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State error estimates for the numerical approximation of sparse distributed control problems in the absence of Tikhonov regularization

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Abstract In this paper, we analyze optimal control problems of semilinear elliptic equations, where the controls are distributed. Box constraints for the controls are imposed and the cost functional does not involve the control itself, except possibly for a non-differentiable sparsity-promoting term. Under appropriate second order sufficient optimality conditions, first we estimate the difference between the discrete and continuous optimal states. Next, under an additional assumption on the optimal adjoint state, we prove error estimates for the controls and improve the estimates for the states.

Keywords optimal control · bang-bang controls · semilinear elliptic equations · optimality conditions · error estimates

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1 Introduction

In this paper, we continue the investigation started in [9,10] about the numerical approximation of problems without Tikhonov regularization term. In the first work, we provide second order conditions for problems with distributed controls and a term in the cost functional that promotes sparsity of the solutions. In the second one, appropriate second order sufficient conditions and error estimates for

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the numerical approximation are provided for optimal control problems governed by semilinear parabolic equations when the control acts only in time.

In [10], both the state and the adjoint state equation are discretized using continuous piecewise linear functions in space and piecewise constant functions in time, while controls are discretized by piecewise constant functions. This coincidence in the time discretization of the state, adjoint state and control, and the fact that the L^2 projection onto the space of piecewise constant functions preserves the admissibility of the controls, leads to some simplifications in the proofs, see [10, Eq. (4.4)]. In the current work, those properties do not apply. Nevertheless, we are able to modify the proofs and to obtain the same kind of error estimates, both for piecewise constant and continuous piecewise linear approximations of the control. Moreover, this is carried out including a nondifferentiable term in the functional that promotes the sparsity.

To our best knowledge, there are two references where the discretization of optimal control problems without Tikhonov regularization term and governed by elliptic equations is studied, [14,15]. In the first one, error estimates for the control variable are derived under a structural assumption on the solution, cf. (6.1) for $\gamma = 1$, which assures that the control is bang-bang. As the authors themselves notice, due to the fact that they study a bilinear control problem, this assumption does not hold in 2D or 3D problems if the control acts in the whole domain, so they chose to restrict the control to act on a subdomain. We are able to derive error estimates for the control and also for the state in the case $\gamma < 1$, so that the structural assumption will hold naturally for bang-bang controls in 2D and 3D domains; see Theorem 9. In [15], the authors study the variational discretization of a control problem governed by a linear elliptic equation. Our result for this kind of discretization and problems governed by semilinear elliptic equations, see Remark 7, is comparable with Corollary 3.3 in [15], and slightly generalizes this result, even for the case of linear equations.

We also provide error estimates in the state variable for control problems whose solution is not bang-bang; see Theorem 7. This situation, not studied in [14], is taken into account in [15] for the variational discretization of the problem.

One of the key points to deduce error estimates when the problem is nonconvex, as is the case where the equation is semilinear, is the obtention of appropriate second order sufficient conditions. We obtained such conditions for strong local minimizers in [9]. In that reference we wrote all the details of the proofs to obtain the second order sufficient optimality condition for a problem governed by a parabolic equation, the translation to the elliptic case being immediate. Though, in the paper at hand, we give the details of the numerical analysis for a problem governed by an elliptic equation, the translation to the parabolic case is straightforward.

The plan of this paper is as follows. In the next section, we introduce the problem, formulate the main assumptions and establish some auxiliary results. The first and second order optimality conditions are studied in section 3, although the details of the proof of Lemma 2 are moved to Appendix A. In section 4 we discretize the control problem, and in section 5 we prove convergence of the discretizations and derive error estimates for the states. In section 6 we prove error estimates for the controls and improve the estimates for the states under the additional assumption (6.1). Finally, we present some numerical results in section 7.

2 Main assumptions and auxiliary results

Let us consider a domain $\Omega \subset \mathbb{R}^n$, $n \le 3$, with $C^{1,1}$ boundary Γ. We will study the following control problem

$$\min_{u \in U_{\rm ad}} J(u), \tag{P}$$

where

$$U_{\rm ad} = \{u \in L^\infty(\Omega) : \alpha \le u(x) \le \beta \ \text{ for a.e. } x \in \Omega\}$$

with $-\infty < \alpha < \beta < +\infty$, and, for $\mu \ge 0$,

$$J(u) = \int_\Omega L(x, y_u(x))\, dx\, dt + \mu \int_\Omega |u(x)| dx.$$

Above y_u denotes the state associated to the control u related by the following state equation

$$\begin{cases} Ay_u + f(x, y_u) = u \text{ in } \Omega, \\ \qquad\qquad\quad y_u = 0 \text{ on } \Gamma. \end{cases} \tag{2.1}$$

On the data A, f, and L we make the following assumptions

(A1) A denotes the elliptic operator

$$Ay = -\sum_{i,j=1}^n \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

where the coefficients $a_{i,j} \in C^{0,1}(\bar\Omega)$ satisfy the uniform ellipticity condition

$$\exists \lambda_A > 0 : \lambda_A |\xi|^2 \le \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \ \text{ for all } \xi \in \mathbb{R}^n \text{ and a.a. } x \in \Omega.$$

(A2) We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable satisfying the following properties:

$$\begin{cases} \frac{\partial f}{\partial y}(x, y) \ge 0 \ \forall y \in \mathbb{R},\ f(\cdot, 0) \in L^\infty(\Omega), \\ \forall M > 0\ \exists C_{f,M} > 0 : \left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \le C_{f,M}\ \forall |y| \le M, \\ \forall \rho > 0 \text{ and } \forall M > 0\ \exists \varepsilon > 0 \text{ such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x, y_1) - \dfrac{\partial^2 f}{\partial y^2}(x, y_2)\right| < \rho\ \forall |y_1|, |y_2| \le M \text{ with } \ |y_1 - y_2| < \varepsilon, \end{cases}$$

for almost all $x \in \Omega$.

(A3) $L : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable satisfying the following properties:

$$\begin{cases} L(\cdot,0) \in L^1(\Omega) \\ \forall M > 0\ \exists C_{L,M} \text{ such that } \left|\frac{\partial L}{\partial y}(x,y)\right| + \left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \le C_{L,M}\ \forall |y| \le M \\ \forall \rho > 0 \text{ and } \forall M > 0\ \exists \varepsilon > 0 \text{ such that} \\ \left|\frac{\partial^2 L}{\partial y^2}(x,y_1) - \frac{\partial^2 L}{\partial y^2}(x,y_2)\right| < \rho\ \forall |y_1|,|y_2| \le M \text{ with } |y_1 - y_2| < \varepsilon, \end{cases}$$

for almost all $x \in \Omega$.

Concerning the state equation, we have the following result on existence, uniqueness and regularity of the solution.

Theorem 1 *For every $u \in L^p(\Omega)$ with $p > n/2$ there exists a unique $y_u \in Y := H_0^1(\Omega)) \cap C(\bar\Omega)$ solution of (2.1). Moreover, there exists a constant $T_p > 0$ independent of u such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar\Omega)} \le T_p(\|u\|_{L^p(\Omega)} + \|f(\cdot,0)\|_{L^\infty(\Omega)}).$$

If $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$, then the strong convergence

$$\|y_{u_k} - y_u\|_{C(\bar\Omega)} + \|y_{u_k} - y_u\|_{H_0^1(\Omega)} \to 0$$

holds. If, further, $u \in L^\infty(\Omega)$ we have that $y_u \in W^{2,p}(\Omega)$ for all $p < \infty$ and

$$\|y_u\|_{W^{2,p}(\Omega)} \le M_0 p\Big(\|u\|_{L^\infty(\Omega)} + \|f(\cdot,0)\|_{L^\infty(\Omega)}\Big) \tag{2.2}$$

holds for a constant M_0 independent of u and p.

This is a well known result. See, for instance, [4]. The continuity property follows directly from [12, Theorem 2.2] taking into account that for $p > n/2$, $L^p(\Omega)$ is compactly imbedded in $W^{-1,q}(\Omega)$ for $q = pn/(n-p) > n$ if $p < n$ and all $q < +\infty$ if $p \ge n$. The regularity follows from [16, Theorem 9.9] and (2.2) can be deduced from [2, Theorem 2.2]. As a consequence of (2.2) and the continuous embedding $H^2(\Omega) \hookrightarrow C(\bar\Omega)$, we can deduce the existence of a constant $M_\infty > 0$ such that

$$\|y_u\|_{C(\bar\Omega)} \le M_\infty\ \forall u \in U_{\rm ad}. \tag{2.3}$$

Given $p > n/2$, let us denote $G : L^p(\Omega) \longrightarrow Y$ the mapping associating to each control the corresponding state $G(u) = y_u$. The next theorem states the differentiability of G, whose proof can be obtained in the standard way by using the implicit function theorem; see e.g. [5, Theorem 1].

Theorem 2 *The control-to-state operator G is of class C^2 and for every $u, v, w \in L^p(\Omega)$, $p > n/2$, we have that $z_v = G'(u)v$ is the solution of*

$$\begin{cases} Az + \frac{\partial f}{\partial y}(x,y_u)z = v \text{ in } \Omega, \\ \qquad\qquad z = 0 \text{ on } \Gamma, \end{cases} \tag{2.4}$$

and $z_{v,w} = G''(u)(v,w)$ *solves the equation*

$$\begin{cases} Az + \dfrac{\partial f}{\partial y}(x, y_u)z + \dfrac{\partial^2 f}{\partial y^2}(x, y_u)z_v z_w = 0 \text{ in } \Omega, \\ \hfill z = 0 \text{ on } \Gamma. \end{cases}$$

Lemma 1 *Let* $u, v \in U_{\rm ad}$. *Then, we have*

$$\|y_u - y_v\|_{H_0^1(\Omega)} \le \frac{1}{\lambda_A}\|u - v\|_{H^{-1}(\Omega)}, \tag{2.5}$$

$$\|y_u - y_v\|_{L^2(\Omega)} \le \hat{C}\|u - v\|_{L^1(\Omega)}, \tag{2.6}$$

$$\|y_u - y_v\|_{C(\bar{\Omega})} \le \hat{C}_p\|u - v\|_{W^{-1,p}(\Omega)}, \tag{2.7}$$

for $p > n$ *and some constants* $\hat{C}, \hat{C}_p > 0$ *independent of* u *and* v.

Proof Define $z = y_v - y_u$. Subtracting the equations satisfied by y_v and y_u and using the mean value theorem, we deduce the existence of a measurable function $0 < \theta(x) < 1$ such that, setting $\hat{y} = y_u + \theta(y_v - y_u)$, we have

$$Az + \frac{\partial f}{\partial y}(x, \hat{y})z = v - u \text{ in } \Omega,\ \ z = 0 \text{ on } \Gamma.$$

Inequality (2.5) follows from the standard variational formulation, the ellipticity of A and Assumption (A2).

Let us prove inequality (2.6). From [23, §9] we know that for every $q < n/(n-1)$ there exists $C_q > 0$ such that

$$\|z\|_{W^{1,q}(\Omega)} \le C_q\|u - v\|_{L^1(\Omega)}.$$

Using the Sobolev imbedding $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$ for $q \ge 2n/(n+2)$ and fixing $q = 6/5$ for instance, we have

$$\|z\|_{L^2(\Omega)} \le C\|z\|_{W^{1,\frac{6}{5}}(\Omega)} \le C_{\frac{6}{5}} C\|u - v\|_{L^1(\Omega)}$$

and (2.6) follows for $\hat{C} = CC_{\frac{6}{5}}$.

Inequality (2.7) is a classical result; see, for instance, [16, Theorem 8.30] and [23, Theorem 4.2].

Next, we state the differentiability properties of the objective functional. We decompose J in two summands

$$J(u) = F(u) + \mu j(u) \text{ with } F(u) = \int_\Omega L(x, y_u(x))dx \text{ and } j(u) = \|u\|_{L^1(\Omega)}.$$

To every u, we relate the adjoint state φ_u that satisfies

$$\begin{cases} A^*\varphi_u + \dfrac{\partial f}{\partial y}(x, y_u)\varphi_u = \dfrac{\partial L}{\partial y}(x, y_u) \text{ in } \Omega, \\ \hfill \varphi_u = 0 \text{ on } \Gamma, \end{cases} \tag{2.8}$$

where A^* denotes the adjoint operator of A. Assumption (A3) together with Theorem 1 imply that $\varphi_u \in H_0^1(\Omega) \cap C(\bar\Omega)$ and analogously to (2.2) we have

$$\|\varphi_u\|_{W^{2,p}(\Omega)} \le M_0 p C_{L,M_\infty} \quad \forall u \in U_{\rm ad} \quad \forall p > 2,$$

where M_∞ is introduced in (2.3) and C_{L,M_∞} is introduced in Assumption (A3). Again using Sobolev embeddings, we deduce the existence of a constant $T_\infty > 0$ such that

$$\|\varphi_u\|_{H_0^1(\Omega)} + \|\varphi_u\|_{C(\bar\Omega)} \le T_\infty \quad \forall u \in U_{\rm ad}. \tag{2.9}$$

The next theorem follows from the chain rule, Theorem 2 and assumptions (A2) and (A3).

Theorem 3 *Given $p > n/2$, the functional $F : L^p(\Omega) \longrightarrow \mathbb{R}$ is of class C^2 and for every $u, v, w \in L^p(\Omega)$*

$$\begin{aligned} F'(u)v &= \int_\Omega \varphi_u v \, dx, \\ F''(u)(v,w) &= \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) \right) z_v z_w \, dx, \end{aligned} \tag{2.10}$$

where φ_u is the solution of (2.8).

Remark 1 The functionals $F'(u)$ and $F''(u)$ can be extended to continuous linear and bilinear forms, respectively, in $L^1(\Omega)$. Notice also that assumptions (A2) and (A3), Theorem 1 and (2.9) imply the existence of some $M_2 > 0$ such that

$$|F''(u)(v,w)| \le M_2 \|z_v\|_{L^2(\Omega)} \|z_w\|_{L^2(\Omega)} \quad \forall u \in U_{\rm ad}, \ \forall v, w \in L^1(\Omega).$$

Finally, we notice that, for given $u \in L^1(\Omega)$, if we denote

$\Omega_u^+ = \{x \in \Omega : \ u(x) > 0\}$, $\Omega_u^- = \{x \in \Omega : \ u(x) < 0\}$ and $\Omega_u^0 = \{x \in \Omega : \ u(x) = 0\}$,

then, the directional derivative of j at u is given by

$$j'(u; v) = \int_{\Omega_u^+} v(x) dx - \int_{\Omega_u^-} v(x) dx + \int_{\Omega_u^0} |v(x)| dx \quad \forall v \in L^1(\Omega).$$

As usual $\partial j(u)$ stands for the convex subdifferential of j at u. In the sequel, we will also denote $J'(u; v) = F'(u)v + \mu j'(u; v)$.

3 First and second order optimality conditions

Existence of a global solution of (P) follows in a standard way. Since (P) is not a convex problem, we have to consider local solutions as well. Let us state precisely the different concepts of local solution.

Definition 1 We say that $\bar u$ is an L^p-weak local minimum of (P), $p \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

$$J(\bar u) \le J(u) \quad \forall u \in U_{\rm ad} \text{ with } \|\bar u - u\|_{L^p(\Omega)} \le \varepsilon.$$

We say that $\bar u$ is a strong local minimum if there exists some $\varepsilon > 0$ such that

$$J(\bar u) \le J(u) \quad \forall u \in U_{\rm ad} \text{ with } \|y_{\bar u} - y_u\|_{L^\infty(\Omega)} \le \varepsilon.$$

We say that $\bar u$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar u$.

Strong local optimality implies weak local optimality. For more details about these definitions, see [9, Lemma 2.8]. First order optimality conditions read as follows.

Theorem 4 *Suppose $\bar u$ is a local solution of (P) in any of the senses given in Definition 1. Then,*

$$J'(\bar u; u - \bar u) \ge 0 \ \forall u \in U_{\rm ad}. \tag{3.1}$$

Moreover, there exist $\bar y$ and $\bar\varphi$ in $H_0^1(\Omega) \cap C(\bar\Omega)$ and $\bar\lambda \in \partial j(\bar u)$ such that

$$\begin{cases} A\bar y + f(x, \bar y) = \bar u \ \text{ in } \Omega, \\ \bar y = 0 \ \text{ on } \Gamma, \end{cases} \tag{3.2a}$$

$$\begin{cases} A^*\bar\varphi + \dfrac{\partial f}{\partial y}(x, \bar y)\bar\varphi = \dfrac{\partial L}{\partial y}(x, \bar y) \ \text{ in } \Omega, \\ \bar\varphi = 0 \ \text{ on } \Gamma, \end{cases} \tag{3.2b}$$

$$\int_\Omega (\bar\varphi + \mu\bar\lambda)(u - \bar u)\,dx \ge 0 \ \forall u \in U_{\rm ad}. \tag{3.2c}$$

The proof of (3.1) is classical. The optimality system (3.2a)–(3.2c) follows easily from (3.1), (2.10), and the fact that the convexity of j implies that $j(u) - j(\bar u) \ge j'(\bar u; u - \bar u)$.

From the conditions (3.2a)–(3.2c), the following relations can be deduced; see, e.g., [4]. For $\mu = 0$

$$\begin{cases} \bar\varphi(x) > 0 \Rightarrow \bar u(x) = \alpha, \\ \bar\varphi(x) < 0 \Rightarrow \bar u(x) = \beta. \end{cases} \tag{3.3}$$

and for $\mu > 0$,

$$\begin{cases} |\bar\varphi(x)| < \mu \ \Rightarrow \bar u(x) = 0, \\ \bar\varphi(x) > +\mu \Rightarrow \bar u(x) = \alpha, \\ \bar\varphi(x) < -\mu \Rightarrow \bar u(x) = \beta, \\ \bar\varphi(x) = +\mu \Rightarrow \bar u(x) \le 0, \\ \bar\varphi(x) = -\mu \Rightarrow \bar u(x) \ge 0. \end{cases} \tag{3.4}$$

Notice that, if $\text{meas}(\{x \in \Omega : \big||\bar\varphi(x)| - \mu\big| = 0\}) = 0$, then, for $\mu = 0$, we recover the classical bang-bang structure of the control, while for $\mu > 0$, the control will only take values in $\{\alpha, \beta, 0\}$, being a so-called bang-bang-bang or bang-off-bang control.

Now, we establish the second order optimality conditions. In what follows, $\bar u$ will denote a control of $U_{\rm ad}$ satisfying (3.1), along with the associated state $\bar y$ and adjoint state $\bar\varphi$, solutions respectively of (3.2a) and (3.2b).

We say that a function $v \in L^2(\Omega)$ satisfies the sign condition if

$$v(x) \begin{cases} \geq 0 & \text{if } \bar u(x) = \alpha, \\ \leq 0 & \text{if } \bar u(x) = \beta. \end{cases} \tag{3.5}$$

We define the cone of critical directions

$$C_{\bar u} = \{v \in L^2(\Omega) \text{ satisfying (3.5) and } J'(\bar u; v) = 0\}.$$

Following [4] we know that the following identities hold

$$C_{\bar u} = \{v \in L^2(\Omega) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar\varphi(x)| > 0\} \quad \text{if } \mu = 0, \tag{3.6}$$

$$C_{\bar u} = \left\{ v \in L^2(\Omega) \text{ satisfying (3.5) and } v(x) \begin{cases} \geq 0 & \text{if } \bar\varphi(x) = -\mu \text{ and } \bar u(x) = 0, \\ \leq 0 & \text{if } \bar\varphi(x) = +\mu \text{ and } \bar u(x) = 0, \\ = 0 & \text{if } \Big| |\bar\varphi(x)| - \mu \Big| > 0, \end{cases} \right\} \quad \text{if } \mu > 0. \tag{3.7}$$

It was proved in [6] that $F''(\bar u)v^2 \geq 0 \ \forall v \in C_{\bar u}$ is a second order necessary condition for local optimality of $\bar u$. However, to formulate a second order sufficient condition we need to extend the cone of critical directions; see [9] for a discussion on it. We have two possible extensions of $C_{\bar u}$. First

$$G^\tau_{\bar u} = \{v \in L^2(Q) \text{ satisfying (3.5) and } J'(\bar u; v) \leq \tau \|z_v\|_{L^1(\Omega)}\}.$$

On the other hand, using the characterizations of the cone $C_{\bar u}$ given by (3.6) and (3.7) the following extensions appear in a natural way as well.

$$\text{If } \mu = 0,\ D^\tau_{\bar u} = \{v \in L^2(\Omega) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar\varphi(x)| > \tau\}.$$

$$\text{If } \mu > 0,\ D^\tau_{\bar u} = \left\{ v \in L^2(\Omega) \text{ satisfying (3.5) and } v(x) \begin{cases} \geq 0 & \text{if } \bar\varphi(x) = -\mu \text{ and } \bar u(x) = 0 \\ \leq 0 & \text{if } \bar\varphi(x) = +\mu \text{ and } \bar u(x) = 0 \\ = 0 & \text{if } \Big| |\bar\varphi(x)| - \mu \Big| > \tau \end{cases} \right\}.$$

In [9], it is proved that the cone

$$C^\tau_{\bar u} = D^\tau_{\bar u} \cap G^\tau_{\bar u}$$

is enough to formulate second order sufficient conditions.

Theorem 5 *Let $\bar u \in U_{\rm ad}$ satisfy (3.1) and*

$$\exists \delta > 0 \text{ and } \exists \tau > 0 : F''(\bar u)v^2 \geq \delta \|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar u}, \tag{3.8}$$

where $z_v = G'(\bar u)v$ is the solution of (2.4) for $y_u = \bar y$, the solution of (3.2a). Then, there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$J(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \leq J(u) \ \forall u \in U_{\rm ad} : \ \|y_u - \bar y\|_{L^\infty(\Omega)} < \varepsilon. \tag{3.9}$$

In Section 6, we will also use the following result.

Lemma 2 *Let $\bar u \in U_{\rm ad}$ satisfy (3.1) and (3.8). Then, there exists $\kappa > 0$ such that for all $\rho > 0$ a number $\varepsilon_\rho > 0$ can be found so that*

$$\rho\left[F'(\bar u)(u-\bar u) + \mu j(u) - \mu j(\bar u)\right] + F''(\bar u + \theta(u-\bar u))(u-\bar u)^2 \geq \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \quad (3.10)$$

for all $\theta \in [0,1]$ and all $u \in U_{\rm ad}$ satisfying $\|y_u - \bar y\|_{L^\infty(\Omega)} < \varepsilon_\rho$.

Proof Following the same scheme of proof as in [9, Theorem 3.1], we can show the existence of $\kappa > 0$ such that

$$\rho J'(\bar u; u-\bar u) + F''(\bar u + \theta(u-\bar u))(u-\bar u)^2 \geq \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \quad \forall \theta \in [0,1]. \quad (3.11)$$

The details of the proof of (3.11) are provided in Appendix A; see [10, Remark 3.6] for a similar situation. Now, using the convexity of j, we know that

$$F'(\bar u)(u-\bar u) + \mu j(u) - \mu j(\bar u) \geq J'(\bar u; u - \bar u)$$

and the proof is complete.

4 Numerical approximation of the control problem (P)

In this section we discretize the control problem (P). To this end, we assume that Ω is convex and consider a quasi-uniform family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar\Omega$, cf. [1, definition (4.4.13)]. We denote $\bar\Omega_h = \cup_{T\in\mathcal{T}_h} T$. We assume that every boundary node of Ω_h is a point of Γ. Additionally we suppose that there exists a constant $C_\Gamma > 0$ independent of h such that the distance $d_\Gamma(x) \leq C_\Gamma h^2$ for every $x \in \Gamma_h = \partial\Omega_h$, which is always satisfied if $n = 2$ and Γ is of class C^2; see, for instance, [20, Section 5.2]. Under this assumption we have that there exists a constant $C_\Omega > 0$ independent of h such that

$$|\Omega \setminus \Omega_h| \leq C_\Omega h^2, \quad (4.1)$$

where $|\cdot|$ denotes the Lebesgue measure.

Now we consider the finite dimensional space

$$Y_h = \{z_h \in C(\bar\Omega) : \ z_{h|T} \in P_1(T) \ \forall T \in \mathcal{T}_h \text{ and } z_h \equiv 0 \text{ on } \Omega\setminus\Omega_h\}.$$

Along this paper $P_i(T)$ denotes the polynomials in T of degree at most i.

For every $u \in L^2(\Omega)$, we define its associated discrete state as the unique element $y_h(u) \in Y_h$ satisfying

$$a(y_h, z_h) + \int_{\Omega_h} f(x, y_h) z_h\,dxdt = \int_{\Omega_h} u z_h\,dx \quad \forall z_h \in Y_h, \quad (4.2)$$

where

$$a(y,z) = \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_{x_i} y \partial_{x_j} z\,dx \quad \forall y, z \in H^1(\Omega).$$

The proof of the existence and uniqueness of a solution for (4.2) is standard; see e.g. [7].

Lemma 3 *there exists a constant $c > 0$, which depends on the data of the problem but is independent of the discretization parameter h, such that for every $u \in U_{\rm ad}$*

$$\|y_h(u) - y_u\|_{L^2(\Omega)} \le ch^2, \tag{4.3}$$

$$\|y_h(u) - y_u\|_{L^\infty(\Omega)} \le ch^2|\log h|^2. \tag{4.4}$$

Proof Estimate (4.3) follows from [7, Lemma 4] and (2.2).

Let us prove (4.4). First of all, notice that from [7, Theorem 1] and (2.3), it is straightforward to deduce that for $h > 0$ small enough

$$\|y_h(u)\|_{L^\infty(\Omega)} \le M_\infty + 1. \tag{4.5}$$

Consider $y^h \in H_0^1(\Omega)$ the unique solution of the linear equation

$$\begin{cases} Ay^h = u - f(x, y_h(u)) \text{ in } \Omega, \\ \quad y^h = 0 \text{ on } \Gamma. \end{cases}$$

Using Theorem 1, Assumption (A2) and (4.5), we know that $y^h \in W^{2,p}(\Omega)$ for all $p < \infty$ and

$$\begin{aligned} \|y^h\|_{W^{2,p}(\Omega)} \le& M_0\, p \Big(\max\{|\alpha|, |\beta|\} + \|f(\cdot, 0)\|_{L^\infty(\Omega)} + C_{f,M_\infty+1}(M_\infty + 1) \Big) \\ =& C_0 p. \end{aligned} \tag{4.6}$$

The difference $y_u - y^h$ satisfies

$$\begin{cases} A(y_u - y^h) = -f(x, y_u) + f(x, y_h(u)) \text{ in } \Omega, \\ \quad y_u - y^h = 0 \text{ on } \Gamma. \end{cases}$$

From the results in Stampacchia [23], Assumption (A2), (2.3), (4.5) and estimate (4.3), we have

$$\begin{aligned} \|y_u - y^h\|_{L^\infty(\Omega)} \le& C_1 \|f(x, y_u) - f(x, y_h(u))\|_{L^2(\Omega)} \\ \le& C_1 C_{f,M_\infty+1} \|y_u - y_h(u)\|_{L^2(\Omega)} \le C_2 h^2. \end{aligned} \tag{4.7}$$

We notice now that $y_h(u)$ satisfies

$$a(y_h, z_h) = \int_{\Omega_h} \big(u - f(x, y_h(u))\big) z_h dx \quad \forall z_h \in Y_h$$

and hence it is the finite element approximation of y^h. Define $I_h y^h$ the continuous piecewise linear Lagrange interpolation of y^h. Applying [22, Theorem 2.1] (see also [19]), the interpolation error (see e.g. [1, Equation (4.4.22)]), and (4.6) we deduce that for all $p < \infty$

$$\begin{aligned} \|y^h - y_h(u)\|_{L^\infty(\Omega)} \le& C_3 h|\log h| \|y^h - I_h y^h\|_{W^{1,\infty}(\Omega)} \\ \le& C_4 h|\log h| h^{1-2/p} \|y^h\|_{W^{2,p}(\Omega)} \\ \le& C_0 C_4 h^{2-2/p} |\log h| p. \end{aligned}$$

Taking now $p = |\log h|$ and using that $h^{-1/|\log h|} = e$ for $h < 1$ we have

$$\|y^h - y_h(u)\| \le C_5 h^2 |\log h|^2.$$

Finally, (4.4) follows from (4.7) and the previous inequality.

The control is discretized using piecewise constant functions, namely

$$U_h = \{u_h \in L^\infty(\Omega_h) : \ u_{h|T} \in P_0(T) \ \forall T \in \mathcal{T}_h\},$$

Since the elements u_h of U_h are not defined on all Ω, we have to specify what we mean when we say that $u_h \rightharpoonup u$ weakly* in $L^\infty(\Omega)$. It means that

$$\int_{\Omega_h} u_h v \, dx \to \int_\Omega uv \, dx \ \forall v \in L^1(\Omega).$$

Due to Assumption (4.1), this is the same as saying that the extension to $\Omega \setminus \Omega_h$ of u_h by any fixed function in $L^\infty(\Omega)$ converges weakly* in $L^\infty(\Omega)$.

We denote π_h the linear projection onto U_h in the $L^2(\Omega_h)$ sense:

$$(\pi_h u)_{|T} = \frac{1}{|T|} \int_T u \, dx, \quad \forall T \in \mathcal{T}_h.$$

Abusing notation, we will sometimes write $\pi_h u = u$ in $\Omega \setminus \Omega_h$. With this notation, we know that $\pi_h u \to u$ in $L^2(\Omega)$.

We will denote

$$j_h(u) = \int_{\Omega_h} |u(x)| dx.$$

The following approximation results will be useful.

Lemma 4 *For all $u \in U_{ad}$,*

$$j_h(\pi_h u) \le j(u) \ \forall h > 0 \tag{4.8}$$

and

$$\lim_{h\to 0} j_h(\pi_h u) = j(u). \tag{4.9}$$

Moreover, given $1 < p < \infty$ there exists a constant $K_p > 0$ that depends on p and Ω but it is independent of h such that

$$\|u - \pi_h u\|_{W^{-1,p}(\Omega_h)} \le K_p h \|u\|_{L^p(\Omega)} \quad \forall u \in L^p(\Omega). \tag{4.10}$$

Proof The first estimate follows from the definition of π_h. The convergence $j(\pi_h u) \to j(u)$ is an immediate consequence of the well known convergence property $\|u - \pi_h u\|_{L^2(\Omega_h)} \to 0$ as $h \to 0$.

Set $p' = p/(p-1)$ the conjugate exponent of p and consider $V_h = \{v \in W_0^{1,p'}(\Omega_h) : \|v\|_{W_0^{1,p'}(\Omega_h)} = 1\}$. By definition of dual norm, the orthogonality property of π_h, [1, Lemma (4.3.8)] (Bramble-Hilbert's Lemma) and the quasi-uniformity of the family of triangulations, we have

$$\begin{aligned}
\|u - \pi_h u\|_{W^{-1,p}(\Omega_h)} &= \sup_{v\in V_h} \langle u - \pi_h u, v\rangle_{W^{-1,p}(\Omega_h), W_0^{1,p'}(\Omega_h)} \\
&= \sup_{v\in V_h} \int_{\Omega_h} (u - \pi_h u) v dx = \sup_{v\in V_h} \int_{\Omega_h} (u - \pi_h u)(v - \pi_h v) dx \\
&= \sup_{v\in V_h} \int_{\Omega_h} u(v - \pi_h v) dx \le \sup_{v\in V_h} \|u\|_{L^p(\Omega)} \|v - \pi_h v\|_{L^{p'}(\Omega_h)} \\
&\le \sup_{v\in V_h} K_p \|v\|_{W_0^{1,p'}(\Omega_h)} \|u\|_{L^p(\Omega)} h = K_p \|u\|_{L^p(\Omega)} h.
\end{aligned}$$

Finally, we define

$$F_h(u) = \int_{\Omega_h} L(x, y_h(u)(x))\, dx, \text{ and } J_h(u) = F_h(u) + \mu j_h(u).$$

and formulate the discrete problem as

$$\min_{u_h \in U_{h,\mathrm{ad}}} J_h(u_h), \tag{P$_h$}$$

where $U_{h,\mathrm{ad}} = U_h \cap U_{\mathrm{ad}}$. Since this set is compact and nonempty, existence of a global solution of (P$_h$) follows immediately from the continuity of J_h in U_h.

For every $u \in L^1(\Omega)$, we define the related discrete adjoint state $\varphi_h(u) \in Y_h$ as the unique solution of

$$a(z_h, \varphi_h) + \int_{\Omega_h} \frac{\partial f}{\partial y}(x, y_h(u))\varphi_h z_h dx = \int_{\Omega_h} \frac{\partial L}{\partial y}(x, y_h(u)) z_h dx \quad \forall z_h \in Y_h. \tag{4.11}$$

With this notation, we have that for every $u, v \in L^1(\Omega)$

$$F_h'(u)v = \int_{\Omega_h} \varphi_h(u) v dx.$$

If $\bar u_h$ is a local solution of (P$_h$), then

$$F_h'(\bar u_h)(u_h - \bar u_h) + \mu j_h(u_h) - \mu j_h(\bar u_h) \ge J_h'(\bar u_h; u_h - \bar u_h) \ge 0 \quad \forall u_h \in U_{h,\mathrm{ad}}. \tag{4.12}$$

5 Error estimates for the optimal states

In this section, we first analyze the convergence of the approximations (P$_h$) of (P) in a sense to be precised below. Then we prove error estimates for the difference between the discrete and the continuous optimal states.

Before stating the convergence theorems, we establish an auxiliary result.

Lemma 5 *Consider $\{u_h\}_{h>0} \subset U_{\mathrm{ad}}$ such that $u_h \stackrel{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ as $h \to 0$. Then,*

$$\lim_{h\to 0} F_h(u_h) = F(u), \tag{5.1}$$

$$j(u) \le \liminf_{h\to 0} j_h(u_h). \tag{5.2}$$

Proof We first write

$$|F_h(u_h) - F(u)| \le |F_h(u_h) - F(u_h)| + |F(u_h) - F(u)|.$$

The convergence to zero of the second term follows from Assumption (A3) and Theorem 1.

For the first term, by the mean value theorem, we know that there exists a measurable function $0 < \theta_h(x) < 1$ such that, if we name $\hat y_h = y_h(u_h) + \theta_h(y_{u_h} -$

$y_h(u_h)$), then, using Theorem 1, assumptions (A3) and (4.1) together with (2.3), we obtain

$$
\begin{aligned}
&|F(u_h) - F_h(u_h)| \\
&=\Big| \int_{\Omega_h} \Big(L(x, y_{u_h}(x)) - L(x, y_h(u_h)(x)) \Big)\, dx + \int_{\Omega\setminus\Omega_h} L(x, y_{u_h}(x))\, dx \Big| \\
&\leq \int_{\Omega_h} \Big| \frac{\partial L}{\partial y}(x, \hat{y}_h(x)) \big(y_{u_h}(x) - y_h(u_h)(x) \big) \Big|\, dx \\
&\quad + \int_{\Omega\setminus\Omega_h} \left(|L(x,0)| + M_\infty C_{L,M_\infty} \right)\, dx
\end{aligned}
$$

The second summand converges to zero due to assumptions (A3) and (4.1). For, the first one, using the finite element estimate (4.3) we obtain

$$
\begin{aligned}
&\int_{\Omega_h} \Big| \frac{\partial L}{\partial y}(x, \hat{y}_h(x)) \big(y_{u_h}(x) - y_h(u_h)(x) \big) \Big|\, dx \\
&\leq \Big\| \frac{\partial L}{\partial y}(\cdot, \hat{y}_h) \Big\|_{L^2(\Omega_h)} \| y_{u_h} - y_h(u_h) \|_{L^2(\Omega_h)} \\
&\leq C_{L,M_\infty} |\Omega|^{\frac{1}{2}} c h^2,
\end{aligned}
$$

which also converges to zero.

To prove (5.2), we notice that if we define $u^h = u_h$ in Ω_h and $u^h = 0$ in $\Omega \setminus \Omega_h$, we have, due to Assumption (4.1), that $u^h \stackrel{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ and also $j_h(u_h) = j(u^h)$. Since $j(\cdot)$ is convex and continuous, it is weakly lower semicontinuous, so

$$
j(u) \leq \liminf_{h\to 0} j(u^h) = \liminf_{h\to 0} j_h(u_h)
$$

and inequality (5.2) follows.

Theorem 6 *Let $\bar{u}$ be a strict strong local minimizer for (P), i.e.,*

$$
\exists \rho > 0 : J(\bar{u}) < J(u) \quad \forall u \in U_{\text{ad}} \setminus \{\bar{u}\} : \|y_u - \bar{y}\|_{L^\infty(\Omega)} \leq \rho. \tag{5.3}
$$

Then, there exists a sequence $\{\bar{u}_h\}_h$ of local minimizers of (P_h) such that $\bar{u}_h \rightharpoonup \bar{u}$ weakly in $L^\infty(\Omega)$. Moreover, there exists $h_0 > 0$ such that*

$$
J_h(\bar{u}_h) \leq J_h(u_h)\ \forall u_h \in U_{h,\text{ad}} \text{ with } \|y_h(u_h) - \bar{y}_h\|_{L^\infty(\Omega_h)} \leq \frac{\rho}{2},\ \forall h \leq h_0. \tag{5.4}
$$

Conversely, let $\{\bar{u}_h\}_h$ be a sequence of local minimizers of (P_h) satisfying (5.4) for some given $\rho > 0$ and such that $\bar{u}_h \stackrel{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$. Then $\bar{u}$ is a strong local solution of (P) satisfying

$$
J(\bar{u}) \leq J(u) \quad \forall u \in U_{\text{ad}} : \|y_u - \bar{y}\|_{L^\infty(\Omega)} < \frac{\rho}{2}. \tag{5.5}
$$

Proof Part I: Consider the set

$$
V_{ad,h,\rho} = \{u_h \in U_{h,\text{ad}} :\ \|y_h(u_h) - \bar{y}\|_{L^\infty(\Omega)} \leq \rho\}.
$$

From (2.7) and (4.10), we have that there exists $h_1 > 0$ such that

$$\|\bar{y} - y_{\pi_h \bar{u}}\|_{L^\infty(\Omega)} \le \frac{\rho}{2} \quad \forall h \le h_1.$$

From the finite element error estimate (4.4) we deduce the existence of h_2 such that

$$\|y_{\pi_h \bar{u}} - y_h(\pi_h \bar{u})\|_{L^\infty(\Omega)} \le \frac{\rho}{2} \quad \forall h \le h_2.$$

So we have that for $0 < h \le h_0 := \min\{h_1, h_2\}$

$$\|\bar{y} - y_h(\pi_h \bar{u})\|_{L^\infty(\Omega)} \le \rho,$$

and we conclude that $\pi_h \bar{u} \in V_{ad,h,\rho}$ $\forall h \le h_0$. Hence, $V_{ad,h,\rho}$ is compact and nonempty. Therefore the problem

$$\min_{u_h \in V_{ad,h,\rho}} J_h(u_h)$$

has a solution $\bar{u}_h$ for every $h \le h_0$. We can extract a subsequence, denoted in the same way, such that $\bar{u}_h \stackrel{*}{\rightharpoonup} \tilde{u}$ in $L^\infty(\Omega)$. Since $U_{h,\rm ad} \subset U_{\rm ad}$ and $U_{\rm ad}$ is weakly* closed in $L^\infty(\Omega)$, we deduce that $\tilde{u} \in U_{\rm ad}$. We also have that $\bar{y}_h = y_h(\bar{u}_h) \to y_{\tilde{u}}$ in $L^\infty(\Omega)$. To check this we write

$$\|\bar{y}_h - y_{\tilde{u}}\|_{L^\infty(\Omega)} \le \|y_h(\bar{u}_h) - y_{\bar{u}_h}\|_{L^\infty(\Omega)} + \|y_{\bar{u}_h} - y_{\tilde{u}}\|_{L^\infty(\Omega)}. \tag{5.6}$$

From (4.4) and Theorem 1 we infer that both terms converge to 0. Since $\bar{u}_h \in V_{ad,h,\rho}$, we have that

$$\begin{aligned}\|y_{\tilde{u}} - \bar{y}\|_{L^\infty(\Omega)} &\le \|y_{\tilde{u}} - \bar{y}_h\|_{L^\infty(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \\ &\le \|y_{\tilde{u}} - \bar{y}_h\|_{L^\infty(\Omega)} + \rho \to \rho \quad \text{ as } h \to 0.\end{aligned}$$

Hence $\|y_{\tilde{u}} - \bar{y}\|_{L^\infty(\Omega)} \le \rho$ holds. Now we use Lemma 5, the optimality of $\bar{u}_h$ and Lemma 4 to infer that

$$J(\tilde{u}) \le \liminf J_h(\bar{u}_h) \le \limsup J_h(\bar{u}_h) \le \limsup J_h(\pi_h \bar{u}) = J(\bar{u}).$$

Due to the strict strong local optimality of $\bar{u}$, cf. (5.3), this is possible only if $\tilde{u} = \bar{u}$, and so (5.6) implies that $\|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \to 0$ as $h \to 0$. Let us take h_0 such that $\|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} < \rho/2$ for any $h \le h_0$. Then for any $u_h \in U_{h,\rm ad}$ such that $\|y_h(u_h) - \bar{y}_h\|_{L^\infty(\Omega)} \le \rho/2$, we get

$$\|y_h(u_h) - \bar{y}\|_{L^\infty(\Omega)} \le \|y_h(u_h) - \bar{y}_h\|_{L^\infty(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} < \rho \;\; \forall h \le h_0.$$

Then $u_h \in V_{ad,h,\rho}$ and, hence, $J_h(\bar{u}_h) \le J_h(u_h)$ for every $h \le h_0$, which proves the first part of the theorem.

Part II: We denote, as above, $\bar{y}_h$ and $\bar{y}$ the discrete and continuous states associated with $\bar{u}_h$ and $\bar{u}$, respectively. Let us take an arbitrary element $u \in U_{\rm ad}$ such that $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \rho/2$. We have to prove that $J(\bar{u}) \le J(u)$.

First, we observe that, proceeding as in (5.6), $\|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \to 0$ as $h \to 0$.

Next, we consider the discrete controls $\pi_h u$. It is obvious that $\pi_h u \in U_{h,\rm ad}$ and, we have, from (4.4), (2.7) and (4.10), that

$$\|y_h(\pi_h u) - y_u\|_{L^\infty(\Omega)} \le \|y_h(\pi_h u) - y_{\pi_h u}\|_{L^\infty(\Omega)} + \|y_{\pi_h u} - y_u\|_{L^\infty(\Omega)} \to 0$$

as $h \to 0$. Therefore, we get

$$\|y_h(\pi_h u) - \bar{y}_h\|_{L^\infty(\Omega)} \to \|y_u - \bar{y}\|_{L^\infty(\Omega)} < \frac{\rho}{2} \quad \text{as } h \to 0.$$

Hence, there exists h_3 with $h_3 \le h_0$ such that

$$\|y_h(\pi_h u) - \bar{y}_h\|_{L^\infty(\Omega)} < \frac{\rho}{2} \quad \forall h \le h_3.$$

Thus, from the convergence $\bar{u}_h \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$, Lemma 5 and the local optimality of $\bar{u}_h$ stated in (5.4), we infer

$$J(\bar{u}) \le \liminf_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(\pi_h u) = J(u),$$

which concludes the second part of proof.

Remark 2 For any fixed $\bar{u}$ strict strong local minimizer of (P), and any sequence $\{\bar{u}_h\}_h$ of local minimizers of (P_h) converging weakly* in $L^\infty(\Omega)$ to $\bar{u}$, we will define without ambiguity $\bar{u}_h = \bar{u}$ in $\Omega \setminus \Omega_h$. This is consistent with our abuse of notation for $\pi_h \bar{u}$ and our definition of weak* convergence in $L^\infty(\Omega)$. Also we can define without problem $j(\bar{u}_h)$ and using (4.12), we can write

$$F_h'(\bar{u}_h)(\pi_h \bar{u} - \bar{u}_h) + \mu j(\pi_h \bar{u}) - \mu j(\bar{u}_h) \ge 0.$$

Remark 3 Let us observe that if $\{\bar{u}_h\}_h$ is a sequence of global minimizers of (P_h), then there exist subsequences converging to elements $\bar{u}$. Any of these controls $\bar{u}$ is a global minimizer of (P). This is an immediate consequence of the second part of Theorem 6. Indeed, it is enough to take ρ sufficiently large.

To obtain error estimates, we will assume in what follows that, besides (A3), there exists a constant $C_{L,0} > 0$ and some $\tilde{h} > 0$ such that for all $h < \tilde{h}$

$$|L(x, 0)| \le C_{L,0} \quad \text{for a.e. } x \in \Omega \setminus \Omega_h. \tag{5.7}$$

Assuming the second order optimality conditions we can prove some error estimates for the difference between the continuous and discrete optimal states.

Theorem 7 *Let $\bar{u}$ be a local solution of* (P) *satisfying the second order sufficient conditions* (3.8). *Let $\{\bar{u}_h\}_h$ be a sequence of local minima of* (P_h) *such that* (5.4) *holds and $\bar{u}_h \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$. Then, there exists a constant $C > 0$ independent of h such that*

$$\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)} \le C\sqrt{h},$$

where $\bar{y}$ is the solution of (3.2b) *and $\bar{y}_h = y_h(\bar{u}_h)$, the solution of* (4.2) *for $\bar{u}_h$.*

Proof By the triangle inequality we have

$$\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)} \le \|\bar{y}_h - y_{\bar{u}_h}\|_{L^2(\Omega)} + \|y_{\bar{u}_h} - \bar{y}\|_{L^2(\Omega)}.$$

The first term in the right hand side is of order $O(h^2)$; see (4.3). We just need to study the second term.

From Theorem 1 we know that $y_{\bar u_h} \to \bar y$ strongly in $L^\infty(\Omega)$, and hence there exists $h_0 > 0$ such that for all $0 < h < h_0$, $\|y_{\bar u_h} - \bar y\|_{L^\infty(\Omega)} < \varepsilon$, where $\varepsilon > 0$ is the one given in (3.9). From Theorem 5, we deduce the existence of $\kappa > 0$ such that

$$\begin{aligned}\frac{\kappa}{2}\|y_{\bar u_h} - \bar y\|^2_{L^2(\Omega)} \leq\ & J(\bar u_h) - J(\bar u)\\ \leq\ & [J(\bar u_h) - J_h(\bar u_h)] + [J_h(\bar u_h) - J_h(\pi_h \bar u)]\\ & + [J_h(\pi_h\bar u) - J(\pi_h \bar u)] + [J(\pi_h\bar u) - J(\bar u)]\\ =\ & \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

Let us estimate the first term. By the mean value theorem, there exists a measurable function $\hat y_h = \bar y_h + \theta(y_{\bar u_h} - \bar y_h)$ with $0 < \theta(x) < 1$ such that

$$\begin{aligned}\mathrm{I} =& J(\bar u_h) - J_h(\bar u_h) = \mu j(\bar u_h) - \mu j_h(\bar u_h)\\ &+ \int_{\Omega\setminus\Omega_h} L(x, y_{\bar u_h}(x))\,dx + \int_{\Omega_h}\big(L(x, y_{\bar u_h}(x)) - L(x, \bar y_h(x))\big)\,dx\\ =& \int_{\Omega\setminus\Omega_h}(\mu|\bar u(x)| + L(x, y_{\bar u_h}(x)))\,dx + \int_{\Omega_h}\frac{\partial L}{\partial y}(x, \hat y_h(x))\big(y_{\bar u_h}(x) - \bar y_h(x)\big)\,dx\\ \leq& (\mu\max\{|\alpha|, |\beta|\} + C_{L,0} + M_\infty C_{L,M_\infty})C_\Omega h^2 + \Big\|\frac{\partial L}{\partial y}(\cdot, \hat y_h)\Big\|_{L^2(\Omega_h)}\|y_{\bar u_h} - \bar y_h\|_{L^2(\Omega_h)}\\ \leq& \Big(\mu\max\{|\alpha|, |\beta|\} + C_{L,0} + M_\infty C_{L,M_\infty})C_\Omega + C_{L,M_\infty}|\Omega|^{\frac12}c\Big)\,h^2,\end{aligned}$$

where we have used assumptions (4.1) together with (5.7) and (A3), Theorem 1, the bound (2.3) and the finite element estimate (4.3). Term III can be estimated exactly in the same way, taking into account that, due to our notation $\pi_h\bar u = \bar u$ in $\Omega\setminus\Omega_h$, and hence

$$\int_{\Omega\setminus\Omega_h}|\pi_h\bar u|dx = \int_{\Omega\setminus\Omega_h}|\bar u|dx \leq \max\{|\alpha|, |\beta|\}C_\Omega h^2$$

thanks to Assumption to (4.1).

Since $\bar u_h$ satisfies (5.4), we have that $\mathrm{II} \leq 0$ for h small enough. Indeed, we can argue as in the second part of the proof of Theorem 6 to deduce the existence of h_1 such that $\pi_h\bar u \in U_{h,\mathrm{ad}}$ and $\|y_h(\pi_h\bar u) - \bar y_h\|_{L^\infty(\Omega)} < \rho/2$ for $h < h_1$.

To estimate term IV, we use again the mean value theorem, together with (2.5), (4.10), our abuse of notation $\pi_h\bar u = \bar u$ in $\Omega\setminus\Omega_h$ and the fact that $\|\pi_h\bar u\|_{L^1(\Omega)} \leq \|\bar u\|_{L^1(\Omega)}$, to obtain

$$\begin{aligned}J(\pi_h\bar u) - J(\bar u) =& \int_\Omega\big(L(x, y_{\pi_h\bar u}(x)) - L(x, \bar y(x))\big)\,dx + \mu j(\pi_h\bar u) - \mu j(\bar u)\\ =& \int_\Omega\frac{\partial L}{\partial y}(x, \hat y_h(x))\big(y_{\pi_h\bar u}(x) - \bar y(x)\big)\,dx + \mu\big(\|\pi_h\bar u\|_{L^1(\Omega)} - \|\bar u\|_{L^1(\Omega)}\big)\\ \leq& \Big\|\frac{\partial L}{\partial y}(\cdot, \cdot, \hat y_h)\Big\|_{L^2(\Omega)}\|y_{\pi_h\bar u} - \bar y\|_{L^2(\Omega)}\\ \leq& \frac{C_{L,M_\infty}|\Omega|^{\frac12}}{\lambda_A}\|\pi_h\bar u - \bar u\|_{H^{-1}(\Omega)} \leq \frac{K_2 C_{L,M_\infty}|\Omega|^{\frac12}}{\lambda_A}h.\end{aligned}$$

Collecting all the estimates, we achieve the desired result.

Remark 4 (Approximation by continuous piecewise linear functions.) If we take

$$U_h = \{v_h \in C(\bar{\Omega}_h) : \ v_{h|T} \in P_1(T) \ \forall T \in \mathcal{T}_h\}$$

and $U_{h,ad} = U_h \cap U_{\rm ad}$, we do not improve the order of convergence. The proof follows the same lines as before, replacing $\pi_h \bar{u}$ by Carstensen's quasi-interpolate; see [3] and [21, Lemma 4.5]. Notice that estimate (4.10) for $p = 2$ is still of order $O(h)$, and although for $p > 2$ the order of convergence is smaller, this case was only used in Theorem 6 to prove convergence, so the proof is still valid.

Remark 5 (Variational discretrization.) If $U_h = L^2(\Omega)$, then the projection is the identity and $\pi_h \bar{u} = \bar{u}$ in Ω. So in the proof of Theorem 7 the term IV disappears and we obtain order $O(h)$. Notice that, unlike the case where the Tikhonov parameter is positive, we cannot express in general the variational optimal control as a function depending on a finite number of parameters.

6 Bang-bang-bang control and control error estimates

In the last section we have used the quadratic growth property of the states (3.9) to prove error estimates between discrete and continuous optimal states. The reader can wonder whether it is possible to get an analogous condition involving a quadratic term for the controls. The answer is negative in general. In [13], the authors prove that if $\bar{u}$ is a local minimizer of (P), which is not bang-bang, then there do not exist $\varepsilon > 0$, $\kappa > 0$, $\gamma > 0$ and $r \geq 1$ such that the inequality

$$J(\bar{u}) + \frac{\kappa}{2}\|u - \bar{u}\|^{\gamma}_{L^r(\Omega)} \leq J(u) \quad \forall u \in U_{\rm ad} : \|u - \bar{u}\|_{L^1(\Omega)} \leq \varepsilon$$

holds. However, if we make a certain structural assumption on the associated adjoint state with $\bar{u}$, which implies the bang-(bang-)bang property of $\bar{u}$, then we can get the desired inequality. Following [13], the next hypothesis will be assumed in the rest of the paper.

$$\exists K > 0, \ \exists \gamma \in (0, 1] : \text{meas}\{x \in \Omega : \Big||\bar{\varphi}(x)| - \mu\Big| \leq \varepsilon\} \leq K\varepsilon^{\gamma}, \ \forall \varepsilon > 0. \qquad (6.1)$$

Notice that a control $\bar{u}$ satisfying first order optimality conditions and (6.1) is a bang-(bang-)bang control. In some papers, the above condition is assumed to be satisfied with $\gamma = 1$. Nevertheless, for dimension $n \geq 2$ and $\mu = 0$, the case $\gamma = 1$ may not hold in certain common situations; see the explanation after [18, Eq. (3.5)] for an example in a polygonal domain.

Assuming that $\mu = 0$, if $\bar{u}$ satisfies the first order optimality conditions and (6.1), then it was proved in [13, Theorem 2.5] that $\bar{u}$ is a local minimum in the $L^\infty(\Omega)$ sense. However, this is not enough to deduce error estimates for the controls. We are going to show that the second order condition (3.8) along with the structural assumption (6.1) are sufficient to obtain some error estimates for the controls. To this end, we first establish the following lemma.

Lemma 6 *Let $\bar u \in U_{\rm ad}$ satisfy the first order condition (3.1) and the structural assumption (6.1). Then*

$$F'(\bar u)(u-\bar u)+\mu j(u)-\mu j(\bar u)\ge \nu\|u-\bar u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \quad \forall u\in U_{\rm ad} \tag{6.2}$$

holds, where $\nu = \frac{1}{2}\Big(2\|\beta-\alpha\|_{L^\infty(\Omega)}\Big)^{-1/\gamma}$.

Proof The inequality $F'(\bar u)(u-\bar u)+\mu j('\bar u;u-\bar u)\ge \nu\|u-\bar u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$ was proved in [17, Lemma 6.3]. Then, it is enough to use that $j(u)-j(\bar u)\ge j('\bar u;u-\bar u)$ to obtain (6.2).

Theorem 8 *Let $\bar u$ be a solution of (P) satisfying the second order sufficient condition (3.8) and the structural assumption (6.1). Then there exist $\varepsilon>0$ and $\kappa>0$ such that*

$$J(\bar u)+\frac{\nu}{2}\|u-\bar u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}+\frac{\kappa}{4}\|y_u-\bar y\|_{L^2(\Omega)}^2\le J(u)\ \forall u\in U_{\rm ad}:\ \|y_u-\bar y\|_{L^\infty(\Omega)}<\varepsilon.$$

Proof First, we make a Taylor expansion and to use (6.2) as follows

$$\begin{aligned}J(u)=&F(u)+\mu j(u) \qquad (6.3)\\ =&F(\bar u)+\mu j(\bar u)+F'(\bar u)(u-\bar u)+\mu j(u)-\mu j(\bar u)+\frac12 F''(u_\theta)(u-\bar u)^2\\ =&J(\bar u)+\frac12\left[F'(\bar u)(u-\bar u)+\mu j(u)-\mu j(\bar u)\right]\\ &+\frac12\left[F'(\bar u)(u-\bar u)+\mu j(u)-\mu j(\bar u)+F''(u_\theta)(u-\bar u)^2\right]\\ \ge&J(\bar u)+\frac{\nu}{2}\|u-\bar u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}+\frac12\left[F'(\bar u)(u-\bar u)+\mu j(u)-\mu j(\bar u)+F''(u_\theta)(u-\bar u)^2\right].\end{aligned}$$

Now, it is enough to estimate the last term with (3.10), taking $\rho=1$, to conclude the proof.

Next, we consider the discrete control problems (P_h) defined in section 4. Let $\bar u$ be a local minimizer of (P) satisfying the second order condition (3.8) and the assumption (6.1). Then, from Theorem 6 we get the existence of a sequence $\{\bar u_h\}_h$ of local minimizers of problems (P_h) such that $\bar u_h \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$, $\|\bar y_h-\bar y\|_{L^\infty(\Omega)}\to 0$ as $h\to 0$, and (5.4) is fulfilled. The goal is to provide error estimates for $\bar u_h-\bar u$. To this end we will need the following approximation properties of $\pi_h\bar u$. Recall that we are defining $\pi_h\bar u=\bar u$ on $\Omega\setminus\Omega_h$.

Lemma 7 *Let $\bar u\in U_{\rm ad}$ satisfy the first order condition (3.1) and the structural assumption (6.1). Then there exists $C_\gamma>0$ independent of h such that*

$$\|\bar u-\pi_h\bar u\|_{L^1(\Omega_h)}\le C_\gamma h^\gamma \tag{6.4}$$

and

$$|F'(\bar u)(\pi_h\bar u-\bar u)+\mu j(\pi_h\bar u)-\mu j(\bar u)|\le C_\gamma h^{1+\gamma}. \tag{6.5}$$

Proof Consider an element T where $|\bar\varphi(x)|-\mu$ changes sign if $\mu>0$, or $\bar\varphi(x)$ changes sign if $\mu=0$. Since $\bar\varphi$ is continuous, there exists $x_0\in T$ such that $|\bar\varphi(x_0)|=\mu$ or $\bar\varphi(x_0)=0$. We also have that $\bar\varphi\in W^{2,p}(\Omega)$ for some $p>n$, so $\bar\varphi$ is Lipschitz, and hence there exists a constant $L_{\bar\varphi}>0$, independent of T, such that for all $x\in T$

$$\begin{aligned}\big||\bar\varphi(x)|-\mu\big| &= \big||\bar\varphi(x)|-|\bar\varphi(x_0)|\big|\le L_{\bar\varphi}h &&\text{if } \mu>0,\\ \big|\bar\varphi(x)\big| &= \big|\bar\varphi(x)-\bar\varphi(x_0)\big|\le L_{\bar\varphi}h &&\text{if } \mu=0.\end{aligned}$$

Denote

$$\begin{aligned}S &= \cup\{T:\ |\bar\varphi(x)|-\mu \text{ changes sign in } T\} &&\text{if } \mu>0,\\ S &= \cup\{T:\ \bar\varphi(x) \text{ changes sign in } T\} &&\text{if } \mu=0.\end{aligned}$$

We have just proved that

$$S\subset\{x\in\Omega_h:\big||\bar\varphi(x)|-\mu\big|\le L_{\bar\varphi}h\}$$

and by Assumption (6.1) we have that $\operatorname{meas} S\le KL_{\bar\varphi}^{\gamma}h^{\gamma}$. From (3.4) we get that $\bar u$ is constant in every triangle T where $|\bar\varphi(x)|-\mu$ has a constant sign if $\mu>0$ or $\bar\varphi(x)$ has a constant sign if $\mu=0$. Hence, the identity $\bar u-\pi_h\bar u=0$ holds in $\Omega\setminus S$. Therefore

$$\|\bar u-\pi_h\bar u\|_{L^1(\Omega_h)}=\|\bar u-\pi_h\bar u\|_{L^1(S)}\le(\beta-\alpha)KL_{\bar\varphi}^{\gamma}h^{\gamma},$$

and (6.4) follows.

Let us prove (6.5). If $\mu=0$, we have

$$\begin{aligned}|F'(\bar u)(\pi_h\bar u-\bar u)| &= \left|\int_{\Omega_h}\bar\varphi(\pi_h\bar u-\bar u)\,dx\right|=\left|\int_S\bar\varphi(\pi_h\bar u-\bar u)\,dx\right|\\ &\le\|\bar\varphi\|_{L^\infty(S)}\|\bar u-\pi_h\bar u\|_{L^1(S)}\le L_{\bar\varphi}h(\beta-\alpha)KL_{\bar\varphi}^{\gamma}h^{\gamma}\le C_\gamma h^{1+\gamma}.\end{aligned}$$

For the case $\mu>0$, we proceed as follows. For $h<\frac{\mu}{L_{\bar\varphi}}$ we have $\bar\varphi(x)\ne0\ \forall x\in S$. Then, for every element $T\subset S$, either $\bar\varphi(x)>0$ or $\bar\varphi(x)<0$ for all $x\in T$. Using (3.4), we deduce in the first case, that both $\bar u(x)\le0$ and $\pi_h\bar u(x)\le0$ for all $x\in T$ and in the second case that $\bar u(x)\ge0$ and $\pi_h\bar u(x)\ge0$ for all $x\in T$. Thus, we have

$$\begin{aligned}|F'(\bar u)(\pi_h\bar u-\bar u)+\mu j(\pi_h\bar u)-\mu j(\bar u)| &= \left|\int_{\Omega_h}\bar\varphi(\pi_h\bar u-\bar u)dx+\mu\int_\Omega|\pi_h\bar u|dx-\mu\int_\Omega|\bar u|dx\right|\\ &=\left|\int_S\bar\varphi(\pi_h\bar u-\bar u)dx+\mu\int_S|\pi_h\bar u|dx-\mu\int_S|\bar u|dx\right|\\ &=\Bigg|\int_{S\cap\{\bar\varphi>0\}}\bar\varphi(\pi_h\bar u-\bar u)dx-\mu\int_{S\cap\{\bar\varphi>0\}}\pi_h\bar u dx+\mu\int_{S\cap\{\bar\varphi>0\}}\bar u dx\\ &\quad+\int_{S\cap\{\bar\varphi<0\}}\bar\varphi(\pi_h\bar u-\bar u)dx+\mu\int_{S\cap\{\bar\varphi<0\}}\pi_h\bar u dx-\mu\int_{S\cap\{\bar\varphi<0\}}\bar u dx\Bigg|\\ &=\left|\int_{S\cap\{\bar\varphi>0\}}(\bar\varphi-\mu)(\pi_h\bar u-\bar u)dx+\int_{S\cap\{\bar\varphi<0\}}(|\bar\varphi|-\mu)(\bar u-\pi_h\bar u)dx\right|\\ &\le\||\bar\varphi|-\mu\|_{L^\infty(S)}\|\bar u-\pi_h\bar u\|_{L^1(S)}\le L_{\bar\varphi}h(\beta-\alpha)KL_{\bar\varphi}^{\gamma}h^{\gamma}\le C_\gamma h^{1+\gamma},\end{aligned}$$

and (6.5) is satisfied.

The next theorem provides an estimate for the difference $\bar u_h - \bar u$ and improves the estimate for the differences of the states provided in Theorem 7 for $\gamma > \phi^{-1} \approx 0.6180...$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the so called *golden ratio.*

Theorem 9 *Let $\bar u$ be a solution of (P) satisfying the second order sufficient conditions (3.8) and the structural assumption (6.1). Let $\{\bar u_h\}_h$ be a sequence of local minima of (P_h) such that (5.4) holds and $\bar u_h \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$. Then, there exists a constant $C > 0$ independent of h such that*

$$\|\bar u_h - \bar u\|_{L^1(\Omega_h)} \le C h^{\gamma^2},$$
$$\|\bar y_h - \bar y\|_{L^2(\Omega)} \le C h^{\frac{\gamma(\gamma+1)}{2}}.$$

Proof Since $\bar u_h \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$, using Theorem 1, we deduce that for any $\varepsilon > 0$ there exists $h_0 > 0$ such that $\|\bar y_h - \bar y\|_{L^\infty(\Omega)} < \varepsilon$ for every $h < h_0$. We extend $\bar u_h$ to Ω by setting $\bar u_h(x) = \bar u(x)$ if $x \in \Omega \setminus \Omega_h$. The same extension is considered for $\pi_h \bar u$. Thus we have $j_h(\pi_h \bar u) - j_h(\bar u_h) = j(\pi_h \bar u) - j(\bar u_h)$. Now, using (6.2), (6.5), the fact that $\pi_h \bar u \in U_{h,ad}$ and the first order optimality condition (4.12) (see also Remark 2) for the discrete problem (P_h) we get

$$\begin{aligned}
&\frac{\nu}{2}\|\bar u_h - \bar u\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + \frac{1}{2}\left[F'(\bar u)(\bar u_h - \bar u) + \mu j(\bar u_h) - \mu j(\bar u)\right] \\
&\le F'(\bar u)(\bar u_h - \bar u) + \mu j(\bar u_h) - \mu j(\bar u) \\
&\le [F'(\bar u)(\bar u_h - \bar u) + \mu j(\bar u_h) - \mu j(\bar u)] + [F_h'(\bar u_h)(\pi_h \bar u - \bar u_h) + \mu j(\pi_h \bar u) - \mu j(\bar u_h)] \\
&= [F'(\bar u) - F_h'(\bar u_h)](\bar u_h - \pi_h \bar u) + [F'(\bar u)(\pi_h \bar u - \bar u) + \mu j(\pi_h \bar u) - \mu j(\bar u)] \\
&\le [F'(\bar u) - F_h'(\bar u_h))](\bar u_h - \pi_h \bar u) + C_\gamma h^{1+\gamma} \\
&= [F'(\bar u) - F'(\bar u_h)](\bar u_h - \pi_h \bar u) + [F'(\bar u_h) - F_h'(\bar u_h)](\bar u_h - \pi_h \bar u) + C_\gamma h^{1+\gamma} \\
&= \mathrm{I} + \mathrm{II} + C_\gamma h^{1+\gamma}.
\end{aligned} \tag{6.6}$$

Let us estimate the terms I and II.

First, we notice that there exists a constant $C_0 > 0$ independent of h such that

$$\|\bar\varphi - \varphi_{\bar u_h}\|_{L^\infty(\Omega)} \le C_0 \|\bar u - \bar u_h\|_{L^1(\Omega_h)}. \tag{6.7}$$

To prove (6.7), define $z = \bar\varphi - \varphi_{\bar u_h}$. Subtracting the equations satisfied by $\bar\varphi$ and $\varphi_{\bar u_h}$, we obtain that z satisfies

$$\left\{\begin{aligned}
&A^* z + \frac{\partial f}{\partial y}(x, \bar y) z \\
&= \Big[\frac{\partial f}{\partial y}(x, y_{\bar u_h}) - \frac{\partial f}{\partial y}(x, \bar y)\Big]\varphi_{\bar u_h} + \Big[\frac{\partial L}{\partial y}(x, \bar y) - \frac{\partial L}{\partial y}(x, y_{\bar u_h})\Big] \ \text{ in } \Omega, \\
&z = 0 \text{ on } \Gamma.
\end{aligned}\right.$$

From assumptions (A2) and (A3), Theorem 1, (2.9) and using the mean value theorem

$$\|z\|_{L^\infty(\Omega)} \le C_1 \|\bar y - y_{\bar u_h}\|_{L^2(\Omega)},$$

where $C_1 = T_2(T_\infty C_{f,M_\infty} + C_{L,M_\infty})$. Now, (6.7) follows from (2.6), and $C_0 = C_1 \hat C$.

For the first term, using the mean value theorem, (6.7), (6.4), and Young's inequality

$$
\begin{aligned}
\mathrm{I} =&(F'(\bar u) - F'(\bar u_h))(\bar u_h - \pi_h \bar u)\\
=&(F'(\bar u) - F'(\bar u_h))(\bar u_h - \bar u) + (F'(\bar u) - F'(\bar u_h))(\bar u - \pi_h \bar u)\\
=& - F''(u_\theta)(\bar u_h - \bar u)^2 + \int_{\Omega_h} (\bar\varphi - \varphi_{\bar u_h})(\bar u - \pi_h \bar u)dx\\
\le& - F''(u_\theta)(\bar u_h - \bar u)^2 + \|\bar\varphi - \varphi_{\bar u_h}\|_{L^\infty(\Omega_h)}\|\bar u - \pi_h \bar u\|_{L^1(\Omega)}\\
\le& - F''(u_\theta)(\bar u_h - \bar u)^2 + C_0\|\bar u - \bar u_h\|_{L^1(\Omega_h)} C_\gamma h^\gamma\\
\le& - F''(u_\theta)(\bar u_h - \bar u)^2 + \frac{\nu}{8}\|\bar u - \bar u_h\|^{\frac{\gamma+1}{\gamma}}_{L^1(\Omega_h)} + C' h^{\gamma(\gamma+1)},
\end{aligned}
\tag{6.8}
$$

where $C' = \frac{1}{\gamma}\left(C_0 C_\gamma \frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{8}{\nu}\right)^\gamma$.

For the second term, taking into account that $\bar\varphi_h \equiv 0$ in $\Omega \setminus \Omega_h$, we can write

$$
\mathrm{II} = \int_{\Omega_h} (\varphi_{\bar u_h} - \bar\varphi_h)(\bar u_h - \pi_h \bar u)dx.
$$

Let us estimate $\varphi_{\bar u_h} - \bar\varphi_h$. To this end, we introduce the function $\varphi^h \in W^{2,p}(\Omega)$ for all $p < \infty$ as the solution of

$$
\left\{
\begin{aligned}
A^*\varphi + \frac{\partial f}{\partial y}(x, \bar y_h)\varphi &= \frac{\partial L}{\partial y}(x, \bar y_h) \text{ in } \Omega,\\
\varphi &= 0 \text{ on } \Gamma.
\end{aligned}
\right.
$$

Obviously, estimate (4.4) can be applied to estimate $\varphi^h - \bar\varphi_h$, hence we have

$$
\|\varphi^h - \bar\varphi_h\|_{L^\infty(\Omega)} \le ch^2 |\log h|^2.
$$

Now, we estimate the difference $z^h = \varphi_{\bar u_h} - \varphi^h$. Subtracting the equations satisfied by $\varphi_{\bar u_h}$ and φ^h we obtain

$$
\left\{
\begin{aligned}
&A^* z^h + \tfrac{\partial f}{\partial y}(x, y_{\bar u_h}) z^h\\
&= \left[\frac{\partial f}{\partial y}(x, \bar y_h) - \frac{\partial f}{\partial y}(x, y_{\bar u_h})\right]\varphi^h + \left[\frac{\partial L}{\partial y}(x, y_{\bar u_h}) - \frac{\partial L}{\partial y}(x, \bar y_h)\right] \text{ in } \Omega,\\
&z^h = 0 \text{ on } \Gamma.
\end{aligned}
\right.
$$

From assumptions (A2) and (A3), Theorem 1, and using the mean value theorem we infer

$$
\|z^h\|_{L^\infty(\Omega)} \le C_1 \|y_{\bar u_h} - \bar y_h\|_{L^2(\Omega)}.
$$

Now, (4.3) and the definition of z^h imply

$$
\|\varphi_{\bar u_h} - \varphi^h\|_{L^\infty(\Omega)} \le C_1 ch^2.
$$

Altogether, and using Young's inequality for $p=\gamma+1$ and $q=(\gamma+1)/\gamma$ along with estimate (6.4), we deduce the existence of constants $C_2, C_3, C_4>0$ independent of h such that

$$\begin{aligned}\text{II} \le & C_2 h^2|\log h|^2(\|\bar u_h-\bar u\|_{L^1(\Omega_h)}+\|\bar u-\pi_h\bar u\|_{L^1(\Omega_h)})\\ \le & C_3(h^2|\log h|^2)^{\gamma+1}+\frac{\nu}{8}\|\bar u-\bar u_h\|^{1+\frac{1}{\gamma}}_{L^1(\Omega_h)}+C_4h^{2+\gamma}|\log h|^2.\end{aligned} \tag{6.9}$$

From (6.6), (6.8) and (6.9), we have that there exists $C_5>0$ independent of h such that

$$\begin{aligned}\frac{\nu}{4}\|\bar u_h-\bar u\|^{1+\frac{1}{\gamma}}_{L^1(\Omega_h)}+&\frac{1}{2}\left[F'(\bar u)(\bar u_h-\bar u)+\mu j(\bar u_h)-\mu j(\bar u)\right]+F''(u_\theta)(\bar u_h-\bar u)^2\\ &\le C_5\big((h^2|\log h|^2)^{\gamma+1}+h^{2+\gamma}|\log h|^2+h^{1+\gamma}+h^{\gamma(\gamma+1)}\big).\end{aligned} \tag{6.10}$$

From Lemma 2 we deduce the existence of $\kappa>0$ such that for ε as above sufficiently small

$$\frac{\kappa}{2}\|y_{\bar u_h}-\bar y\|^2_{L^2(\Omega)}\le\frac{1}{2}\left[F'(\bar u)(\bar u_h-\bar u)+\mu j(\bar u_h)-\mu j(\bar u)\right]+F''(u_\theta)(\bar u_h-\bar u)^2 \quad \forall h\le h_0. \tag{6.11}$$

Combining (6.10) and (6.11) and taking into account that $\gamma\le 1$, we get

$$\frac{\nu}{4}\|\bar u_h-\bar u\|^{1+\frac{1}{\gamma}}_{L^1(\Omega_h)}+\frac{\kappa}{2}\|y_{\bar u_h}-\bar y\|^2_{L^2(\Omega)}\le C_6h^{\gamma(\gamma+1)}$$

for some $C_6>0$ independent of h.

The proof concludes observing that $\|y_{\bar u_h}-\bar y_h\|_{L^2(\Omega)}\le ch^2$; see (4.3).

Remark 6 (Approximation by continuous piecewise linear functions.) If we take

$$U_h=\{v_h\in C(\bar\Omega_h):\ v_{h|T}\in P_1(T)\ \forall T\in\mathcal{T}_h\}$$

and $U_{h,ad}=U_h\cap U_{\rm ad}$, we do not improve the order of convergence. The proof follows the same lines as before, replacing $\pi_h\bar u$ by any control $u_h^*\in U_{h,ad}$ such that $u^*_{h|T}=\alpha$ if $\bar\varphi(x)>\mu$ for all $x\in T$, $u^*_{h|T}=\beta$ if $\bar\varphi(x)<-\mu$ for all $x\in T$ and $u^*_{h|T}=0$ if $|\bar\varphi(x)|<\mu$ for all $x\in T$.

Remark 7 (Variational discretrization.) If $U_h=L^2(\Omega)$, then the projection is the identity and $\pi_h\bar u=\bar u$ in Ω. So in the proof of Theorem 9 the terms of order $h^{2+\gamma}|\log h|^2$, $h^{1+\gamma}$ and $h^{\gamma(1+\gamma)}$ in (6.10) disappear, and finally we get

$$\|\bar u-\bar u_h\|_{L^1(\Omega_h)}\le C(h|\log h|)^{2\gamma},\ \|\bar y-\bar y_h\|_{L^2(\Omega)}\le C(h|\log h|)^{1+\gamma}.$$

This order of convergence was obtained for problems governed by linear equations in [15, Lemma 3.3] assuming that $\mu=0$, $\gamma=1$, and $\bar\varphi\in W^{2,\infty}(\Omega)$.

7 Numerical experiment

Consider $n = 1$, $\Omega = (-1, 1)$, $A = -\Delta = -\partial_{xx}$. We will take $f(x, y) = y|y|^3$, $\mu = 0$ and $L(x, y) = \frac{1}{2}(y - y_d(x))^2$, where y_d is defined later. The state equation is given by

$$-\partial^2_{xx} y + y|y|^3 = u \text{ in } (-1, 1),\ y(-1) = y(1) = 0,$$

and the adjoint state equation is given by

$$-\partial^2_{xx}\varphi + 4|y|^3\varphi = y - y_d \text{ in } (-1, 1),\ \varphi(-1) = \varphi(1) = 0.$$

We define $\alpha = -1$, $\beta = 1$ and

$$\bar\varphi(x) = (1 - x^2)(x_0 - x)|x_0 - x|^{q-1},$$

where the switching point x_0 is $x_0 = 2^{-\delta}/3$ for some $\delta > 0$ and $q \geq 1$. We test examples for $\delta = 9$, $q = 1$ and $\delta = 3$, $q = 2$. This function clearly satisfies the boundary conditions of the adjoint state equation and Assumption (6.1) holds for $\gamma = 1/q \in (0, 1]$.

Taking into account the sign of $\bar\varphi$, we define

$$\bar u(x) = \begin{cases} \alpha \text{ if } x < x_0, \\ \beta \text{ if } x > x_0. \end{cases}$$

Using this control, we compute $\bar y = y_{\bar u}$. Since we cannot solve the state equation exactly, we solve it for a uniform mesh with constant step size $h = 2^{-15}/3$, so that x_0 is a mesh node.

Finally, we define $y_d(x) = \bar\varphi''(x) - 4|\bar y(x)|^3\bar\varphi(x) + \bar y(x)$.

With this data, we have that first order optimality conditions are satisfied and Assumptions (A1), (A2), (A3) and (6.1) hold for $q = 1$ and all $q \geq 2$. Experimentally, it is clear that $\frac{\partial^2 L}{\partial y^2}(x, \bar y(x)) - \bar\varphi(x)\frac{\partial^2 L}{\partial y^2}(x, \bar y(x)) \geq 1$ for all $x \in \Omega$, so second order sufficient condition (3.8) is also satisfied; see (2.10).

We discretize the problem using a family of uniform meshes with constant step size $h_i = 2^{-i}$, $i = 3, \ldots$. To solve the discrete problems, we do a Tikhonov regularization, cf. [17], i.e., we solve the finite element approximation of

$$(\mathrm{P}^{\nu_j})\ \min_{u \in U_{\mathrm{ad}}} J_{\nu_j}(u) = J(u) + \frac{\nu_j}{2}\|u\|^2_{L^2(\Omega)}.$$

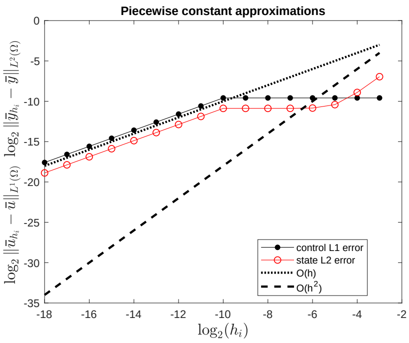

Fig. 1 Experimental order of convergence. Piecewise constant approximations of the control. $\gamma = 1$

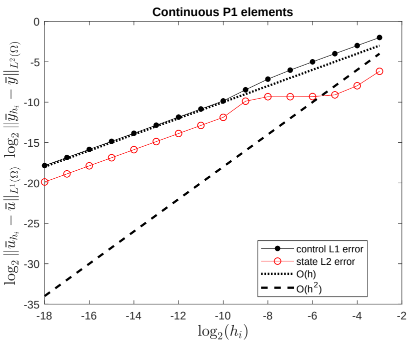

Fig. 2 Experimental order of convergence. Continuous piecewise linear approximations of the control. $\gamma = 1$

for a sequence $\nu_j \searrow 0$. This problem is solved using a semismooth Newton method as described in [8, Section 14]. We use algorithm 7.1, with parameters $\nu_0 = 1$, $r = 0.1$, $\varepsilon = 10^{-15}$ and $\nu_{\min} = 10^{-11}$.

Algorithm 7.1: Optimization algorithm

1 Set $j = 0$, an initial $\nu_0 > 0$ and an initial guess u_0, y_0, φ_0. Fix $0 < r < 1$ and a tolerance $\varepsilon > 0$
2 Solve (P^{ν_j}) with initial guess u_j, y_j, φ_j
3 Name the result u_{j+1}, y_{j+1}, φ_{j+1}
4 Set $\nu_{j+1} = \max\{r\nu_j, \nu_{\min}\}$
5 **if** $\|u_{j+1} - u_j\|_{L^1(0,T)} + \|y_{j+1} - y_j\|_{L^2(\Omega)} + \|\varphi_{j+1} - \varphi_j\|_{L^2(\Omega)} < \varepsilon$ **then**
6 | stop
7 **else**
8 | Set $j = j + 1$ and go to 2
9 **end**

For $\gamma = 1$, we obtain the results summarized in Figure 1 (piecewise constant approximations), Figure 2 (piecewise linear approximations) and Figure 3 (variational approximation). The experimental results are quite in agreement with the results in Section 6.

For $\gamma = 1/2$, we are only able to observe the predicted error estimate for the error control in the variational approach. See Figure 4. For the other approximation

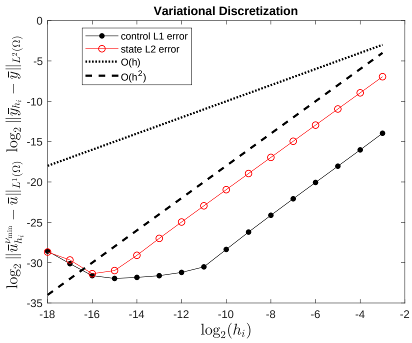

Fig. 3 Experimental order of convergence. Variational approximation of the control. $\gamma = 1$

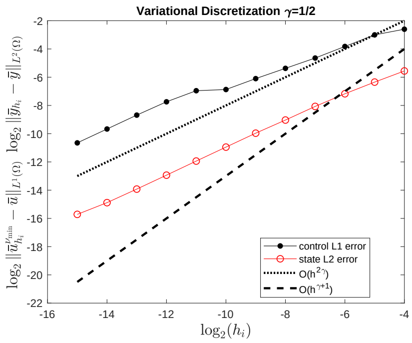

Fig. 4 Experimental order of convergence. Variational approximation of the control. $\gamma = 1/2$

we observe $O(h)$, despite expecting only $O(h^{1/4})$ for the controls and $O(\sqrt{h})$ for the states.

A Appendix: sketch of the proof of equation (3.11)

For a given $\theta \in [0,1]$, we denote $u_\theta = \bar{u} + \theta(u - \bar{u})$. We will use the following property, proved in [10, Lemma 3.5-2]. For every $\gamma > 0$ there exists $\varepsilon > 0$ such that if $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon$, then

$$|(F''(u_\theta) - F''(\bar{u}))z_v^2| \le \gamma \|z_v\|^2_{L^2(\Omega)} \;\forall v \in L^2(\Omega). \tag{A.1}$$

Firstly, we prove that there exists a constant $C > 0$ such that

$$\rho J'(\bar{u}; u - \bar{u}) + F''(u_\theta)(u - \bar{u})^2 \ge C\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}. \tag{A.2}$$

As in [9, Theorem 3.1], we distinguish three cases

Case 1: $u - \bar{u} \in C^\tau_{\bar{u}}$. On one hand, from [6, Lemma 2.5], we have that

$$J'(\bar{u}; u - \bar{u}) \ge 0.$$

On the other hand, taking ε small enough, using (A.1) with $\gamma = \delta/2$ and the second order condition (3.8), we have

$$F''(u_\theta)(u - \bar{u})^2 \ge \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}$$

and (A.2) follows.

Case 2: $u - \bar{u} \notin G^\tau_{\bar{u}}$. In this case it is the first derivative the one that dominates. For any given $\rho > 0$, it is proved as in [9, eq. (3.8)] that for $\varepsilon > 0$ small enough and $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon$, we have

$$\rho J'(\bar{u}; u - \bar{u}) \geq \frac{\rho\tau}{2\varepsilon}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}.$$

By continuity properties of the second derivative, there exists $M > 0$ such that

$$F''(u_\theta)(u - \bar{u})^2 \leq M\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}.$$

So for $\varepsilon > 0$ small enough (and maybe depending on ρ), equation (A.2) holds.

Case 3: $u - \bar{u} \notin D^\tau_{\bar{u}}$ and $u - \bar{u} \in G^\tau_{\bar{u}}$. First of all, let us define $C_{\Omega,1}$ as the continuity constant of the mapping $v \mapsto z_v$ in $L^1(\Omega)$, i.e.,

$$\|z_v\|_{L^1(\Omega)} \leq C_{\Omega,1}\|v\|_{L^1(\Omega)}\ \forall v \in L^1(\Omega),$$

and $\tau^* = \tau/\max\{1, C_{\Omega,1}\}$. If $u - \bar{u} \notin G^{\tau^*}_{\bar{u}}$, then Case 2 applies. Otherwise, we define the set $V \subset \Omega$ in the following way:

$$\text{If } \mu = 0, \quad V = \{x \in \Omega : \ u(x) - \bar{u}(x) = 0 \text{ if } |\bar{\varphi}(x)| > \tau\}.$$

$$\text{If } \mu > 0, \quad V = \left\{ x \in \Omega : u(x) - \bar{u}(x) \begin{cases} \geq 0 & \text{if } \big(\bar{\varphi}(x) = -\mu \text{ and } \bar{u}(x) = 0\big) \\ \leq 0 & \text{if } \big(\bar{\varphi}(x) = +\mu \text{ and } \bar{u}(x) = 0\big) \\ = 0 & \text{if } \Big||\bar{\varphi}(x)| - \mu\Big| > \tau \end{cases} \right\}.$$

Let us also denote $W = \Omega \setminus V$ and define now $v = (u - \bar{u})\chi_V$ and $w = (u - \bar{u})\chi_W$. On one hand we have (see [11, Proposition 3.6] and [9, eq. (3.11)]) that

$$J'(\bar{u}; u - \bar{u}) \geq \tau\|w\|_{L^1(\Omega)}. \tag{A.3}$$

On the other hand, by construction, v belongs to $D^\tau_{\bar{u}}$, and it is proved in [9] that $v \in G^{\tau^*}_{\bar{u}}$. So $v \in C^\tau_{\bar{u}}$. Using now that $u - \bar{u} = v + w$, (A.1) and (3.8), we have that there exists a constant $c > 0$ such that

$$F''(u_\theta)(u - \bar{u})^2 \geq \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} - c\|z_w\|^2_{L^2(\Omega)}. \tag{A.4}$$

Next taking into account that for $\varepsilon > 0$ small enough, $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon$ implies $\|z_w\|_{L^\infty(\Omega)} < 2\varepsilon$ (which follows from [9, Lemma 2.4], the definition of w and the maximum principle), we deduce $\|z_w\|^2_{L^2(\Omega)} \leq 2\varepsilon\|z_w\|_{L^1(\Omega)} \leq 2\varepsilon C_{\Omega,1}\|w\|_{L^1(\Omega)}$, and hence for $\varepsilon > 0$ small enough (maybe depending on $\rho > 0$),

$$\rho\tau\|w\|_{L^1(Q)} - c\|z_w\|^2_{L^2(\Omega)} \geq 0,$$

and (A.2) follows from this inequality, (A.3) and (A.4).

To conclude the proof of (3.11), it is enough to notice that (see [9, eq. (2.9)]) for $\varepsilon > 0$ small enough

$$\|z_{u-\bar{u}}\|_{L^2(\Omega)} \geq \frac{1}{2}\|y_u - \bar{y}\|_{L^2(\Omega)}.$$

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