_\gamma).$$

Setting $S = U_{\gamma_k} \cap B_\varepsilon(u_\gamma)$ we get from the above inequalities

$$u_{\gamma_k} = \mathrm{Proj}_S\left(-\frac{1}{\kappa}\varphi_{\gamma_k}\right),$$

where Proj_S denotes the $L^2(Q)$ projection on S. Let us prove that

$$\|u_{\gamma_k}\|_{L^\infty(Q)} \leq 2\left(\frac{1}{\kappa}\|\varphi_{\gamma_k}\|_{L^\infty(Q)} + \|u_\gamma\|_{L^\infty(Q)}\right). \tag{6.2}$$

For this purpose we define

$$Q_0 = \left\{(x, t) \in Q : |u_{\gamma_k}(x, t)| > 2\left(\frac{1}{\kappa}|\varphi_{\gamma_k}(x, t)| + |u_\gamma(x, t)|\right)\right\}.$$

Put

$$u(x, t) = \begin{cases} -\frac{1}{\kappa}\varphi_{\gamma_k}(x, t) + u_\gamma(x, t) & \text{if } (x, t) \in Q_0, \\ u_{\gamma_k}(x, t) & \text{otherwise.} \end{cases}$$

Then, it is obvious that

$$\|u(t)\|_{L^1(\Omega)} \leq \|u_{\gamma_k}(t)\|_{L^1(\Omega)} \leq \gamma_k,$$
$$\|u - u_\gamma\|_{L^r(0,T;L^2(\Omega))} \leq \|u_{\gamma_k} - u_\gamma\|_{L^r(0,T;L^2(\Omega))} \leq \varepsilon,$$
$$\left\|u + \frac{1}{\kappa}\varphi_{\gamma_k}\right\|_{L^2(Q)} < \left\|u_{\gamma_k} + \frac{1}{\kappa}\varphi_{\gamma_k}\right\|_{L^2(Q)} \text{ if } |Q_0| \neq 0,$$

The first two inequalities show that $u \in S$ and, consequently, the third one contradicts the fact that u_{γ_k} is the $L^2(Q)$ projection of $-\frac{1}{\kappa}\varphi_{\gamma_k}$ unless $|Q_0| = 0$. Now, the boundedness of $\{\varphi_{\gamma_k}\}_{k=1}^\infty$ in $L^\infty(Q)$ and (6.2) imply the boundedness of $\{u_{\gamma_k}\}_{k=1}^\infty$. Therefore, there exists a subsequence, denoted in the same way, such that $u_{\gamma_k} \overset{*}{\rightharpoonup} \hat{u}$ in $L^\infty(Q)$. Using the functions $\{\hat{u}_k\}_{k=1}^\infty$ defined in (6.1) and arguing as above, we deduce that $\hat{u} \in U_\gamma$. Moreover, is is also immediate that $\hat{u} \in B_\varepsilon(u_\gamma)$. Let us consider the functions $\{u_k\}_{k=1}^\infty$ defined in (6.1). Since

$$\|u_k - u_\gamma\|_{L^\infty(Q)} = \begin{cases} 0 & \text{if } \gamma \leq \gamma_k, \\ \frac{\gamma - \gamma_k}{\gamma}\|u_\gamma\|_{L^\infty(Q)} & \text{otherwise,} \end{cases}$$

we have that $u_k \to u_\gamma$ in $L^\infty(Q)$ as $k \to \infty$ and $u_k \in U_{\gamma_k} \cap B_\varepsilon(u_\gamma)$ for every k large enough. Then, using the optimality of u_γ and u_{γ_k}, and the fact that u_k and $\hat{u}$ are feasible controls for (PB_{γ_k}) and (PB_γ), respectively, we infer

$$J(u_\gamma) \le J(\hat{u}) \le \liminf_{k\to\infty} J(u_{\gamma_k}) \le \limsup_{k\to\infty} J(u_{\gamma_k}) \le \limsup_{k\to\infty} J(u_k) = J(u_\gamma).$$

This implies that $J(u_\gamma) = J(\hat{u})$ and, hence, $\hat{u}$ is also a solution of (PB_γ). Due to the uniqueness of solution of (PB_γ) we conclude that $u_\gamma = \hat{u}$. The strong convergence $u_{\gamma_k} \to u_\gamma$ in $L^p(Q)$ follows as above. We have proved that every subsequence converge to u_γ, then the whole sequence does. In particular, the convergence $u_{\gamma_k} \to u_\gamma$ in $L^r(0, T; L^2(\Omega))$ implies that u_{γ_k} is in the interior of the ball $B_\varepsilon(u_\gamma)$ for all k sufficiently large. Hence, u_{γ_k} is an $L^r(0, T; L^2(\Omega))$ local minimizer of (PB_{γ_k}). $\square$

Remark 6.1 Given an $L^r(0, T; L^2(\Omega))$ strict local minimizer of (P_γ), from the above theorem we deduce the existence of a family $\{u_{\gamma'}\}_{\gamma'>0}$ of $L^r(0, T; L^2(\Omega))$ local minimizers of problems $(P_{\gamma'})$ such that $u_{\gamma'} \to u_\gamma$ in $L^p(Q)$ as $\gamma' \to \gamma$ for every $p < \infty$. Looking at the definition of the elements u_{γ_k} in the previous proof we have that

$$J(u_{\gamma'}) \le J(u) \ \ \forall u \in U_{\gamma'} \cap B_\varepsilon(u_\gamma) \quad \text{and} \quad J(u_\gamma) \le J(u) \ \ \forall u \in U_\gamma \cap B_\varepsilon(u_\gamma). \tag{6.3}$$

Theorem 6.2 *Let $\{u_{\gamma'}\}_{\gamma'}$ be a family of local minimizers of problems $(P_{\gamma'})$ such that $u_{\gamma'} \to u_\gamma$ in $L^r(0, T; L^2(\Omega))$ as $\gamma' \to \gamma$ with u_γ a local minimizer of (P_γ) satisfying* (5.3). *We also assume that* (6.3) *holds. Then, there exists a constant L such that*

$$\|u_{\gamma'} - u_\gamma\|_{L^2(Q)} \le L|\gamma' - \gamma|^{\frac{1}{2}}. \tag{6.4}$$

Proof The first part of the theorem follows from Remark 6.1. We only have to prove (6.4). For every γ' we define

$$\hat{u}_{\gamma'} = \begin{cases} u_\gamma & \text{if } \gamma < \gamma', \\ \frac{\gamma'}{\gamma} u_\gamma & \text{if } \gamma > \gamma', \end{cases} \quad \text{and} \quad \hat{v}_{\gamma'} = \begin{cases} u_{\gamma'} & \text{if } \gamma' < \gamma, \\ \frac{\gamma}{\gamma'} u_{\gamma'} & \text{if } \gamma' > \gamma. \end{cases} \tag{6.5}$$

Then we have

$$\hat{u}_{\gamma'}, \hat{v}_{\gamma'} \in U_\gamma \cap U_{\gamma'}, \ \ \hat{u}_{\gamma'} \to u_\gamma \text{ in } L^\infty(Q) \text{ and } \hat{v}_{\gamma'} \to u_\gamma \text{ in } L^r(0, T; L^2(\Omega)). \tag{6.6}$$

From here we infer that $v_{\gamma'} \in U_\gamma \cap B_\varepsilon(u_\gamma)$ for γ' close enough to γ with $B_\varepsilon(u_\gamma)$ defined in (5.3). Therefore, we get

$$\frac{\delta}{2}\|\hat{v}_{\gamma'} - u_\gamma\|^2_{L^2(Q)} \le J(\hat{v}_{\gamma'}) - J(u_\gamma). \tag{6.7}$$

In the case $\gamma' < \gamma$, using (6.7), the optimality of $u_{\gamma'}$, and the definition of $\hat{v}_{\gamma'}$ we obtain with the mean value theorem

$$\|u_{\gamma'} - u_\gamma\|^2_{L^2(Q)} \leq \frac{2}{\delta}\big[(J(u_{\gamma'}) - J(\hat{u}_{\gamma'})) + (J(\hat{u}_{\gamma'}) - J(u_\gamma))\big]$$

$$\leq \frac{2}{\delta}(J(\hat{u}_{\gamma'}) - J(u_\gamma)) \leq C_1\|\hat{u}_{\gamma'} - u_\gamma\|_{L^\infty(Q)} = \frac{C_1}{\gamma}\|u_\gamma\|_{L^\infty(Q)}|\gamma' - \gamma|.$$

In the case $\gamma' > \gamma$ we proceed as follows

$$\|\hat{v}_{\gamma'} - u_\gamma\|^2_{L^2(Q)} \leq \frac{2}{\delta}\big[(J(\hat{v}_{\gamma'}) - J(u_{\gamma'})) + (J(u_{\gamma'}) - J(u_\gamma))\big]$$

$$\leq \frac{2}{\delta}\big(J(\hat{v}_{\gamma'}) - J(u_{\gamma'})\big) \leq C_2\|\hat{v}_{\gamma'} - u_{\gamma'}\|_{L^r(0,T;L^2(\Omega))}$$

$$= \frac{C_2}{\gamma'}\|u_{\gamma'}\|_{L^r(0,T;L^2(\Omega))}|\gamma' - \gamma| \leq C_3|\gamma' - \gamma|.$$

From here we get

$$\|u_{\gamma'} - u_\gamma\|_{L^2(Q)} \leq \|u_{\gamma'} - \hat{v}_{\gamma'}\|_{L^2(Q)} + \|\hat{v}_{\gamma'} - u_\gamma\|_{L^2(Q)}$$

$$\leq \frac{|\gamma' - \gamma|}{\gamma'}\|u_{\gamma'}\|_{L^2(Q)} + \sqrt{C_3}|\gamma' - \gamma|^{\frac{1}{2}} \leq C_4|\gamma' - \gamma|^{\frac{1}{2}},$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorems 5.2 and 6.2 imply Hölder stability with respect to γ of the optimal controls if the sufficient second order condition $J''(u_\gamma)v^2 > 0 \ \forall v \in C_{\bar{u}} \setminus \{0\}$ holds. Now, we are interested in proving Lipschitz stability. To this end we need to make a stronger assumption, namely

$$J''(u_{\bar{\gamma}})v^2 > 0 \quad \forall v \in L^2(Q) \setminus \{0\}, \quad y_0 \in C_0(\Omega), \quad \text{and} \quad \hat{r} > \frac{4}{4-n}, \quad (6.8)$$

where $\bar{\gamma} > 0$ is fixed and $C_0(\Omega)$ denotes the space of continuous real valued functions on $\bar{\Omega}$ vanishing on Γ. From the first assumption in (6.8) we deduce the existence of strictly positive numbers ρ and ν such that

$$J''(u)v^2 \geq \nu\|v\|^2_{L^2(Q)} \quad \forall v \in L^2(Q) \text{ and } \forall u \in B_\rho(u_{\bar{\gamma}}), \quad (6.9)$$

where $B_\rho(u_{\bar{\gamma}})$ denotes the $L^{\hat{r}}(0,T;L^2(\Omega))$ closed ball. Indeed, if (6.9) does not hold, then we can take sequences $\{u_k\}^\infty_{k=1} \subset L^{\hat{r}}(0,T;L^2(\Omega))$ and $\{v_k\}^\infty_{k=1} \subset L^2(Q)$ satisfying

$$\lim_{k\to\infty} \|u_k - u_{\bar{\gamma}}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} = 0, \ \|v_k\|_{L^2(Q)} = 1, \ v_k \rightharpoonup v \text{ in } L^2(Q), \ J''(u_k)v^2_k \leq \frac{1}{k}.$$

It is easy to pass to the limit and to deduce

$$J''(u_{\bar{\gamma}})v^2 \le \liminf_{k\to\infty} J''(u_k)v_k^2 \le 0.$$

This inequality and (6.8) yield $v = 0$. But, arguing as in the proof of Theorem 5.3 we infer

$$\kappa = \liminf_{k\to\infty} \kappa \|v_k\|_{L^2(Q)}^2 = \liminf_{k\to\infty} J''(u_k)v_k^2 \le 0,$$

which contradicts our assumption $\kappa > 0$.

We finish this section by proving the next theorem.

Theorem 6.3 *Let $u_{\bar{\gamma}}$ be a local minimizer of $(P_{\bar{\gamma}})$. We assume that (6.8) holds and that ρ satisfies (6.9). Then, there exists $\bar{\varepsilon} \in (0, \bar{\gamma})$ such that (P_γ) has a unique local minimizer u_γ in the interior of the $L^{\hat{r}}(0, T; L^2(\Omega))$ ball $B_\rho(u_{\bar{\gamma}})$ for every $\gamma \in (\bar{\gamma} - \bar{\varepsilon}, \bar{\gamma} + \bar{\varepsilon})$. Moreover, there exists a constant L such that*

$$\|u_\gamma - u_{\bar{\gamma}}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} \le L|\gamma - \bar{\gamma}| \quad \forall \gamma \in (\bar{\gamma} - \bar{\varepsilon}, \bar{\gamma} + \bar{\varepsilon}). \tag{6.10}$$

Proof Let us take $\rho > 0$ such that (6.9) holds. Then, J has at most one local (and global) minimizer u_γ in the closed set $B_\rho(u_{\bar{\gamma}}) \cap U_{ad}$. This is a consequence of the strict convexity of J in the ball $B_\rho(u_{\bar{\gamma}})$; see (6.9). We will prove that this local minimizer belongs to the interior of the $L^{\hat{r}}(0, T; L^2(\Omega))$ ball $B_\rho(u_{\bar{\gamma}})$ if γ is close enough to $\bar{\gamma}$, and consequently it is a local minimizer of (P_γ). In order to prove this, as well as (6.10), we reformulate the control problem (P_γ) as follows

$$(Q_\gamma) \quad \inf_{u \in K_{ad}} J_\gamma(u) := \frac{1}{2}\int_Q (y_{\gamma,u}(x,t) - y_d(x))^2 \,dx\,dt + \frac{\kappa\gamma^2}{2}\int_Q u(x,t)^2 \,dx\,dt,$$

where

$$K_{ad} = \{u \in L^{\hat{r}}(0, T; L^2(\Omega)) : \|u(t)\|_{L^1(\Omega)} \le 1 \text{ for a.a. } t \in (0, T)\}$$

and $y_{\gamma,u}$ is the solution of the semilinear parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + a(x,t,y) = \gamma u \text{ in } Q = \Omega \times (0, T), \\ y = 0 \text{ on } \Sigma = \Gamma \times (0, T), \quad y(0) = y_0 \text{ in } \Omega. \end{cases} \tag{6.11}$$

It is obvious that the problems (P_γ) and (Q_γ) are equivalent for every γ. This equivalence is understood in the sense that u is a local (global) minimizer of (Q_γ) if and only if $u_\gamma = \gamma u$ is a local (global) minimizer of (P_γ), and $J(u_\gamma) = J_\gamma(u)$; recall Remark 4.2.

Take $\varepsilon \in (0, \bar{\gamma})$ and $\bar{\rho} \in (0, \rho]$ such that $(\bar{\gamma} + \varepsilon)\bar{\rho} + \varepsilon \|\bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} < \rho$. Then, we have with the notation $u_{\bar{\gamma}} = \bar{\gamma}\bar{u}$ and $u_{\gamma} = \gamma u$

$$\|u_{\gamma} - u_{\bar{\gamma}}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} \leq \gamma \|u - \bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} + |\gamma - \bar{\gamma}|\|\bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))}$$
$$\leq (\bar{\gamma} + \varepsilon)\bar{\rho} + \varepsilon \|u_{\bar{\gamma}}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} < \rho \quad \forall u \in B_{\bar{\rho}}(\bar{u}) \text{ and } \forall \gamma \in (\bar{\gamma} - \varepsilon, \bar{\gamma} + \varepsilon).$$

Due to (6.9) and the fact that $J_{\gamma}''(u)v^2 = \gamma^2 J''(u_{\gamma})v^2$, we deduce that

$$J_{\gamma}''(u)v^2 \geq \gamma^2 \nu \|v\|^2_{L^{\hat{r}}(0,T;L^2(\Omega))} \geq (\bar{\gamma} - \varepsilon)^2 \nu \|v\|^2_{L^{\hat{r}}(0,T;L^2(\Omega))} \quad \forall u \in B_{\bar{\rho}}(\bar{u}).$$

Therefore, J_{γ} is strictly convex on the ball $B_{\bar{\rho}}(\bar{u})$. Hence, a control u is a local solution of (Q_{γ}) in the interior of $B_{\bar{\rho}}(\bar{u})$ if and only if u satisfies the optimality system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + a(x, t, y) = \gamma u \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(0) = y_0 \text{ in } \Omega, \end{cases} \tag{6.12}$$

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial a}{\partial y}(x, t, y)\varphi = y - y_d \text{ in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = 0 \text{ in } \Omega, \end{cases} \tag{6.13}$$

$$\int_Q \mu(v - u) \, dx \, dt \leq 0 \quad \forall v \in K_{ad}, \tag{6.14}$$

$$\gamma \varphi + \kappa \gamma^2 u + \mu = 0. \tag{6.15}$$

Denote by $\bar{y}$ and $\bar{\varphi}$ the state and adjoint state associated to $\bar{u}$. Our goal is to apply [10,Theorem 2.4] to the previous optimality system. To this end we define the spaces:

$$\mathcal{Y} = \{y \in W(0, T) \cap C(\bar{Q}) : \frac{\partial y}{\partial t} + Ay \in L^{\hat{r}}(0, T; L^2(\Omega))\},$$

$$\Phi = \{\varphi \in H^1(Q) \cap C(\bar{Q}) : -\frac{\partial \varphi}{\partial t} + A^*\varphi \in L^{\hat{r}}(0, T; L^2(\Omega)) \text{ and } \varphi(T) = 0\},$$

$$X = \mathcal{Y} \times \Phi \times L^{\hat{r}}(0, T; L^2(\Omega)), \quad Y = \mathbb{R}, \quad Z = C_0(\Omega) \times L^{\hat{r}}(0, T; L^2(\Omega))^3.$$

On $\mathcal{Y}$ and Φ we consider the graph norms

$$\|y\|_{\mathcal{Y}} = \|y\|_{W(0,T)} + \|y\|_{C(\bar{Q})} + \left\|\frac{\partial y}{\partial t} + Ay\right\|_{L^{\hat{r}}(0,T;L^2(\Omega))},$$

$$\|\varphi\|_{\Phi} = \|\varphi\|_{H^1(Q)} + \|\varphi\|_{C(\bar{Q})} + \left\|-\frac{\partial \varphi}{\partial t} + A^*\varphi\right\|_{L^{\hat{r}}(0,T;L^2(\Omega))}.$$

Thus, X is a Banach space. Moreover, we introduce the mapping $f : X \times Y \longrightarrow Z$ and the multivalued function $F : X \longrightarrow Z$

$$
f((y, \varphi, u), \gamma) = \begin{pmatrix} y(0) - y_0 \\ \dfrac{\partial y}{\partial t} + Ay + a(\cdot, \cdot, y) - \gamma u \\ -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial a}{\partial y}(\cdot, \cdot, y)\varphi - y + y_d \\ \gamma \varphi + \gamma^2 \kappa u \end{pmatrix},
$$

$$
F(y, \varphi, u) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_0(u) \end{pmatrix},
$$

where the multivalued function $F_0 : L^{\hat{}}(0, T; L^2(\Omega)) \longrightarrow L^{\hat{}}(0, T; L^2(\Omega))$ is defined by

$$
F_0(u) = \begin{cases} \varnothing & \text{if } u \notin K_{ad}, \\ \left\{ \mu \in L^{\hat{}}(0, T; L^2(\Omega)) : \displaystyle\int_Q \mu(v - u)\, dx\, dt \le 0 \ \forall v \in K_{ad} \right\}, & \text{otherwise.} \end{cases}
$$

Due to the regularity $y_0 \in C_0(\Omega)$, see assumption (6.8), we deduce from (6.12) that $\bar{y} \in \mathcal{Y}$. Therefore, we have that $(\bar{y}, \bar{\varphi}, \bar{u}) \in X$. Moreover, $(\bar{y}, \bar{\varphi}, \bar{u})$ satisfies the optimality system (6.12)–(6.15), which implies that $0 \in f((\bar{y}, \bar{\varphi}, \bar{u}), \bar{\gamma}) + F(\bar{y}, \bar{\varphi}, \bar{u})$. Using our assumptions on a and the continuous embedding $\mathcal{Y} \subset C(\bar{Q})$ we deduce that the function f is of class C^1. Then, the function $g : X \longrightarrow Z$, defined by

$$
g(y, \varphi, u) = f((\bar{y}, \bar{\varphi}, \bar{u}), \bar{\gamma}) + D_{(y,\varphi,u)} f((\bar{y}, \bar{\varphi}, \bar{u}), \bar{\gamma})(y - \bar{y}, \varphi - \bar{\varphi}, u - \bar{u}),
$$

strongly approximates f at $((\bar{y}, \bar{\varphi}, \bar{u}), \bar{\gamma})$, and $g(\bar{y}, \bar{\varphi}, \bar{u}) = f((\bar{y}, \bar{\varphi}, \bar{u}), \bar{\gamma})$; see [19] for the definition of a strong approximation.

We will apply [10,Theorem 2.4] to deduce the existence of $\bar{\varepsilon} \in (0, \varepsilon]$ and $\tilde{\rho} \in (0, \bar{\rho}]$ such that (6.12)–(6.15) has a unique solution u in the interior of the ball $B_{\tilde{\rho}}(\bar{u})$ for every $\gamma \in (\bar{\gamma} - \bar{\varepsilon}, \bar{\gamma} + \bar{\varepsilon})$. Moreover, these solutions satisfy

$$
\|u - \bar{u}\|_{L^{\hat{}}(0,T;L^2(\Omega))} \le \lambda |\gamma - \bar{\gamma}|. \tag{6.16}
$$

for some $\lambda > 0$. For this purpose it is enough to prove that the equation

$$
\beta \in g(y, \varphi, u) + F(y, \varphi, u) \tag{6.17}
$$

has a unique solution $(y_\beta, \varphi_\beta, u_\beta) \in X$ for every $\beta = (\beta_i)_{i=1}^4 \in Z$ and the Lipschitz property

$$
\|(y_{\hat{\beta}}, \varphi_{\hat{\beta}}, u_{\hat{\beta}}) - (y_\beta, \varphi_\beta, u_\beta)\|_X \le \lambda \|\hat{\beta} - \beta\|_Z \tag{6.18}
$$

holds for some $\lambda > 0$ and all $\hat{\beta}, \beta \in Z$. First, we prove the existence of a unique solution. To this end we consider the optimal control problem

$$(\mathrm{P}_\beta) \quad \inf_{u \in K_{ad}} \mathcal{J}_\beta(u),$$

where

$$\mathcal{J}_\beta(u) := \frac{1}{2} \int_Q \Big[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi}\Big] y^2 \,dx\,dt + \frac{\kappa\bar{\gamma}^2}{2} \int_Q u(x, t)^2 \,dx\,dt$$

$$+ \int_Q \beta_3 y \,dx\,dt + \int_Q (\bar{\gamma}\bar{\varphi} + \bar{\gamma}^2\kappa\bar{u} - \beta_4)u \,dx\,dt,$$

and y satisfies the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + \dfrac{\partial a}{\partial y}(x, t, \bar{y})y = \bar{\gamma}u + \beta_2 \;\text{ in } Q, \\ y = 0 \text{ on } \Sigma, \;\; y(0) = \beta_1 \text{ in } \Omega. \end{cases} \tag{6.19}$$

Let us consider the solution $\xi_\beta \in \mathcal{Y}$ of the equation

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + A\xi + \dfrac{\partial a}{\partial y}(x, t, \bar{y})\xi = \beta_2 \;\text{ in } Q, \\ \xi = 0 \text{ on } \Sigma, \;\; \xi(0) = \beta_1 \text{ in } \Omega. \end{cases} \tag{6.20}$$

According to (2.17) we have that $y = \bar{\gamma}G'(u_{\bar{\gamma}})u + \xi_\beta = \bar{\gamma}z_u + \xi_\beta$. Inserting this identity in the cost functional we get

$$\mathcal{J}_\beta(u) = \frac{\bar{\gamma}^2}{2} \left\{ \int_Q \Big[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi}\Big] z_u^2 \,dx\,dt + \kappa \int_Q u^2 \,dx\,dt \right\}$$

$$+ \bar{\gamma} \int_Q \Big(\Big[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi}\Big]\xi_\beta + \beta_3 \Big) z_u \,dx\,dt$$

$$+ \int_Q (\bar{\gamma}\bar{\varphi} + \bar{\gamma}^2\kappa u_{\bar{\gamma}} - \beta_4)u \,dx\,dt$$

$$+ \int_Q \Big(\frac{1}{2}\Big[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi}\Big]\xi_\beta^2 + \beta_3\xi_\beta \Big) \,dx\,dt.$$

From (2.21), (6.9), and the continuity of the mapping $u \to z_u$ in $L^2(Q)$ we deduce the existence of two constants C_1 and C_2 such that

$$\mathcal{J}_\beta(u) \geq \frac{\bar{\gamma}^2}{2}\nu\|u\|^2_{L^2(Q)} + C_1\|u\|_{L^2(Q)} + C_2.$$

Therefore, $\mathcal{J}_\beta$ is a coercive, continuous, and strictly convex quadratic functional on $L^2(Q)$. As a consequence, we infer the existence and uniqueness of a minimizer $\tilde{u}_\beta$

of $\mathcal{J}_\beta$ on the set

$$\tilde{K}_{ad} = \{u \in L^2(Q) : \|u(t)\|_{L^1(\Omega)} \leq 1 \text{ for a.a. } t \in (0, T)\}.$$

Similarly as in Theorem 3.1, we deduce the existence of elements $\tilde{y}_\beta \in W(0, T)$, $\tilde{\varphi}_\beta \in H^1(Q)$, and $\tilde{\mu}_\beta \in L^2(Q)$ satisfying

$$\begin{cases} \dfrac{\partial \tilde{y}_\beta}{\partial t} + A\tilde{y}_\beta + \dfrac{\partial a}{\partial y}(x, t, \bar{y})\tilde{y}_\beta = \bar{\gamma}\tilde{u}_\beta + \beta_2 \text{ in } Q, \\ \tilde{y}_\beta = 0 \text{ on } \Sigma, \quad \tilde{y}_\beta(0) = \beta_1 \text{ in } \Omega, \end{cases} \quad (6.21)$$

$$\begin{cases} -\dfrac{\partial \tilde{\varphi}_\beta}{\partial t} + A^*\tilde{\varphi}_\beta + \dfrac{\partial a}{\partial y}(x, t, \bar{y})\tilde{\varphi}_\beta = \Big[1 - \dfrac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi}\Big]\tilde{y}_\beta + \beta_3 \text{ in } Q, \\ \tilde{\varphi}_\beta = 0 \text{ on } \Sigma, \quad \tilde{\varphi}_\beta(T) = 0 \text{ in } \Omega, \end{cases} \quad (6.22)$$

$$\int_Q \tilde{\mu}_\beta(u - \tilde{u}_\beta) \, dx \, dt \leq 0 \quad \forall u \in \tilde{K}_{ad}, \quad (6.23)$$

$$\bar{\gamma}\tilde{\varphi}_\beta + \bar{\gamma}^2\kappa\tilde{u}_\beta + \bar{\gamma}\bar{\varphi} + \bar{\gamma}^2\kappa\bar{u} - \beta_4 + \tilde{\mu}_\beta = 0. \quad (6.24)$$

Arguing similarly as in the proof of Theorem 4.4 we deduce that $\tilde{u}_\beta$ and $\tilde{\mu}_\beta$ belong to the space $L^{\hat{r}}(0, T; L^2(\Omega))$. Thus, $\tilde{u}_\beta$ is the unique solution of (P_β). Moreover, from (6.21) and (6.22) along with (6.8) we infer that $\tilde{y}_\beta \in \mathcal{Y}$ and $\tilde{\varphi}_\beta \in \Phi$. Hence, we have that $(\tilde{y}_\beta, \tilde{\varphi}_\beta, \tilde{u}_\beta) \in X$ and (6.23) holds for every $u \in K_{ad}$. Due to the convexity of (P_β), we know that (6.21)–(6.24) are necessary and sufficient conditions of optimality for (P_β). This fact and the strict convexity of $\mathcal{J}_\beta$ imply that the system (6.21)–(6.24) has a unique solution $(\tilde{y}_\beta, \tilde{\varphi}_\beta, \tilde{u}_\beta, \tilde{\mu}_\beta)$. If we set $y_\beta = \tilde{y}_\beta + \bar{y}$, $\varphi_\beta = \tilde{\varphi}_\beta + \bar{\varphi}$, $u_\beta = \tilde{u}_\beta + \bar{u}$, and $\mu_\beta = \tilde{\mu}_\beta$, (6.21)–(6.24) yields that $(y_\beta, \varphi_\beta, u_\beta)$ is the unique element of X satisfying (6.17).

Now, we prove that this solution is Lipschitz with respect to β. First, we observe that (6.24) can be written as

$$\bar{\gamma}\varphi_\beta + \bar{\gamma}^2\kappa u_\beta - \beta_4 + \mu_\beta = 0. \quad (6.25)$$

Given $\beta, \hat{\beta} \in Z$, we infer from (6.23)-(6.24) and (6.25) for β and $\hat{\beta}$

$$\int_\Omega (\bar{\gamma}\varphi_\beta(t) + \bar{\gamma}^2\kappa u_\beta(t) - \beta_4(t))(u_{\hat{\beta}}(t) - u_\beta(t)) \, dx \, dt \leq 0,$$

$$\int_\Omega (\bar{\gamma}\varphi_{\hat{\beta}}(t) + \bar{\gamma}^2\kappa u_{\hat{\beta}}(t) - \hat{\beta}_4(t))(u_\beta(t) - u_{\hat{\beta}}(t)) \, dx \, dt \leq 0.$$

Adding these inequalities we get

$$\bar{\gamma}^2\kappa\|u_{\hat{\beta}}(t) - u_\beta(t)\|^2_{L^2(\Omega)} \leq \bar{\gamma}\int_\Omega (\varphi_\beta(t) - \varphi_{\hat{\beta}}(t))(u_{\hat{\beta}}(t) - u_\beta(t)) \, dx$$
$$+ \|\hat{\beta}_4(t) - \beta_4(t)\|_{L^2(\Omega)}\|u_{\hat{\beta}}(t) - u_\beta(t)\|_{L^2(\Omega)} \quad (6.26)$$

for almost every $t \in (0, T)$. Now, taking into account that $y_{\hat{\beta}} - y_\beta = \tilde{y}_{\hat{\beta}} - \tilde{y}_\beta$, $\varphi_{\hat{\beta}} - \varphi_\beta = \tilde{\varphi}_{\hat{\beta}} - \tilde{\varphi}_\beta$, and $u_{\hat{\beta}} - u_\beta = \tilde{u}_{\hat{\beta}} - \tilde{u}_\beta$, subtracting the equations (6.21) satisfied by $y_{\hat{\beta}}$ and y_β, and the equations (6.22) for $\varphi_{\hat{\beta}}$ and φ_β, respectively, we obtain

$$
\begin{aligned}
&\bar{\gamma} \int_Q (\varphi_\beta(t) - \varphi_{\hat{\beta}}(t))(u_{\hat{\beta}}(t) - u_\beta(t)) \, \mathrm{d}x \\
&= \int_Q \left\{ \left(\frac{\partial}{\partial t} + A + \frac{\partial a}{\partial y}(x, t, \bar{y}) \right)(y_{\hat{\beta}} - y_\beta)(\varphi_\beta - \varphi_{\hat{\beta}}) - (\hat{\beta}_2 - \beta_2)(\varphi_\beta - \varphi_{\hat{\beta}}) \right\} \mathrm{d}x \, \mathrm{d}t \\
&= \int_Q \left\{ \left(-\frac{\partial}{\partial t} + A^* + \frac{\partial a}{\partial y}(x, t, \bar{y}) \right)(\varphi_\beta - \varphi_{\hat{\beta}})(y_{\hat{\beta}} - y_\beta) - (\hat{\beta}_2 - \beta_2)(\varphi_\beta - \varphi_{\hat{\beta}}) \right\} \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_\Omega (\hat{\beta}_1 - \beta_1)(\varphi_\beta(0) - \varphi_{\hat{\beta}}(0)) \, dx \\
&= -\int_Q \left\{ \left[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi} \right](y_{\hat{\beta}} - y_\beta)^2 + (\hat{\beta}_2 - \beta_2)(\varphi_\beta - \varphi_{\hat{\beta}}) \right\} \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_Q (\hat{\beta}_3 - \beta_3)(y_{\hat{\beta}} - y_\beta) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\hat{\beta}_1 - \beta_1)(\varphi_\beta(0) - \varphi_{\hat{\beta}}(0)) \, \mathrm{d}x.
\end{aligned}
$$

Let us denote by ξ_β and $\xi_{\hat{\beta}}$ the solutions of (6.20) corresponding to (β_1, β_2) and $(\hat{\beta}_1, \hat{\beta}_2)$, respectively. Then, we have that $y_{\hat{\beta}} - y_\beta = \bar{\gamma} G'(\bar{u})(u_{\hat{\beta}} - u_\beta) + \xi_{\hat{\beta}} - \xi_\beta = \bar{\gamma} z_{u_{\hat{\beta}} - u_\beta} + (\xi_{\hat{\beta}} - \xi_\beta)$. Inserting this identity in the above equality we infer

$$
\begin{aligned}
&\bar{\gamma} \int_Q (\varphi_\beta(t) - \varphi_{\hat{\beta}}(t))(u_{\hat{\beta}}(t) - u_\beta(t)) \, \mathrm{d}x \\
&\quad = -\bar{\gamma}^2 \int_Q \left[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi} \right] z_{u_{\hat{\beta}} - u_\beta}^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_Q \left[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi} \right] [(\xi_{\hat{\beta}} - \xi_\beta)^2 + 2 z_{u_{\hat{\beta}} - u_\beta}(\xi_{\hat{\beta}} - \xi_\beta)] \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_Q \left\{ (\hat{\beta}_2 - \beta_2)(\varphi_{\hat{\beta}} - \varphi_\beta) + (\hat{\beta}_3 - \beta_3)(y_{\hat{\beta}} - y_\beta) \right\} \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_\Omega (\hat{\beta}_1 - \beta_1)(\varphi_\beta(0) - \varphi_{\hat{\beta}}(0)) \, \mathrm{d}x \\
&\quad \leq -\bar{\gamma}^2 \int_Q \left[1 - \frac{\partial^2 a}{\partial y^2}(x, t, \bar{y})\bar{\varphi} \right] z_{u_{\hat{\beta}} - u_\beta}^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + C_3 \Big\{ \|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)}^2 + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)}^2 \\
&\qquad + \|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)} \big[\|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)} + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)} \big] \\
&\qquad + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)} \|\varphi_\beta - \varphi_{\hat{\beta}}\|_{L^2(Q)} + \|\hat{\beta}_3 - \beta_3\|_{L^2(Q)} \|y_{\hat{\beta}} - y_\beta\|_{L^2(Q)} \\
&\qquad + \|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)} \|\varphi_\beta(0) - \varphi_{\hat{\beta}}(0)\|_{L^2(\Omega)} \Big\}.
\end{aligned}
$$

Now, from the equations satisfied by $y_{\hat{\beta}} - y_\beta$ and $\varphi_{\hat{\beta}} - \varphi_\beta$ we get

$$\|y_{\hat{\beta}} - y_\beta\|_{W(0,T)} \leq C_4 \Big(\|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)} + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)} + \|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)} \Big), \tag{6.27}$$

$$\|\varphi_{\hat{\beta}} - \varphi_\beta\|_{H^1(Q)} \leq C_5 \Big(\|y_{\hat{\beta}} - y_\beta\|_{L^2(Q)} + \|\hat{\beta}_3 - \beta_3\|_{L^2(Q)} \Big). \tag{6.28}$$

Using the continuous embeddings $W(0,T) \subset L^2(Q)$ and $H^1(Q) \subset C([0,T]; L^2(\Omega))$, and the estimates (6.27) and (6.28), we infer

$$\bar{\gamma} \int_Q (\varphi_\beta(t) - \varphi_{\hat{\beta}}(t))(u_{\hat{\beta}}(t) - u_\beta(t)) \, \mathrm{d}x$$

$$\leq -\bar{\gamma}^2 \int_Q \Big[1 - \frac{\partial^2 a}{\partial y^2}(x,t,\bar{y})\bar{\varphi} \Big] z_{u_{\hat{\beta}} - u_\beta}^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ C_6 \Big\{ \|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)} \big[\|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)} + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)} + \|\hat{\beta}_3 - \beta_3\|_{L^2(Q)} \big]$$

$$+ \|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)}^2 + \|\hat{\beta}_2 - \beta_2\|_{L^2(Q)}^2 + \|\hat{\beta}_3 - \beta_3\|_{L^2(Q)}^2 \Big\}.$$

Combining this inequality with (6.26) and using (6.9) we deduce

$$\bar{\gamma}^2 v \|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)}^2 \leq \bar{\gamma}^2 J''(\bar{u})(u_{\hat{\beta}} - u_\beta)^2$$

$$= \bar{\gamma}^2 \Big\{ \int_Q \Big[1 - \frac{\partial^2 a}{\partial y^2}(x,t,\bar{y})\bar{\varphi} \Big] z_{u_{\hat{\beta}} - u_\beta}^2 \, \mathrm{d}x \, \mathrm{d}t + \kappa \|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)}^2 \Big\}$$

$$\leq C_7 \Big\{ \|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)} \Big(\|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)} + \sum_{j=2}^4 \|\hat{\beta}_j - \beta_j\|_{L^2(Q)} \Big)$$

$$+ \|\hat{\beta}_1 - \beta_1\|_{L^2(\Omega)}^2 + \sum_{j=2}^3 \|\hat{\beta}_j - \beta_j\|_{L^2(Q)}^2 \Big\}.$$

This yields

$$\|u_{\hat{\beta}} - u_\beta\|_{L^2(Q)} \leq C_8 \|\hat{\beta} - \beta\|_Z. \tag{6.29}$$

Using (6.27) and (6.29) it follows that

$$\|y_{\hat{\beta}} - y_\beta\|_{W(0,T)} \leq C_9 \|\hat{\beta} - \beta\|_Z. \tag{6.30}$$

Now, (6.28) and (6.30) lead to

$$\|\varphi_{\hat{\beta}} - \varphi_\beta\|_{H^1(Q)} + \|\varphi_{\hat{\beta}} - \varphi_\beta\|_{C(\bar{Q})} \leq C_{10} \|\hat{\beta} - \beta\|_Z. \tag{6.31}$$

Getting back to (6.26), and using (6.31), we get

$$\|u_{\hat{\beta}} - u_{\beta}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} \leq C_{11}\|\hat{\beta} - \beta\|_Z. \tag{6.32}$$

Using this in the equation satisfied by $y_{\hat{\beta}} - y_{\beta}$ we also obtain

$$\|y_{\hat{\beta}} - y_{\beta}\|_{C(\bar{Q})} \leq C_{12}\|\hat{\beta} - \beta\|_Z. \tag{6.33}$$

Now, (6.30)–(6.33) imply (6.18). Hence, we apply [10,Theorem 2.4] to deduce the existence of $\bar{\varepsilon} \in (0, \varepsilon]$ and $\tilde{\rho} \in (0, \bar{\rho}]$ such that for every $\gamma \in (\bar{\gamma} - \bar{\varepsilon}, \bar{\gamma} + \bar{\varepsilon})$ the system (6.12)–(6.15) has a solution (y, φ, u) with u in the interior of the ball $B_{\tilde{\rho}}(\bar{u})$ satisfying (6.16). Since $\bar{\varepsilon} \leq \epsilon$ and $\tilde{\rho} \leq \bar{\rho}$, we know that J_{γ} is strictly convex on $B_{\tilde{\rho}}(\bar{u})$, hence u is the unique local minimizer of (Q_{γ}) in this ball. Moreover, $u_{\gamma} = \gamma u$ belongs to the interior of the ball $B_{\rho}(u_{\bar{\gamma}})$ and u_{γ} is the unique local minimizer of (P_{γ}) in $B_{\rho}(u_{\bar{\gamma}})$. Moreover, from (6.16) we infer

$$\begin{aligned}
\|u_{\gamma} - u_{\bar{\gamma}}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} &\leq \gamma\|u - \bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} + |\gamma - \bar{\gamma}|\|\bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} \\
&< (\bar{\gamma} + \bar{\varepsilon})\lambda|\gamma - \bar{\gamma}| + |\gamma - \bar{\gamma}|\|\bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))} = L|\gamma - \bar{\gamma}|
\end{aligned}$$

for $L = (\bar{\gamma} + \bar{\varepsilon})\lambda + \|\bar{u}\|_{L^{\hat{r}}(0,T;L^2(\Omega))}$. This ends the proof. □

References

1. Casas, E., Chrysafinos, K.: Analysis of the velocity tracking control problem for the 3D evolutionary Navier–Stokes equations. SIAM J. Control. Optim. **54**(1), 99–128 (2016)
2. Casas, E., Clason, C., Kunisch, K.: Parabolic control problems in measure spaces with sparse solutions. SIAM J. Control. Optim. **51**(1), 28–63 (2013)
3. Casas, E., Herzog, R., Wachsmuth, G.: Analysis of spatio-temporally sparse optimal control problems of semilinear parabolic equations. ESAIM Control Optim. Calc. Var. **23**, 263–295 (2017)
4. Casas, E., Kunisch, K.: Parabolic control problems in space-time measure spaces. ESAIM Control Optim. Calc. Var. **22**(2), 355–370 (2016)
5. Casas, E., Kunisch, K.: Stabilization by sparse controls for a class of semilinear parabolic equations. SIAM J. Control. Optim. **55**(1), 512–532 (2017)
6. Casas, E., Kunisch, K.: Optimal control of the 2d evolutionary Navier–Stokes equations with measure valued controls. SIAM J. Control. Optim. **59**(3), 2223–2246 (2021)
7. Casas, E., Mateos, M., Rösch, A.: Approximation of sparse parabolic control problems. Math. Control Relat. Fields **7**(3), 393–417 (2017)

8. Casas, E., Ryll, C., Tröltzsch, F.: Second order and stability analysis for optimal sparse control of the FitzHugh–Nagumo equation. SIAM J. Control. Optim. **53**(4), 2168–2202 (2015)
9. Casas, E., Vexler, B., Zuazua, E.: Sparse initial data identification for parabolic PDE and its finite element approximations. Math. Control Relat. Fields **5**(3), 377–399 (2015)
10. Dontchev, A.L.: Implicit function theorems for generalized equations. Math. Program. **70**, 91–106 (1995)
11. Garcke, H., Lam, K.F., Signori, A.: Sparse optimal control of a phase field tumor model with mechanical effects. SIAM J. Control. Optim. **52**(2), 1555–1580 (2021)
12. Gugat, M., Mateos, M., Tröltzsch, F.: Exponential stability for the Schlögl system by Pyragas feedback. Vietnam J. Math. **48**, 769–790 (2020)
13. Gunzburger, M., Manservisi, S.: The velocity tracking problem for Navier–Stokes flows with bounded distributed control. SIAM J. Control. Optim. **37**(6), 1913–1945 (1999)
14. Herzog, R., Stadler, G., Wachsmuth, G.: Directional sparsity in optimal control of partial differential equations. SIAM J. Control. Optim. **50**(2), 943–963 (2012)
15. Kunisch, K., Pieper, K., Vexler, B.: Measure valued directional sparsity for parabolic optimal control problems. SIAM J. Control. Optim. **52**(5), 3078–3108 (2014)
16. Kunisch, K., Trautmann, P., Vexler, B.: Optimal control of the undamped linear wave equation with measure valued controls. SIAM J. Control. Optim. **54**(3), 1212–1244 (2016)
17. Ladyzhenskaya, O., Solonnikov, V., Ural'tseva, N.: Linear and Quasilinear Equations of Parabolic Type. American Mathematical Society, Providence (1988)
18. Leykekhman, D., Vexler, B., Walter, D.: Numerical analysis of sparse initial data identification for parabolic problems. ESAIM Math. Model. Numer. Anal. **54**(4), 1139–1180 (2020)
19. Robinson, S.M.: An implicit-function theorem for a class of nonsmooth functions. Math. Oper. Res. **16**(2), 292–309 (1991)
20. Tröltzsch, F.: Optimal Control of Partial Differential Equations: Theory, Methods and Applications, Graduate Studies in Mathematics, vol. 112. American Mathematical Society, Philadelphia (2010)