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SECOND ORDER ANALYSIS FOR BANG-BANG CONTROL PROBLEMS OF PDEs*

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Abstract. In this paper, we derive some sufficient second order optimality conditions for control problems of partial differential equations (PDEs) when the cost functional does not involve the usual quadratic term for the control or higher nonlinearities for it. Though not always, in this situation the optimal control is typically bang-bang. Two different control problems are studied. The second differs from the first in the presence of the L^1 norm of the control. This term leads to optimal controls that are sparse and usually take only three different values (we call them bang-bang-bang controls). Though the proofs are detailed in the case of a semilinear elliptic state equation, the approach can be extended to parabolic control problems. Some hints are provided in the last section to extend the results.

Key words. optimal control, semilinear partial differential equation, second order optimality conditions, bang-bang controls, sparse controls

AMS subject classifications. 49K20, 49K30, 35J61

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1. Introduction.

The aim of this paper is to prove some sufficient second order optimality conditions for optimal control problems of elliptic partial differential equations (PDEs) when the cost functional does not involve the control in an explicit form. In particular, the Hessian of the Hamiltonian with respect to the control vanishes so that the classical Legendre–Clebsch condition does not hold. In these situations, the optimal control is usually bang-bang. In the case that the Legendre–Clebsch condition holds, there are several papers providing sufficient second order conditions (see [3], [4], [6], [10], [11] [13], [14]). The results that we present here cover the case of bang-bang controls. The main difference with the usual second order conditions is that the inequality $J''(\bar{u})v^2 \geq \delta\|v\|^2$ for every v in some cone of critical directions does not hold in general, and it has to be replaced for a weaker assumption, but one that is still strong enough to warrant the strict local optimality of the controls.

As far as we know, there is no second order analysis for bang-bang controls within the framework of PDEs. However, the case of ODEs has been extensively studied; see, for instance, [18], [22], [23], [24], [25], [26], [27]. The analysis for control problems of ODEs is based on the assumption of a finite number of switching points, and at those points the derivative of the switching function does not vanish. The extension of this approach to the case of PDEs is not clear at all. Here we will present a different approach. We give a sufficient second order condition for strict local optimality in the L^2 sense, with a quadratic growth. Both of them, the sufficient condition and the quadratic growth, are based on the L^2 norm of the linearized state. This is in contrast to the usual situation, where the L^2 norm of the control is involved in both terms. One of the referees drew our attention to the Goh transformation [20], which has been recently used in [2] to give some sufficient second order conditions for control

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problems governed by ODEs. Though the Goh transformation is different from our approach because he considered the primitive of a given control v and we use the linearized state in the direction v, both approaches coincide in changing the usual L^2 norm of the control by the L^2 norm of the primitive or the linearized state.

In this paper, we will consider two different control problems. The first problem is studied in section 2, and it is a control problem associated with a semilinear elliptic equation where all the functions involved in the problem are of class C^2 with respect to the state. In the second problem, studied in section 3, the cost functional includes the L^1 norm of the control so that it is not differentiable. To deal with this case we will combine the approach developed in section 2 and the ideas of [5]. In this second problem, the structure of the optimal control is typically bang-bang-bang because there are three possible values for the optimal control instead of the usual two extreme values corresponding to a bang-bang control problem.

It is well known that the solution stability with respect to data perturbations and conditions for strict local optimality are closely related facts. This justifies the attention paid to the second order analysis for control problems. On the other hand, the sufficient second order conditions are essential in the numerical analysis of the nonconvex control problems. They have been used to derive error estimates for the discretization of control problems; see, for instance, [1], [5], [7], [8], [9], [12]. In all the precedent papers, the Legendre–Clebsch condition was satisfied, which excludes the case of bang-bang controls. For linear-quadratic control problems some error estimates for the approximation of bang-bang controls were obtained in [16]. However, for nonlinear PDEs there are no results. It is our aim to use the sufficient second order conditions obtained in this paper to prove these error estimates.

2. A bang-bang control problem. Let us consider the control problem

$$(\mathrm{P}_1) \qquad \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\, dx, \\ \alpha \le u(x) \le \beta, \end{cases}$$

where y_u is the solution of the Dirichlet problem

$$\begin{cases} Ay + f(x,y) &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma, \end{cases} \tag{2.1}$$

$-\infty < \alpha < \beta < +\infty$, and $L, f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions of class C^2 with respect to the second variable satisfying the following.

(A1) $f(\cdot, 0) \in L^{\bar p}(\Omega)$, with $\bar p > n/2$,

$$\frac{\partial f}{\partial y}(x,y) \ge 0 \quad \text{for a.e. } x \in \Omega,$$

and for all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \le C_{f,M} \quad \text{for a.e. } x \in \Omega \text{ and } |y| \le M.$$

For every $M > 0$ and $\varepsilon > 0$ there exists $\delta > 0$, depending on M and ε, such that

$$\left|\frac{\partial^2 f}{\partial y^2}(x,y_2) - \frac{\partial^2 f}{\partial y^2}(x,y_1)\right| < \varepsilon \quad \text{if } |y_1|, |y_2| \le M,\ |y_2 - y_1| \le \delta, \text{ and for a.e. } x \in \Omega.$$

(A2) $L(\cdot,0) \in L^1(\Omega)$, and for all $M > 0$ there are a constant $C_{L,M} > 0$ and a function $\psi_M \in L^{\bar p}(\Omega)$ such that for every $|y| \le M$ and almost all $x \in \Omega$

$$\left|\frac{\partial L}{\partial y}(x,y)\right| \le \psi_M(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \le C_{L,M}.$$

For every $M > 0$ and $\varepsilon > 0$ there exists $\delta > 0$, depending on M and ε, such that

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y_2) - \frac{\partial^2 L}{\partial y^2}(x,y_1)\right| < \varepsilon \ \text{ if } |y_1|,|y_2| \le M,\ |y_2 - y_1| \le \delta, \text{ and for a.e. } x \in \Omega.$$

(A3) We also assume that Ω is an open and bounded domain in $\mathbb{R}^n$, $n \le 3$, with a Lipschitz boundary Γ, and A denotes a second order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x));$$

the coefficients $a_{ij} \in C(\bar\Omega)$ satisfy

$$\lambda_A|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } x \in \Omega,$$

for some $\lambda_A > 0$.

Hereafter, we will denote

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Omega) : \alpha \le u(x) \le \beta \ \text{ for a.e. } x \in \Omega\}.$$

For every $u \in L^p(\Omega)$, with $p > n/2$, the state equation (2.1) has a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar\Omega)$. The proof of this result is a quite standard combination of the Schauder fixed point theorem and the $L^\infty(\Omega)$ estimates [28]. For the continuity of the solution in $\bar\Omega$, see, for instance, [19, Theorem 8.30]. Moreover, the mapping $G : L^p(\Omega) \longrightarrow H_0^1(\Omega) \cap C(\bar\Omega)$ is of class C^2. In what follows, we will take $p = 2$, and we will denote $z_v = G'(u)v$, which is the solution of

$$\text{(2.2)} \qquad \begin{cases} Az + \dfrac{\partial f}{\partial y}(x,y)z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

As usual, we consider the adjoint state equation associated with a control u

$$\text{(2.3)} \qquad \begin{cases} A^*\varphi + \dfrac{\partial f}{\partial y}(x,y)\varphi = \dfrac{\partial L}{\partial y}(x,y) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma, \end{cases}$$

where $y = G(u)$ is the state corresponding to u. Because of the assumptions on L, we have that $\varphi \in H_0^1(\Omega) \cap C(\bar\Omega)$. Moreover, there exists $M > 0$ such that

$$\text{(2.4)} \qquad \|y_u\|_\infty + \|\varphi_u\|_\infty \le M \quad \forall u \in \mathcal{U}_{ad}.$$

Under the above assumptions, the problem (P$_1$) has at least one solution $\bar u$ with an associated state $\bar y \in H_0^1(\Omega) \cap C(\bar\Omega)$. The cost functional $J : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2, and the first and second derivatives are given by

$$\text{(2.5)} \qquad J'(u)v = \int_\Omega \varphi(x)v(x)\,dx$$

and

$$J''(u)(v_1, v_2) = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y(x)) - \varphi(x) \frac{\partial^2 f}{\partial y^2}(x, y(x)) \right) z_{v_1}(x) z_{v_2}(x)\, dx, \tag{2.6}$$

where y is the state associated with u, solution of (2.1), and φ is the adjoint state, solution of (2.3), and $z_{v_i} = G'(u)v_i$ is the solution of (2.2) for $v = v_i$, $i = 1, 2$.

Any local solution $\bar{u}$ satisfies the optimality system

$$\left\{ \begin{array}{rcll} A\bar{y} + f(x, \bar{y}) & = & \bar{u} & \text{in } \Omega, \\ \bar{y} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{rcll} A^* \bar{\varphi} + \dfrac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} & = & \dfrac{\partial L}{\partial y}(x, \bar{y}) & \text{in } \Omega, \\ \bar{\varphi} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{2.8}$$

$$\int_\Omega \bar{\varphi}(x)(u(x) - \bar{u}(x))\, dx \geq 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{2.9}$$

From the last condition, we deduce, as usual,

$$\bar{u}(x) \left\{ \begin{array}{ll} = \alpha & \text{if } \bar{\varphi}(x) > 0, \\ = \beta & \text{if } \bar{\varphi}(x) < 0, \end{array} \right. \quad \text{and} \quad \bar{\varphi}(x) \left\{ \begin{array}{ll} \geq 0 & \text{if } \bar{u}(x) = \alpha, \\ \leq 0 & \text{if } \bar{u}(x) = \beta, \\ = 0 & \text{if } \alpha < \bar{u}(x) < \beta. \end{array} \right. \tag{2.10}$$

The cone of critical directions associated with $\bar{u}$ is defined by

$$C_{\bar{u}} = \left\{ v \in L^2(\Omega) : v(x) \left\{ \begin{array}{ll} \geq 0 & \text{if } \bar{u}(x) = \alpha \\ \leq 0 & \text{if } \bar{u}(x) = \beta \\ = 0 & \text{if } \bar{\varphi}(x) \neq 0 \end{array} \right\} \right\}.$$

Then, the necessary second order conditions satisfied by the local minimum $\bar{u}$ can be written in the form

$$J''(\bar{u})v^2 \geq 0 \quad \forall v \in C_{\bar{u}}. \tag{2.11}$$

For the above results the reader is referred to [6] for the analogous case of a distributed control problem associated with a semilinear Neumann boundary problem or to [11] for the Dirichlet case associated with a quasilinear equation.

Let us remark that in the case where the set of zeros of $\bar{\varphi}$ has a zero Lebesgue measure, $\bar{u}(x)$ is either α or β for almost all points $x \in \Omega$; i.e., $\bar{u}$ is a bang-bang control. Moreover, in this case, $C_{\bar{u}} = \{0\}$; therefore, (2.11) does not provide any information. Consequently, it is unlikely that the sufficient second order conditions could be based on the set $C_{\bar{u}}$. To overcome this drawback we are going to increase the set $C_{\bar{u}}$. For every $\tau \geq 0$ we define

$$C^\tau_{\bar{u}} = \left\{ v \in L^2(\Omega) : v(x) \left\{ \begin{array}{ll} \geq 0 & \text{if } \bar{u}(x) = \alpha \\ \leq 0 & \text{if } \bar{u}(x) = \beta \\ = 0 & \text{if } |\bar{\varphi}(x)| > \tau \end{array} \right\} \right\}.$$

It is obvious that $C^0_{\bar{u}} = C_{\bar{u}}$. The next example, due to Dunn [17], proves that, in general, the second order condition based on the cone $C_{\bar{u}}$ is not sufficient for the local optimality.

Example 2.1. Given the function $a(x) = 1 - 2x$, let us consider the optimization problem

$$\begin{cases} \min J(u) = \int_0^1 [2a(x)u(x) - \mathrm{sign}(a(x))u(x)^2]dx, \\ u \in L^2(0,1),\ u(x) \geq 0 \text{ for a.e. } x \in [0,1]. \end{cases}$$

Let us set $\bar{u}(x) = \max\{0, -a(x)\}$. Then we have that

$$J'(\bar{u})(u-\bar{u}) = \int_0^1 2[a(x)-\mathrm{sign}(a(x))\bar{u}(x)](u(x)-\bar{u}(x))(x)dx = \int_0^{1/2} 2a(x)u(x)dx \geq 0$$

holds for all $u \in L^2(0,1)$, with $u(x) \geq 0$. Taking into account that $J'(\bar{u}) = 2[a(x) - \mathrm{sign}(a(x))\bar{u}(x)]$, which vanishes in $[1/2, 1]$, and $\bar{u}(x) > 0$ for $x > 1/2$, we see that the cone $C_{\bar{u}}$ is given by

$$C_{\bar{u}} = \left\{ v \in L^2(0,1) : v(x) = 0 \text{ for a.e. } x \in \left[0, \frac{1}{2}\right] \right\}.$$

Now, for $v \in C_{\bar{u}}$ we have

$$J''(\bar{u})v^2 = 2\int_{1/2}^1 v^2(x)dx = 2\|v\|^2_{L^2(0,1)}.$$

Thus, $\bar{u}$ satisfies the first order optimality conditions and an apparently reasonable sufficient second order condition. However, $\bar{u}$ is not a local minimum in $L^2(0,1)$. Indeed, let us take for $0 < \varepsilon < \frac{1}{2}$

$$u_\varepsilon(x) = \begin{cases} \bar{u}(x) + 3\varepsilon & \text{if } x \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2}\right], \\ \bar{u}(x) & \text{otherwise.} \end{cases}$$

Then, we have $J(u_\varepsilon) - J(\bar{u}) = -3\varepsilon^3 < 0$.

Before formulating the sufficient second order condition for the problem (P_1), let us take a look at the Tikhonov regularization of (P_1). For any $\Lambda > 0$, let us consider the problem

$$(\mathrm{P}_{1,\Lambda}) \qquad \min_{u \in \mathcal{U}_{ad}} J_\Lambda(u) = \int_\Omega L(x, y_u(x))\, dx + \frac{\Lambda}{2}\int_\Omega u^2(x)\, dx.$$

Then, we have

$$J'_\Lambda(u)v = \int_\Omega (\varphi_u + \Lambda u)v\, dx$$

and

$$J''_\Lambda(u)(v_1, v_2) = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) \right) z_{v_1} z_{v_2} dx + \Lambda \int_\Omega v_1 v_2 dx.$$

For ($\mathrm{P}_{1,\Lambda}$) the following theorem holds; see [13].

Theorem 2.2. *Let $\bar{u} \in \mathcal{U}_{ad}$ satisfy that*

$$J'_\Lambda(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in \mathcal{U}_{ad} \quad \textit{and}$$
$$J''_\Lambda(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}.$$

Then, there exist $\delta > 0$ *and* $\varepsilon > 0$ *such that*

$$J_\Lambda(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le J_\Lambda(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}.$$

In the above theorem and hereafter, $B_\varepsilon(\bar{u})$ denotes the $L^2(\Omega)$-ball of center at $\bar{u}$ and radius ε. Let us remark that the presence of $\Lambda > 0$ in the cost functional implies that the positivity of $J''(\bar{u})$ on $C_{\bar{u}}$ is enough to deduce that $\bar{u}$ is a strict local minimum. It seems that we do not need to assume the strict positivity of the quadratic form on the extended cone $C^\tau_{\bar{u}}$. However, this is not completely true as the following result shows.

Theorem 2.3. *Let* $\bar{u} \in \mathcal{U}_{ad}$ *satisfy* $J'_\Lambda(\bar{u})(u - \bar{u}) \ge 0$ *for every* $u \in \mathcal{U}_{ad}$. *Then, the following assumptions are equivalent:*

1. $J''_\Lambda(\bar{u})v^2 > 0\ \forall v \in C_{\bar{u}} \setminus \{0\}$.
2. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \ge \nu\|v\|^2_{L^2(\Omega)}\ \forall v \in C^\tau_{\bar{u}}$.
3. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \ge \nu\|z_v\|^2_{L^2(\Omega)}\ \forall v \in C^\tau_{\bar{u}}$,

where $z_v = G'(\bar{u})v$.

The most delicate proof is $1 \Rightarrow 2$, but this is already a known result; see, for instance, [3] or [13]. The implications of $2 \Rightarrow 3 \Rightarrow 1$ are immediate, and they hold even if $\Lambda = 0$. As Dunn's example shows, 1 is not enough, in general, to ensure the local optimality of $\bar{u}$. We will see later that 2 does not hold for $\Lambda = 0$. Then, it remains to analyze if the assumption 3 is enough for the local optimality of $\bar{u}$ when $\Lambda = 0$. The next theorem proves that it is sufficient.

Theorem 2.4. *Let us assume that* $\bar{u}$ *is a feasible control for problem* (P$_1$) *satisfying the first order optimality conditions* (2.7)–(2.9) *and suppose that there exist* $\delta > 0$ *and* $\tau > 0$ *such that*

$$J''(\bar{u})v^2 \ge \delta\|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}, \tag{2.12}$$

where $z_v = G'(\bar{u})v$ *is the solution of* (2.6) *for* $y = \bar{y}$. *Then, there exists* $\varepsilon > 0$ *such that*

$$J(\bar{u}) + \frac{\delta}{8}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} \le J(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}, \tag{2.13}$$

with $z_{u-\bar{u}} = G'(\bar{u})(u - \bar{u})$.

The proof of this result requires some technical lemmas. For convenience, we introduce the space $Y = H^1_0(\Omega) \cap C(\bar{\Omega})$ endowed with the norm

$$\|y\|_Y = \|y\|_{H^1_0(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

Lemma 2.5. *There exists a constant* $C_1 > 0$ *such that*

$$\|y_u - \bar{y}\|_Y + \|\varphi_u - \bar{\varphi}\|_Y \le C_1\|u - \bar{u}\|_{L^2(\Omega)} \quad \forall u \in \mathcal{U}_{ad}, \tag{2.14}$$

where y_u *and* φ_u *denote the state and adjoint state associated with* u.

The proof of this lemma is a straightforward consequence from the the assumptions on A, f, and L.

Lemma 2.6. *For any* $u \in \mathcal{U}_{ad}$ *and* $v \in L^2(\Omega)$, *denote* $z_{u,v} = G'(u)v$. *Also, we set* $z_v = G'(\bar{u})v$. *There exist constants* $C_2 > 0$ *and* $C_3 > 0$ *such that*

$$\|z_{u,v} - z_v\|_Y \le C_2\|u - \bar{u}\|_{L^2(\Omega)}\|z_v\|_{L^2(\Omega)} \quad \forall v \in L^2(\Omega), \tag{2.15}$$

$$\|z_{u,v}\|_{L^2(\Omega)} \le C_3\|v\|_{L^1(\Omega)} \quad \forall v \in L^1(\Omega), \tag{2.16}$$

for every $u \in \mathcal{U}_{ad}$.

Proof. Subtracting the equations satisfied by $z_{u,v}$ and z_v and using the mean value theorem in the nonlinear term, we get

$$A(z_{u,v} - z_v) + \frac{\partial f}{\partial y}(x, y_u)(z_{u,v} - z_v) + \frac{\partial^2 f}{\partial y^2}(x, \hat{y})(y_u - \bar{y})z_v = 0,$$

where $\hat{y} = \bar{y} + \theta(y_u - \bar{y})$ for some measurable function $0 \le \theta(x) \le 1$. Using (2.4), the assumptions on f, and (2.14), we deduce from the above equation

$$\|z_{u,v} - z_v\|_Y \le C\|y_u - \bar{y}\|_{L^\infty(\Omega)}\|z_v\|_{L^2(\Omega)} \le CC_1\|u - \bar{u}\|_{L^2(\Omega)}\|z_v\|_{L^2(\Omega)},$$

which implies (2.15) with $C_2 = CC_1$.

Inequality (2.16) follows from a regularity result for (2.2). Jerison and Kenig [21] proved that the Laplace operator defines an isomorphism between $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ for p in some range $[2, p_M)$, with $p_M > 3$ depending on Ω. Moreover, it is well known that $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ and $W_0^{1,p'}(\Omega) = W^{-1,p}(\Omega)^*$, where $p' = p/(p-1)$ is the conjugate of p. Then, we can argue by transposition to deduce that $-\Delta : W_0^{1,p'}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is also an isomorphism for $p'_M < p' \le 2$, where p'_M is the conjugate of p_M. The result of Jerison and Kenig can be extended to elliptic operators with coefficients $a_{ij} \in C(\bar{\Omega})$. It is enough to use the classical technique of freezing the coefficients around a grid of points. Since $L^1(\Omega) \subset W^{-1,p}(\Omega)$ for every $p < n/(n-1)$, the inclusion being continuous, and $p'_M < n/(n-1) < p_M$, then we have

$$\|z_{u,v}\|_{W_0^{1,p}(\Omega)} \le C_p\|v\|_{L^1(\Omega)}.$$

The constant C_p is independent of u because $\mathcal{U}_{ad}$ is bounded in $L^\infty(\Omega)$. Finally, it is enough to take p close enough to $n/(n-1)$ to have the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$, and (2.16) follows from the above inequality. ∎

LEMMA 2.7. *For every $\varepsilon > 0$, there exists $\rho > 0$ such that if $\|u - \bar{u}\|_{L^2(\Omega)} \le \rho$ and $u \in \mathcal{U}_{ad}$, then the following inequality holds:*

$$|J''(u)v^2 - J''(\bar{u})v^2| \le \varepsilon\|z_v\|^2_{L^2(\Omega)} \quad \forall v \in L^2(\Omega). \tag{2.17}$$

Proof. Let us define the function $F : \Omega \times \mathcal{U}_{ad} \longrightarrow L^\infty(\Omega)$ by

$$F(x, u) = \frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u\frac{\partial^2 f}{\partial y^2}(x, y_u),$$

where y_u and φ_u are the solutions of (2.1) and (2.3), respectively. The assumptions on f and L and (2.4) imply that F is well defined. Moreover, using again the assumptions on f and L and (2.14), we know that given $\varepsilon > 0$ there exists $\rho_1 \in (0, 1)$ such that

$$\|F(x, u) - F(x, \bar{u})\|_\infty \le \frac{\varepsilon}{2} \text{ if } \|u - \bar{u}\|_{L^2(\Omega)} \le \rho_1. \tag{2.18}$$

Also we have that

$$\|F(x, u)\|_\infty \le K_M \quad \forall u \in \mathcal{U}_{ad}. \tag{2.19}$$

From (2.6) we obtain

$$J''(u)v^2 - J''(\bar{u})v^2 = \int_\Omega \left\{F(x, u(x))z^2_{u,v} - F(x, \bar{u}(x))z_v^2\right\} dx$$
$$= \int_\Omega F(x, u(x))(z^2_{u,v} - z^2_v)dx + \int_\Omega [F(x, u(x)) - F(x, \bar{u}(x))]z_v^2\, dx.$$

Using (2.18) and (2.19), we get

$$|J''(u)v^2 - J''(\bar{u})v^2| \le K_M(\|z_{u,v}\|_{L^2(\Omega)} + \|z_v\|_{L^2(\Omega)})\|z_{u,v} - z_v\|_{L^2(\Omega)} + \frac{\varepsilon}{2}\|z_v\|^2_{L^2(\Omega)}$$
$$\le K_M(\|z_{u,v} - z_v\|_{L^2(\Omega)} + 2\|z_v\|_{L^2(\Omega)})\|z_{u,v} - z_v\|_{L^2(\Omega)} + \frac{\varepsilon}{2}\|z_v\|^2_{L^2(\Omega)},$$

and from (2.15) we get

$$\le K_M(C_2\|u - \bar{u}\|_{L^2(\Omega)}\|z_v\|_{L^2(\Omega)} + 2\|z_v\|_{L^2(\Omega)})C_2\|u - \bar{u}\|_{L^2(\Omega)}\|z_v\|_{L^2(\Omega)} + \frac{\varepsilon}{2}\|z_v\|^2_{L^2(\Omega)}$$
$$\le \left\{K_M(C_2\rho_1 + 2)C_2\rho_1 + \frac{\varepsilon}{2}\right\}\|z_v\|^2_{L^2(\Omega)}.$$

Taking

$$\rho = \min\left\{\rho_1, \frac{\varepsilon}{2K_M C_2(C_2 + 2)}\right\}$$

and recalling that $\rho_1 < 1$, we deduce (2.17) from the above inequality. ∎

Proof of Theorem 2.4. First, let us observe that for every $v, w \in L^2(\Omega)$ and any $u \in \mathcal{U}_{ad}$ the following inequality follows from (2.19):

$$(2.20)\qquad |J''(u)(v,w)| = \left|\int_\Omega F(x, u(x))z_{u,v}z_{u,w}\,dx\right| \le K_M\|z_{u,v}\|_{L^2(\Omega)}\|z_{u,w}\|_{L^2(\Omega)}.$$

From Lemma 2.7, we deduce the existence of $\varepsilon_0 > 0$ such that

$$(2.21)\qquad |J''(u)v^2 - J''(\bar{u})v^2| \le \frac{\delta}{4}\|z_v\|^2_{L^2(\Omega)} \quad \text{if} \quad \|u - \bar{u}\|_{L^2(\Omega)} \le \varepsilon_0,$$

where δ is given in (2.12). Now, we take $0 < \varepsilon < \varepsilon_0$ such that

$$(2.22)\qquad \frac{\tau}{C_3^2\varepsilon\sqrt{|\Omega|}} - \frac{K_M(C_2\varepsilon + 1)^2}{2} - \frac{2K_M^2(C_2\varepsilon + 1)^4}{\delta} \ge \frac{\delta}{4},$$

where C_2, C_3, and τ are given in Lemma 2.6 and Theorem 2.4. Above, $|\Omega|$ denotes the Lebesgue measure of Ω.

Now, given an element $u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}$, we define

$$v(x) = \begin{cases} u(x) - \bar{u}(x) & \text{if } |\bar{\varphi}(x)| \le \tau, \\ 0 & \text{otherwise,} \end{cases}$$

and $w = (u - \bar{u}) - v$. Obviously, we have that $v \in C^\tau_{\bar{u}}$.

Making a Taylor expansion of second order, we obtain for some $\hat{u} = \bar{u} + \theta(u - \bar{u})$, with $\theta \in (0, 1)$,

$$J(u) = J(\bar{u}) + J'(\bar{u})(u - \bar{u}) + \frac{1}{2}J''(\hat{u})(u - \bar{u})^2$$

using (2.10) and that $u - \bar{u} = v + w$

$$= J(\bar{u}) + \int_\Omega |\bar{\varphi}||u - \bar{u}|\,dx + \frac{1}{2}J''(\bar{u})v^2 + \frac{1}{2}[J''(\hat{u})v^2 - J''(\bar{u})v^2] + \frac{1}{2}J''(\hat{u})w^2$$
$$+ J''(\hat{u})(v, w)$$

with (2.12), (2.20), and (2.21)

$$\geq J(\bar{u}) + \int_\Omega |\bar{\varphi}||w|\,dx + \frac{\delta}{2}\|z_v\|^2_{L^2(\Omega)} - \frac{\delta}{8}\|z_v\|^2_{L^2(\Omega)}$$

$$-\frac{K_M}{2}\|z_{\hat{u},w}\|^2_{L^2(\Omega)} - K_M\|z_{\hat{u},v}\|_{L^2(\Omega)}\|z_{\hat{u},w}\|_{L^2(\Omega)}$$

$$\geq J(\bar{u}) + \tau\|w\|_{L^1(\Omega)} + \frac{3\delta}{8}\|z_v\|^2_{L^2(\Omega)}$$

$$-\frac{K_M}{2}\|z_{\hat{u},w}\|^2_{L^2(\Omega)} - K_M\|z_{\hat{u},v}\|_{L^2(\Omega)}\|z_{\hat{u},w}\|_{L^2(\Omega)}. \tag{2.23}$$

From (2.15), we get

$$\|z_{\hat{u},w}\|_{L^2(\Omega)} \leq \|z_{\hat{u},w} - z_w\|_{L^2(\Omega)} + \|z_w\|_{L^2(\Omega)} \leq C_2\|\hat{u} - u\|_{L^2(\Omega)}\|z_w\|_{L^2(\Omega)} + \|z_w\|_{L^2(\Omega)}.$$

Since

$$\|\hat{u} - \bar{u}\|_{L^2(\Omega)} = \|\theta(u - \bar{u})\|_{L^2(\Omega)} \leq \|u - \bar{u}\|_{L^2(\Omega)} < \varepsilon,$$

we conclude

$$\|z_{\hat{u},w}\|_{L^2(\Omega)} \leq (C_2\varepsilon + 1)\|z_w\|_{L^2(\Omega)} \quad \forall w \in L^2(\Omega). \tag{2.24}$$

An analogous estimate holds for $\|z_{\hat{u},v}\|_{L^2(\Omega)}$. Now, by the Schwarz inequality we obtain

$$\|w\|_{L^1(\Omega)} \leq \sqrt{|\Omega|}\|w\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}\|u - \bar{u}\|_{L^2(\Omega)} < \varepsilon\sqrt{|\Omega|},$$

and therefore

$$\frac{1}{\varepsilon\sqrt{|\Omega|}}\|w\|^2_{L^1(\Omega)} \leq \|w\|_{L^1(\Omega)}.$$

Inserting (2.16) into this inequality, we obtain

$$\frac{1}{C_3^2\varepsilon\sqrt{|\Omega|}}\|z_w\|^2_{L^2(\Omega)} \leq \|w\|_{L^1(\Omega)}. \tag{2.25}$$

Combining (2.23), (2.24), and (2.25) and using the Young inequality, we have

$$J(u) \geq J(\bar{u}) + \frac{\tau}{C_3^2\varepsilon\sqrt{|\Omega|}}\|z_w\|^2_{L^2(\Omega)} + \frac{3\delta}{8}\|z_v\|^2_{L^2(\Omega)}$$

$$-\frac{K_M(C_2\varepsilon + 1)^2}{2}\|z_w\|^2_{L^2(\Omega)} - K_M(C_2\varepsilon + 1)^2\|z_w\|_{L^2(\Omega)}\|z_v\|_{L^2(\Omega)}$$

$$\geq J(\bar{u}) + \frac{3\delta}{8}\|z_v\|^2_{L^2(\Omega)} - \frac{\delta}{8}\|z_v\|^2_{L^2(\Omega)}$$

$$+\left\{\frac{\tau}{C_3^2\varepsilon\sqrt{|\Omega|}} - \frac{K_M(C_2\varepsilon + 1)^2}{2} - \frac{2K_M^2(C_2\varepsilon + 1)^4}{\delta}\right\}\|z_w\|^2_{L^2(\Omega)}$$

with (2.22)

$$\geq J(\bar u)+\frac{\delta}{4}\|z_v\|^2_{L^2(\Omega)}+\frac{\delta}{4}\|z_w\|^2_{L^2(\Omega)}\geq J(\bar u)+\frac{\delta}{8}\|z_v+z_w\|^2_{L^2(\Omega)}=J(\bar u)+\frac{\delta}{8}\|z_{u-\bar u}\|^2_{L^2(\Omega)},$$

which concludes the proof. ∎

COROLLARY 2.8. *Under the assumptions of Theorem* 2.4*, there exists* $\varepsilon>0$ *such that*

$$J(\bar u)+\frac{\delta}{9}\|y_u-\bar y\|^2_{L^2(\Omega)}\leq J(u)\quad\forall u\in B_\varepsilon(\bar u)\cap\mathcal{U}_{ad}. \tag{2.26}$$

Proof. It is enough to prove the estimate

$$\frac{\sqrt{8}}{3}\|y_u-\bar y\|_{L^2(\Omega)}\leq\|z_{u-\bar u}\|_{L^2(\Omega)} \tag{2.27}$$

for all $\|u-\bar u\|_{L^2(\Omega)}<\varepsilon$, for a convenient $\varepsilon>0$, and then apply (2.13).

Let us set $z=y_u-\bar y-z_{u-\bar u}$. From the equations satisfied by y_u, $\bar y$, and $z_{u-\bar u}$ we obtain

$$Az+f(x,y_u)-f(x,\bar y)-\frac{\partial f}{\partial y}(x,\bar y)z_{u-\bar u}=0\quad\text{in }\Omega.$$

Making a second order Taylor expansion of $f(x,y_u)$ around $\bar y$, we deduce for some $\hat y=\bar y+\theta(y_u-\bar y)$, with $0\leq\theta(x)\leq 1$,

$$Az+\frac{\partial f}{\partial y}(x,\bar y)z+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,\hat y)(y_u-\bar y)^2=0\quad\text{in }\Omega.$$

Now, from (2.16) and taking into account (2.4) and the assumption (A1), we obtain

$$\|z\|_{L^2(\Omega)}\leq\frac{C_3C_{f,M}}{2}\|y_u-\bar y\|^2_{L^2(\Omega)}=C_4\|y_u-\bar y\|^2_{L^2(\Omega)}.$$

Hence,

$$\|y_u-\bar y\|_{L^2(\Omega)}\leq\|z_{u-\bar u}\|_{L^2(\Omega)}+\|z\|_{L^2(\Omega)}\leq\|z_{u-\bar u}\|_{L^2(\Omega)}+C_4\|y_u-\bar y\|^2_{L^2(\Omega)},$$

which implies

$$\left(1-C_4\|y_u-\bar y\|_{L^2(\Omega)}\right)\|y_u-\bar y\|_{L^2(\Omega)}\leq\|z_{u-\bar u}\|_{L^2(\Omega)}.$$

Using (2.14) and taking $\varepsilon\leq(3-\sqrt{8})/(3C_1C_4)$ such that (2.13) holds for this value, we deduce

$$1-C_4\|y_u-\bar y\|_{L^2(\Omega)}\geq 1-C_1C_4\|u-\bar u\|_{L^2(\Omega)}\geq 1-C_1C_4\varepsilon\geq\frac{\sqrt{8}}{3}.$$

The last two inequalities prove (2.27). ∎

We finish this section by proving that statement 2 of Theorem 2.3 does not hold for $\Lambda=0$. Indeed, let us assume that it holds. Then, a simple modification of the proof of Theorem 2.4 leads to the inequality

$$J(\bar u)+\frac{\nu}{2}\|u-\bar u\|^2_{L^2(\Omega)}\leq J(u)\quad\forall u\in B_\varepsilon(\bar u)\cap\mathcal{U}_{ad}, \tag{2.28}$$

for some $\nu > 0$ and $\varepsilon > 0$. Then, $\bar{u}$ is a solution of the problem

$$(\mathrm{P}_\nu) \qquad \min_{u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}} J(u) - \frac{\nu}{2} \int_\Omega (u - \bar{u})^2 \, dx.$$

The Hamiltonian of this control problem is given by

$$H(x, y, u, \varphi) = L(x, y) + \varphi(u - f(x, y)) - \frac{\nu}{2}(u - \bar{u}(x))^2.$$

From the Pontryagin principle we deduce

$$H(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \min_{t \in [\alpha, \beta]} H(x, \bar{y}(x), t, \bar{\varphi}(x)) \text{ for a.e. } x \in \Omega.$$

However, invoking (2.10), we obtain that this is a contradiction with the following facts that can be easily checked:

$$\begin{cases} \text{if } 0 < \bar{\varphi}(x) < \dfrac{\nu}{2}(\beta - \alpha) \Rightarrow H(x, \bar{y}(x), \beta, \bar{\varphi}(x)) < H(x, \bar{y}(x), \alpha, \bar{\varphi}(x)), \\ \text{if } 0 > \bar{\varphi}(x) > \dfrac{\nu}{2}(\alpha - \beta) \Rightarrow H(x, \bar{y}(x), \alpha, \bar{\varphi}(x)) < H(x, \bar{y}(x), \beta, \bar{\varphi}(x)). \end{cases}$$

3. A bang-bang-bang control problem.

In this section we will analyze the control problem

$$(\mathrm{P}_2) \qquad \begin{cases} \min I(u) = J(u) + \mu j(u) = \displaystyle\int_\Omega L(x, y_u(x)) \, dx + \mu \int_\Omega |u| \, dx, \\ \alpha \le u(x) \le \beta, \end{cases}$$

where y_u is the solution of the Dirichlet problem (2.1), $\mu > 0$, and $-\infty < \alpha < 0 < \beta < +\infty$. We also assume the hypotheses (A1)–(A3) introduced in section 2, and $\mathcal{U}_{ad}$ will stand for the set of feasible controls. The motivation to include the L^1 norm of the control in the cost functional is the following. In many cases, it is not desirable or not even possible to control the system from the whole domain Ω; we do not want or we cannot put controls at every point of domain. Instead we prefer to select a small domain ω where we put the controllers. The issue is to decide which is the most convenient domain ω to localize the controllers. The solution of (P_2) is sparse; the bigger μ is, the smaller the support ω of the optimal control is. Therefore, solving (P_2) for a convenient μ we discover the most convenient place ω, to put the controllers as well as the power of these controllers.

It is obvious that (P_2) has at least one solution $\bar{u}$. Moreover, using that I is the sum of a smooth function and a convex function, we deduce the existence of $\bar{\lambda} \in \partial j(\bar{u})$ such that $J'(\bar{u})(u - \bar{u}) + \langle \bar{\lambda}, u - \bar{u} \rangle \ge 0$ for every $u \in \mathcal{U}_{ad}$. Recall that j is Lipschitz and convex; then the generalized gradient (see [15]) and the subdifferential in the sense of convex analysis coincide. Now, from (2.5), we obtain the optimality system

$$(3.1) \qquad \begin{cases} A\bar{y} + f(x, \bar{y}) = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases}$$

$$(3.2) \qquad \begin{cases} A^* \bar{\varphi} + \dfrac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \dfrac{\partial L}{\partial y}(x, \bar{y}) & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma, \end{cases}$$

$$(3.3) \qquad \int_\Omega (\bar{\varphi}(x) + \mu \bar{\lambda}(x))(u(x) - \bar{u}(x)) \, dx \ge 0 \quad \forall u \in \mathcal{U}_{ad}.$$

From (3.3) we deduce the following properties for $\bar{u}$ and $\bar{\lambda}$.

THEOREM 3.1. *Let $\bar{u}$, $\bar{\varphi}$, and $\bar{\lambda}$ satisfy* (3.1)–(3.2)*; then the following relations hold*

$$|\bar{\varphi}(x)| < \mu \Rightarrow \bar{u}(x) = 0, \tag{3.4}$$
$$\bar{\varphi}(x) > +\mu \Rightarrow \bar{u}(x) = \alpha, \tag{3.5}$$
$$\bar{\varphi}(x) < -\mu \Rightarrow \bar{u}(x) = \beta, \tag{3.6}$$
$$\bar{\varphi}(x) = +\mu \Rightarrow \bar{u}(x) \leq 0, \tag{3.7}$$
$$\bar{\varphi}(x) = -\mu \Rightarrow \bar{u}(x) \geq 0, \tag{3.8}$$
$$\bar{\lambda}(x) = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\bar{\varphi}(x)\Big). \tag{3.9}$$

Moreover, from the last representation formula it follows that $\bar{\lambda} \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\bar{\lambda}$ is unique for any fixed local minimum $\bar{u}$.

Proof. First, we recall that the fact $\bar{\lambda} \in \partial j(\bar{u})$ implies that

$$\bar{\lambda}(x) \begin{cases} = & +1 & \text{if } \bar{u}(x) > 0, \\ = & -1 & \text{if } \bar{u}(x) < 0 \\ \in & [-1,+1] & \text{if } \bar{u}(x) = 0. \end{cases} \quad \text{a.e.,} \tag{3.10}$$

Let us consider the following different cases.

1. $\bar{\varphi}(x) > \mu$. Then, we have that $\bar{\varphi}(x) + \mu\bar{\lambda}(x) > 0$. Hence, (3.3) implies that $\bar{u}(x) = \alpha < 0$, and (3.5) holds. Finally, (3.10) leads to $\bar{\lambda}(x) = -1$, which proves that (3.9) is fulfilled in this case.

2. $\bar{\varphi}(x) < -\mu$. Then we can argue analogously to conclude that (3.6) and (3.9) hold.

3. $|\bar{\varphi}(x)| < \mu$. Let us check that $\bar{u}(x) = 0$. Indeed, if, for instance, $\bar{u}(x) > 0$, then (3.10) implies that $\bar{\lambda}(x) = 1$; consequently $\bar{\varphi}(x) + \mu\bar{\lambda}(x) > 0$. Hence, according to (3.3), $\bar{u}(x) = \alpha < 0$, which contradicts the assumed positivity. Analogously we can prove that $\bar{u}(x) < 0$ leads to a contradiction; therefore, $\bar{u}(x) = 0$ and (3.4) is proved. Using once again (3.3) and the fact that $\alpha < \bar{u}(x) = 0 < \beta$, we get that $\bar{\varphi}(x) + \mu\bar{\lambda}(x) = 0$, and therefore (3.9) is fulfilled in this case, too.

4. $\bar{\varphi}(x) = +\mu$. If $\bar{\lambda}(x) > -1$, then $\bar{\varphi}(x) + \mu\bar{\lambda}(x) > 0$, and from (3.3) we get that $\bar{u}(x) = \alpha < 0$, which contradicts (3.10). Therefore, $\bar{\lambda}(x) = -1$, and (3.9) holds. Moreover, according to (3.10) we have that $\bar{u}(x) \leq 0$, which proves (3.7).

5. $\bar{\varphi}(x) = -\mu$. Arguing as above we prove that (3.8) and (3.9) are also satisfied in this case. ∎

Remark 3.2. In most of the cases, the identity $|\bar{\varphi}(x)| = \mu$ is satisfied in a set of zero Lebesgue measure; then (3.4)–(3.6) imply that $\bar{u}(x) \in \{\alpha, \beta, 0\}$ almost everywhere, which justifies our denomination of bang-bang-bang control.

To write the necessary second order conditions we follow [5] and introduce the cone of critical directions:

$$C_{\bar{u}} = \{v \in L^2(\Omega) \text{ satisfying (3.11) and } J'(\bar{u})\,v + \mu\, j'(\bar{u}; v) = 0\}$$

with

$$\begin{cases} v(x) \geq 0 & \text{if } \bar{u}(x) = \alpha, \\ v(x) \leq 0 & \text{if } \bar{u}(x) = \beta, \end{cases} \tag{3.11}$$

and

$$j'(\bar u; v) = \lim_{\rho\searrow 0}\frac{j(\bar u+\rho v)-j(\bar u)}{\rho} = \int_{\Omega_+} v(x)\,dx - \int_{\Omega_-} v(x)\,dx + \int_{\Omega_0}|v(x)|\,dx, \tag{3.12}$$

where

$$\Omega_+ = \{x\in\Omega : \bar u(x) > 0\},\quad \Omega_- = \{x\in\Omega : \bar u(x) < 0\},\ \text{ and }\ \Omega_0 = \{x\in\Omega : \bar u(x) = 0\}.$$

As proved in [5, Theorem 3.6], if $\bar u$ is a local minimum of (P$_2$), then $J''(\bar u)\,v^2 \ge 0$ for every $v \in C_{\bar u}$. However, we observe that this condition is empty in many cases because $C_{\bar u}$ is frequently reduced to $\{0\}$. This is a consequence of the following proposition that characterizes $C_{\bar u}$.

PROPOSITION 3.3. *$C_{\bar u}$ is the the cone formed by the elements $v \in L^2(\Omega)$ satisfying*

$$v(x)\begin{cases} \ge 0 & \text{if } [\bar u(x) = \alpha \text{ and } \bar\varphi(x) = +\mu] \text{ or } [\bar u(x) = 0 \text{ and } \bar\varphi(x) = -\mu],\\ \le 0 & \text{if } [\bar u(x) = \beta \text{ and } \bar\varphi(x) = -\mu] \text{ or } [\bar u(x) = 0 \text{ and } \bar\varphi(x) = +\mu],\\ = 0 & \text{if } |\bar\varphi(x)| \ne \mu.\end{cases} \tag{3.13}$$

Proof. Given $v \in L^2(\Omega)$, from (3.4)–(3.8) and (3.12) we deduce that

$$J'(\bar u)v + \mu j'(\bar u; v) = \int_\Omega g(x)\,dx,$$

where g is defined by

$$g(x) = \begin{cases} (\bar\varphi(x)-\mu)v(x) & \text{if } \bar\varphi(x) > +\mu,\\ (\bar\varphi(x)+\mu)v(x) & \text{if } \bar\varphi(x) < -\mu,\\ \bar\varphi(x)v(x)+\mu|v(x)| & \text{if } |\bar\varphi(x)| < +\mu,\\ \mu(v(x)+|v(x)|) & \text{if } \bar\varphi(x) = +\mu \text{ and } \bar u(x) = 0,\\ \mu(-v(x)+|v(x)|) & \text{if } \bar\varphi(x) = -\mu \text{ and } \bar u(x) = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Now, if v satisfies (3.11), then $g(x) \ge 0$ for almost every $x \in \Omega$. Therefore, $v \in C_{\bar u}$ if and only if $g(x) = 0$ for almost every $x \in \Omega$, which is equivalent to (3.13). ∎

As for the problem (P$_1$), we cannot base a sufficient second order condition on the cone $C_{\bar u}$. Recall also Example 2.1. We extend the cone of critical directions as follows. Given $\tau \ge 0$, we define

$$C^\tau_{\bar u} = \{v \in L^2(\Omega) \text{ satisfying (3.11) and } J'(\bar u)\,v + \mu\, j'(\bar u; v) \le \tau\|z_v\|_{L^2(\Omega)}\}.$$

Analogously to the problem (P$_1$), let us take a look at the Tikhonov regularization of (P$_2$). For any $\Lambda > 0$, we consider the problem

$$(\mathrm{P}_{2,\Lambda})\qquad \min_{u\in\mathcal U_{ad}} I_\Lambda(u) = J_\Lambda(u) + \mu j(u).$$

This problem was studied in [5]. This problem has at least one solution, and the first order optimality conditions are given by (3.1), (3.2), and

$$\int_\Omega (\bar\varphi(x) + \Lambda\bar u(x) + \mu\bar\lambda(x))(u(x)-\bar u(x))\,dx \ge 0 \quad \forall u \in \mathcal U_{ad}, \tag{3.14}$$

where $\bar\lambda \in \partial j(\bar u)$; see [5, Theorem 3.1]. Now, we have the following results analogous to Theorems 2.2 and 2.3.

THEOREM 3.4. *Let $\bar{u} \in \mathcal{U}_{ad}$ satisfy* (3.1), (3.2), (3.14), *and the second order condition*

$$J''_\Lambda(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}.$$

Then, there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$I_\Lambda(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \leq I_\Lambda(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}.$$

THEOREM 3.5. *Let $\bar{u} \in \mathcal{U}_{ad}$ satisfy the first order optimality conditions given by* (3.1), (3.2), *and* (3.14). *Then, the following assumptions are equivalent:*

1. $J''_\Lambda(\bar{u})v^2 > 0 \ \forall v \in C_{\bar{u}} \setminus \{0\}$.
2. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \geq \nu\|v\|^2_{L^2(\Omega)} \ \forall v \in C^\tau_{\bar{u}}$.
3. $\exists \nu > 0$ *and* $\tau > 0$ *s.t.* $J''_\Lambda(\bar{u})v^2 \geq \nu\|z_v\|^2_{L^2(\Omega)} \ \forall v \in C^\tau_{\bar{u}}$,

where $z_v = G'(\bar{u})v$.

Theorem 3.4 was proved in [5, Theorem 3.7]. The only delicate point in the proof of Theorem 3.5 is the implication $1 \Rightarrow 2$. Indeed, since $C_{\bar{u}} \subset C^\tau_{\bar{u}}$ for every $\tau > 0$, then it is obvious that $2 \Rightarrow 3 \Rightarrow 1$, which holds even for $\Lambda = 0$. The proof of $1 \Rightarrow 2$ can be found in [5, Theorem 3.8], and it requires Λ to be strictly positive. Looking at the precedent two theorems, we will take a decision on the convenient formulation for the sufficient second order condition for problem (P_2). As in the case of problem (P_1), condition 1 is not convenient because of Example 2.1 and the fact that $C_{\bar{u}}$ can be reduced to $\{0\}$ as a consequence of Proposition 3.3. Condition 2 does not hold for $\Lambda = 0$ as we will prove later. Therefore, condition 3 remains. The next theorem states the sufficiency of this condition.

THEOREM 3.6. *Let us assume that $\bar{u}$ is a feasible control for problem* (P_2) *satisfying the first order optimality conditions* (3.1), (3.2), *and* (3.3). *We also suppose that there exist $\delta > 0$ and $\tau > 0$ such that*

$$J''(\bar{u})v^2 \geq \delta\|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}, \tag{3.15}$$

where $z_v = G'(\bar{u})v$ is the solution of (2.2) *for $y = \bar{y}$. Then, there exists $\varepsilon > 0$ such that*

$$I(\bar{u}) + \frac{\delta}{4}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} \leq I(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}, \tag{3.16}$$

with $z_{u-\bar{u}} = G'(\bar{u})(u - \bar{u})$.

Proof. From Lemma 2.7 we deduce the existence of $\varepsilon_1 > 0$ such that

$$|[J''(\bar{u}) - J''(u)]v^2| \leq \frac{\delta}{2}\|z_v\|^2_{L^2(\Omega)} \ \forall u \in B_{\varepsilon_1}(\bar{u}) \cap \mathcal{U}_{ad} \text{ and } v \in L^2(\Omega). \tag{3.17}$$

From (2.2) we infer the existence of a constant $C_4 > 0$ such that

$$\|z_v\|_{L^2(\Omega)} \leq C_4\|v\|_{L^2(\Omega)} \quad \forall v \in L^2(\Omega).$$

Now, for τ fixed in the statement of theorem, we take

$$\varepsilon = \min\left\{\varepsilon_1, \frac{2\tau}{(\delta + K_M)C_4}\right\},$$

where $\delta > 0$ is also given in the theorem and K_M is defined in (2.19). Then, if $\|u-\bar{u}\|_{L^2(\Omega)} \le \varepsilon$ and $u \in \mathcal{U}_{ad}$,

$$\|z_{u-\bar{u}}\|_{L^2(\Omega)} \le C_4\|u-\bar{u}\|_{L^2(\Omega)} \le C_4\varepsilon \le \frac{2\tau}{\delta + K_M}. \tag{3.18}$$

Let $u \in \mathcal{U}_{ad} \cap B_\varepsilon(\bar{u})$; then we will prove that (3.16) holds. We will distinguish two cases.

Case I. $(u-\bar{u}) \in C_{\bar{u}}^\tau$. We recall that the convexity of j implies that $j(u)-j(\bar{u}) \ge j'(\bar{u}; u-\bar{u})$. Moreover, since $u-\bar{u}$ satisfies (3.11), then $J'(\bar{u})(u-\bar{u})+\mu j'(\bar{u}; u-\bar{u}) \ge 0$. Hence

$$\begin{aligned} I(u)-I(\bar{u}) &\ge J(u)-J(\bar{u})+\mu j'(\bar{u}; u-\bar{u}) = J'(\bar{u})(u-\bar{u})+\mu j'(\bar{u}; u-\bar{u})+\frac{1}{2}J''(\hat{u})(u-\bar{u})^2 \\ &\ge \frac{1}{2}J''(\bar{u})(u-\bar{u})^2 + \frac{1}{2}[J''(\hat{u}) - J''(\bar{u})](u-\bar{u})^2 \end{aligned}$$

for some point $\hat{u} = \bar{u} + \theta(u-\bar{u}) \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}$. Then, we invoke (3.15) and (3.17) to get

$$I(u) - I(\bar{u}) \ge \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} - \frac{\delta}{4}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} = \frac{\delta}{4}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)},$$

which proves (3.16).

Case II. $(u-\bar{u}) \notin C_{\bar{u}}^\tau$. Since $u-\bar{u}$ satisfies (3.11), we have that $J'(\bar{u})(u-\bar{u}) + \mu j'(\bar{u}; u-\bar{u}) > \tau\|z_{u-\bar{u}}\|_{L^2(\Omega)}$. Arguing as above and using (2.19) it follows that

$$\begin{aligned} I(u) - I(\bar{u}) &\ge J'(\bar{u})(u-\bar{u}) + \mu j'(\bar{u}; u-\bar{u}) + \frac{1}{2}J''(\hat{u})(u-\bar{u})^2 \\ &\ge \tau\|z_{u-\bar{u}}\|_{L^2(\Omega)} - \frac{K_M}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, from (3.18),

$$I(u) - I(\bar{u}) \ge \frac{\delta + K_M}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} - \frac{K_M}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} = \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)},$$

which concludes the proof. ∎

An obvious modification of the proof of Corollary 2.8 leads to the analogous result.

COROLLARY 3.7. *Under the assumptions of Theorem* 2.4*, there exists* $\varepsilon > 0$ *such that*

$$I(\bar{u}) + \frac{\delta}{5}\|y_u - \bar{y}\|^2_{L^2(\Omega)} \le I(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}. \tag{3.19}$$

We conclude this section by showing that condition 2 from Theorem 3.5 is never fulfilled in case $\Lambda = 0$. To this end, we first observe that if it holds, then an obvious modification of the proof of Theorem 3.6 leads to the existence of $\varepsilon > 0$ and $\nu > 0$ such that

$$I(\bar{u}) + \frac{\nu}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le I(u) \quad \forall u \in B_\varepsilon(\bar{u}) \cap \mathcal{U}_{ad}.$$

This inequality implies that $\bar{u}$ is the solution of the problem

$$(\mathrm{P}_\nu) \qquad \min_{u\in B_\varepsilon(\bar u)\cap \mathcal{U}_{ad}} I(u) - \frac{\nu}{2}\int_\Omega (u-\bar u)^2\,dx.$$

The Hamiltonian of this control problem is given by

$$H(x,y,u,\varphi) = L(x,y) + \mu|u| + \varphi(u - f(x,y)) - \frac{\nu}{2}(u-\bar u(x))^2.$$

From the Pontryagin principle we deduce

$$H(x,\bar y(x),\bar u(x),\bar\varphi(x)) = \min_{t\in[\alpha,\beta]} H(x,\bar y(x),t,\bar\varphi(x)) \text{ for almost all } x\in\Omega.$$

However, this contradicts the following inequalities that can be easily checked:

$$\begin{aligned}
&\text{if } \mu < \bar\varphi(x) < \mu + \frac{\nu}{2}|\alpha| \Rightarrow H(x,\bar y(x),0,\bar\varphi(x)) < H(x,\bar y(x),\alpha,\bar\varphi(x)),\\
&\text{if } \mu < -\bar\varphi(x) < \mu + \frac{\nu}{2}\beta \Rightarrow H(x,\bar y(x),0,\bar\varphi(x)) < H(x,\bar y(x),\beta,\bar\varphi(x)),\\
&\text{if } 0 < \bar\varphi(x) < \mu \text{ and } \mu - \bar\varphi(x) < \frac{\nu}{2}|\alpha| \Rightarrow H(x,\bar y(x),\alpha,\bar\varphi(x)) < H(x,\bar y(x),0,\bar\varphi(x)),\\
&\text{if } -\mu < \bar\varphi(x) < 0 \text{ and } \mu + \bar\varphi(x) < \frac{\nu}{2}\beta \Rightarrow H(x,\bar y(x),\beta,\bar\varphi(x)) < H(x,\bar y(x),0,\bar\varphi(x)).
\end{aligned}$$

4. Final remarks. The reader should notice that the approach followed to define the extended cone $C^\tau_{\bar u}$ in section 2 is different from the one of section 3. Indeed, in section 2 we could consider the cone

$$E^\tau_{\bar u} = \{v\in L^2(\Omega) \text{ satisfying (3.11) and } J'(\bar u)v \le \tau\|z_v\|_{L^2(\Omega)}\}.$$

Observe that (2.10) and (3.11) imply that $J'(\bar u)v \le \tau\|z_v\|_{L^2(\Omega)} \ge 0$. The cones $E^\tau_{\bar u}$ and $C^\tau_{\bar u}$ are different, and they both contain $C_{\bar u}$. Furthermore, Theorems 2.3 and 2.4 remain valid if we change $C^\tau_{\bar u}$ by $E^\tau_{\bar u}$. Additionally, we could consider a third cone

$$S^\tau_{\bar u} = \{v\in L^2(\Omega) \text{ satisfying (3.11) and } J'(\bar u)v \le \tau\|v\|_{L^2(\Omega)}\},$$

and Theorems 2.3 and 2.4 still would hold. However, the cones $S^\tau_{\bar u}$ are bigger than $C^\tau_{\bar u}$ and $E^\tau_{\bar u}$. More precisely, it is immediate to check that for any $\tau > 0$ there exists $\tau_0 > 0$ such that $C^{\tau'}_{\bar u} \subset S^\tau_{\bar u}$ and $E^{\tau'}_{\bar u} \subset S^\tau_{\bar u}$ for every $\tau' \le \tau_0$. If we replace $C^\tau_{\bar u}$ by $E^\tau_{\bar u}$ or $S^\tau_{\bar u}$ in the second order condition (2.12), then the proof of Theorem 2.4 can be simplified following the same approach of the proof of Theorem 3.5. In particular, the estimate (2.16) is not necessary, which is very important if we want to extend our proofs to the control of parabolic equations. Indeed, the estimate (2.16) does not hold in the parabolic case, and consequently Theorem 2.4 fails. There are two ways to overcome this difficulty. The first and most interesting way is just to replace $C^\tau_{\bar u}$ by $E^\tau_{\bar u}$ or $S^\tau_{\bar u}$ in the condition (2.12). The second alternative is to maintain (2.12) in the same form and to conclude the strict local optimality of $\bar u$ in an L^∞ ball $B_\varepsilon(\bar u)$, which is less interesting from the application point of view.

Unlike what happens for the parabolic case, Theorem 2.4 is also valid for the case of Neumann controls because the $L^2(\Omega)$ estimates of z_v in terms of $\|v\|_{L^1(\Gamma)}$ hold. Of course, the alternative of considering the cones $E^\tau_{\bar u}$ or $S^\tau_{\bar u}$ is also valid.

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