



Global Existence and Exponential Decay to Equilibrium for DLSS-Type Equations

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Abstract

In this paper, we deal with two logarithmic fourth order differential equations: the extended one-dimensional DLSS equation and its multi-dimensional analog. We show the global existence of solution in critical spaces, its convergence to equilibrium and the gain of spatial analyticity for these two equations in a unified way.

Keywords Derrida–Lebowitz–Speer–Spohn equation · Wiener space · Existence of solution · Asymptotic behavior · Analyticity

Mathematics Subject Classification 35K30 · 35B40 · 35Q40 · 42B37

1 Introduction

Fourth order differential equations appear in many applications such as thin films (see for instance the works by Constantin et al. [15] and Bertozzi and Pugh [3]), crystal surface models (see the works by Krug et al. [31] and Marzuola and Weare [33]) and quantum semiconductors (see the papers by Ancona [2] and Gasser et al. [22]). In particular, Derrida et al. [17] derived the following logarithmic fourth order equation (DLSS in short) as a model of interface fluctuations in a certain spin system

$$w_t + \partial_x^2(w \partial_x^2(\log w)) = 0. \quad (1)$$

This equation is a nonlinear parabolic equation. Although the theory of second-order diffusion equations is well known, there are few mathematical results for higher-order equations, and even less results addressing high-order equation in the case of several spatial dimensions. In

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this paper we first consider the multi-dimensional DLSS equation taking the form [2,16,22],

$$w_t + \sum_{i,j=1}^d \partial_i \partial_j (w \partial_i \partial_j (\log w)) = 0, \quad (2)$$

where the dimension satisfies $d = 1, 2$ or $d = 3$ and the spatial variable x lies in the d -dimensional flat torus, or equivalently, on $[-\pi, \pi]^d$ with the periodic boundary conditions. We also study the extended 1D DLSS equation derived by Bordenave et al. [5]

$$w_t - \frac{\mu \Gamma}{3} \partial_x^3 w - \mu \Gamma \partial_x (w \partial_x^2 \log w) = \epsilon (2\Gamma^2 - 2\Gamma) \partial_x^2 (w \partial_x^2 (\log w)), \quad (3)$$

where $0 \leq \Gamma \leq \frac{1}{4}$ and $-1 \leq \mu \leq 1$. We note that (3) is reduced to the 1D (1) for $\mu = 0$, $\epsilon = \frac{8}{3}$, and $\Gamma = \frac{1}{4}$, while (3) becomes [18, eq. (10)] for $\epsilon = 0$.

Besides the applications to quantum semiconductors and spin systems mentioned above, (1) is also interesting due to the fact that, as shown by Gianazza et al. [23], it is a Wasserstein gradient flow for the Fisher information

$$\mathcal{F}(u) = - \int \Delta u \log(u) dx$$

and so (1) can be written in the form

$$u_t = \nabla \cdot \left(u \nabla \frac{\delta \mathcal{F}}{\delta u} \right).$$

This somehow establishes a link between the DLSS equation and the heat equation [26]. Indeed, the heat equation is a Wasserstein gradient flow for the entropy

$$\mathcal{H}(u) = \int u \log(u) dx,$$

and its entropy production is the widely appearing Fisher information.

The available literature studying (1) and (2) is large. First, Bleher et al. [4] proved that (1) has local classical solutions starting from $H^1(\mathbb{T})$ positive initial data. These solutions remain smooth as long as the solution remains positive. We note that, although positivity preservation is a basic property for second order diffusions, it is no longer true for higher order diffusions. In fact, we can see numerical simulations violating the maximum principle in [26]. This leaves the door open to possible singularity formation as u approaches zero at some point. There are some papers showing the global existence of *non-negative* weak solutions. These results are mainly based on some appropriate Lyapunov functionals. In particular, both the entropy \mathcal{H} and Fisher information \mathcal{F} decay and can be used to obtain a priori estimates. For (1), this approach was exploited by Jüngel and Pinnau [28] (see also Pia Gualdani et al. [34]) where the authors used the functional

$$\mathcal{G}(t) = \int u(x, t) - \log(u(x, t)) dx.$$

In the multi-dimensional case (2), the global existence of *non-negative* weak solution was obtained by Gianazza et al. [23] and Jüngel and Matthes [27].

In addition to the existence results, there are also some works studying the decay to the equilibrium. In the one-dimensional case, Cáceres et al. [11], Dolbeault et al. [19], Jüngel and Violet [30] and Jüngel and Matthes [27] proved the decay of appropriate functionals in the case of periodic conditions. The case of Dirichlet and Neumann boundary conditions in

one-dimension has been studied by Jüngel and Pinnau [28] and Jüngel and Toscani [29]. In the multi-dimensional setting, the reader is referred to [23,27]. To the best of our knowledge, the only decay result in the multi-dimensional setting with periodic boundary conditions is [27], where the authors proved decay of L^p -norm-like functionals for

$$1 \leq p < \frac{(\sqrt{d} + 1)^2}{d + 2} < \infty. \quad (4)$$

The uniqueness question has also been studied. Fischer [20] proved that weak solutions in a certain class are unique. Remarkably, this class contains the weak solutions constructed in [27]. We would like to remark that the question of uniqueness is rather subtle. As it was pointed out in [26], there are non-negative explicit functions that are steady solutions of (2). These non-negative functions lead, after invoking the previously mentioned existence results, to a time-dependent weak solution converging to the homogeneous steady state. In other words, starting from smooth initial data we can have two different weak solutions.

The results for equation (3) are more scarce. This equation was derived and studied by Bordenave et al. [5]. In particular, the authors found a number of Lyapunov functionals and used them to study the asymptotic behavior.

Some other related works are [7–9] (and the references therein) where the DLSS equation and some generalization are studied.

In this paper, we deal with (2) and (3) in a unified way. Assuming that the initial data satisfy certain *explicit* size restrictions in the Wiener algebra \mathcal{A}_0 (defined in (5)), we prove the global existence of solutions, its convergence to the steady state in L^∞ and instantaneous gain of analyticity. We will present our results in Sect. 2. Then, we prove Theorem 1 and 2 in Sects. 3 and 4, respectively.

We also want to mention that our approach has the following advantages:

- We would like to emphasize that the smallness conditions of initial data are given *explicitly* in terms of the parameters in the equation.
- If w is a solution to (2), then the rescaled function $w_\lambda(t, x) = w(\lambda^4 t, \lambda x)$ is another solution for every $\lambda > 0$. Some spaces with scaling-invariant norm are

$$L^\infty, H^{d/2}, \mathcal{A}_0, \text{ etc.}$$

Thus, our results involve a critical space for (2). We should compare our global existence results with the one contained in [4] where the size constrain is at the level of $\dot{H}^1(\mathbb{T}^1)$. In particular, the initial data that we consider can be arbitrarily large in the space \dot{H}^1 while still leading to a global solution. Moreover, to the best of our knowledge, our results are the first results showing analyticity (not merely C^∞ as in [4]) of solutions of (2) and (3).

- We prove the exponential decay to equilibrium in the L^∞ norm which generalizes the result in [27]. In particular, the decay (4) is extended to $1 \leq p \leq \infty$.
- Our approach is very flexible and can be implemented to other high order semi-linear or quasi-linear equations in an arbitrary dimension d . We refer the reader to [13,14,21] for a free boundary problem arising in the dynamics of a fluid in a porous medium, to [25] for a nonlocal quasilinear diffusion, to [10] for the doubly parabolic Keller-Segel system, to [6] for thin film equations, and to [1,24,32] for the evolution of crystal surfaces. These results are obtained by mainly observing that the Wiener algebra norm is a Lyapunov functional regardless of the order of the diffusion, the local/nonlocal character of nonlinear terms and the dimension d .

2 Main Results

2.1 Notation and Definitions

Before stating our results, we fix some notation and introduce the functional spaces that we will use.

The spatial derivatives are denoted by

$$\begin{aligned}\partial_i f &= f_{,i}, \\ \partial_i \partial_j f &= f_{,ij},\end{aligned}$$

and so on, where we also used Einstein convention for the repeated indices. In particular,

$$\Delta f = f_{,ii}$$

and

$$\Delta^2 f = f_{,ijij}.$$

In the one dimensional case, we also write

$$f_x = f_{,1}.$$

Similarly, the time derivative is denoted as

$$f_t = \partial_t f.$$

For $n \in \mathbb{N}$ we denote by

$$W^{n,p} = \{u \in L^p \text{ such that } \|u\|_{L^p}^p + \|\partial_x^n u\|_{L^p}^p < \infty\}$$

the standard L^p -based Sobolev spaces. Then we define the norm as

$$\|u\|_{W^{n,p}}^p := \|u\|_{L^p}^p + \|\partial_x^n u\|_{L^p}^p.$$

When $p = 2$, we use $W^{n,2} = H^n$.

The k th Fourier coefficients of a 2π -periodic function on \mathbb{T}^d are

$$\widehat{u}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-ik \cdot x} dx,$$

and the Fourier series expansion of u is given by

$$u(x) = \sum_{k \in \mathbb{Z}^d} \widehat{u}(k) e^{ik \cdot x}.$$

Using this, we define the Wiener spaces: for $s \geq 0$

$$\mathcal{A}_s = \left\{ u \in L^1(\mathbb{T}^d) \text{ such that } \sum_{k \in \mathbb{Z}^d} |k|^s |\widehat{u}(k)| < \infty \right\}. \quad (5)$$

Then we define the following norm in this space

$$|u|_s := \sum_{k \in \mathbb{Z}^d} |k|^s |\widehat{u}(k)|$$

We note that \mathcal{A}_0 is a Banach algebra. Moreover,

$$\mathcal{A}_s \subset C^s \subset H^s.$$

Being ∂ a differential operator of order 1 we also have the following properties

$$\begin{aligned} \left| \partial^l u \right|_0 &\leq |u|_l, \\ \left| \partial^l u \partial^{l'} u \right|_0 &\leq |u|_l |u|_{l'}, \\ |u|_l &\leq |u|_{l'} \quad \text{if } l' \geq l > 0. \end{aligned} \quad (6)$$

We finally have the following interpolation relationships:

$$|u|_s \leq |u|_0^{1-\theta} |u|_r^\theta \quad \text{for all } 0 \leq s \leq r \text{ with } \theta = \frac{s}{r}. \quad (7)$$

Let X be a Banach space. Then, $L_T^p X$ denotes the Banach set of Bochner measurable functions f from $(0, T)$ to X such that

$$\|f(t)\|_X \in L^p(0, T)$$

endowed with the norm

$$\left(\int_0^T \|f(\cdot, t)\|_X^p dt \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

or

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \|f(\cdot, t)\|_X \quad \text{for } p = \infty.$$

2.2 Setting

We assume that the spatial variable x lies in the d -dimensional flat torus $[-\pi, \pi]^d$ with the periodic boundary conditions. Together with (2) and (3), we have to consider *non-negative* initial data $w(x, 0) = w_0(x)$. We note that both (2) and (3) preserve the mean $\langle w \rangle$:

$$\langle w(t) \rangle = \int_{\mathbb{T}^d} w(t, x) dx = \int_{\mathbb{T}^d} w_0(x) dx = \langle w_0 \rangle > 0$$

In this setting, this mean will then correspond to the steady state. Without loss of generality, we assume $\langle w_0 \rangle = 1$.

2.3 The Multi-dimensional DLSS Equation

We now reformulate (2) in a new variable which is more convenient to show our results. Let $u = w - 1$. Then, u satisfies

$$u_t + ((u + 1)(\log(u + 1)))_{,ij} = 0, \quad 1 \leq i, j \leq d, \quad (8)$$

with initial data $u(x, 0) = u_0(x) = w_0(x) - 1$. We note that

$$\langle u(t) \rangle = \int_{\mathbb{T}^d} u(t, x) dx = \int_{\mathbb{T}^d} u_0(x) dx = \langle u_0 \rangle = 0.$$

This property will be used several times in proving our results.

Definition 1 We say that $u \in L_T^\infty L^\infty \cap L_T^2 H^1 \cap L_T^1 W^{2,1}$ is a strong solution of (8) if

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^d} \partial_t \phi(t, x) u(t, x) dx dt - \int_{\mathbb{T}^d} \phi(0, x) u_0(x) dx \\ & + \int_0^T \int_{\mathbb{T}^d} \log(u(t, x) + 1)_{,ij} (u(t, x) + 1) \phi_{,ij}(t, x) dx dt = 0 \end{aligned} \quad (9)$$

for all $\phi \in C^\infty([0, \infty) \times \mathbb{T}^d)$.

We note

$$\log(u + 1)_{,ij} (u + 1) = u_{,ij} - \frac{u_{,i} u_{,j}}{1 + u}.$$

From this, the regularity condition $u \in L_T^\infty L^\infty \cap L_T^2 H^1 \cap L_T^1 W^{2,1}$ seems *minimal* to define (9) by the following reasons:

- $u \in L_T^1 W^{2,1}$ is required to define $u_{,ij}$ weakly,
- $u \in L_T^\infty L^\infty$, which will be sufficiently small, to make $1 + u > 0$;
- $u \in L_T^2 H^1$ to define $u_{,i} u_{,j}$ weakly.

This required regularity of u will be achieved by taking a sufficiently small $u_0 \in \mathcal{A}_0$. The smallness condition of u is defined explicitly by the following the rational function:

$$P_1(z) = \frac{4z}{1-z} + \frac{5z^2}{(1-z)^2} + \frac{4z^3}{(1-z)^3}. \quad (10)$$

Theorem 1 Let $u_0 \in \mathcal{A}_0$ be a zero mean function that satisfies $|u_0|_0 < 1$ and consider $P_1(z)$ as in (10). Assume that the initial data satisfies

$$d^2 P_1(|u_0|_0) < 1.$$

Then there is a solution

$$u \in L^\infty([0, T]; L^\infty) \cap L^p\left(0, T; \mathcal{A}_{\frac{4}{p}}\right), \quad 1 < p < \infty,$$

of (8) for all $T > 0$. Moreover, u converges uniformly to 0 exponentially in time:

$$\|u(t)\|_{L^\infty} \leq |u_0|_0 e^{-(1-d^2 P_1(|u_0|_0))t}.$$

Finally, u becomes instantaneously spatial analytic with increasing radius of analyticity, namely,

$$e^{\sigma t |k|} \widehat{u}(t, k) \in L^\infty(0, T; \ell^1)$$

for a sufficiently small $\sigma > 0$ such that $1 - \sigma - d^2 P_1(|u_0|_0) > 0$.

2.4 The Extended 1D DLSS Equation

As before, we define $u = w - 1$. Then, u satisfies the following equation:

$$\begin{aligned} u_t - \frac{\mu \Gamma}{3} u_{xxx} - \mu \Gamma ((u + 1)(\log(u + 1))_{xx})_x \\ = -\epsilon (2\Gamma - 2\Gamma^2) ((u + 1)(\log(u + 1))_{xx})_{xx}. \end{aligned} \quad (11)$$

Definition 2 We say that $u \in L_T^\infty L^\infty \cap L_T^2 H^1 \cap L_T^1 W^{2,1}$ is a strong solution of (11) if

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}} \partial_t \phi(t, x) u(t, x) dx dt + \frac{\mu \Gamma}{3} \int_0^T \int_{\mathbb{T}} u(t, x) \phi_{xxx}(t, x) dx dt \\ & - \int_{\mathbb{T}} \phi(0, x) u_0(x) dx \\ & + \mu \Gamma \int_0^T \int_{\mathbb{T}} (\log(u(t, x) + 1))_{xx} (1 + u(t, x)) \phi_x(t, x) dx dt \\ & + \epsilon (2\Gamma^2 - 2\Gamma) \int_0^T \int_{\mathbb{T}} (\log(u(t, x) + 1))_{xx} (1 + u(t, x)) \phi_{xx}(t, x) dx dt = 0 \end{aligned} \quad (12)$$

for all $\phi \in C^\infty([0, \infty) \times \mathbb{T}^d)$.

We then follow the arguments used for (8) to state the following result. Let

$$P_2(z) = \frac{z}{1-z} + \frac{z^2}{(1-z)^2}. \quad (13)$$

Theorem 2 Let $u_0 \in \mathcal{A}_0$ be a zero mean function that satisfies $|u_0|_0 < 1$ and consider $P_j(z)$, $j = 1, 2$, as in (10) and (13). Assume that the initial data satisfies

$$\mu \Gamma P_2(|u_0|_0) + \epsilon (2\Gamma^2 - 2\Gamma) P_1(|u_0|_0) < \epsilon (2\Gamma^2 - 2\Gamma).$$

Then there is a solution

$$u \in L^\infty([0, T]; L^\infty) \cap L^p\left(0, T; \mathcal{A}_{\frac{4}{p}}\right), \quad 1 < p < \infty,$$

of (11) for all $T > 0$. Moreover, u converges uniformly to 0 exponentially in time

$$\|u(t)\|_{L^\infty} \leq |u_0|_0 e^{-(\epsilon(2\Gamma^2 - 2\Gamma) - \mu \Gamma P_2(|u_0|_0) - \epsilon(2\Gamma^2 - 2\Gamma) P_1(|u_0|_0))t}.$$

Finally, u becomes instantaneously spatial analytic with increasing radius of analyticity, namely

$$e^{\sigma t |k|} \widehat{u}(t, k) \in L^\infty(0, T; \ell^1)$$

for a sufficiently small $\sigma > 0$ such that $\epsilon(2\Gamma^2 - 2\Gamma) - \sigma - \mu \Gamma P_2(|u_0|_0) - \epsilon(2\Gamma^2 - 2\Gamma) P_1(|u_0|_0) > 0$.

3 Proof of Theorem 1

3.1 A Priori Estimates

We compute

$$\begin{aligned} u_t &= -((u + 1)(\log(u + 1)))_{,ij},_{ij} \\ &= -\left((u + 1) \left(\frac{u_{,ij}}{1 + u} - \frac{u_{,j} u_{,i}}{(1 + u)^2} \right)\right)_{,ij} \\ &= -u_{,ijij} + \left(\frac{u_{,j} u_{,i}}{1 + u} \right)_{,ij} \end{aligned}$$

$$\begin{aligned}
 &= -u_{,ijij} + \frac{u_{,jj} u_{,ii} + u_{,j} u_{,iij} + u_{,ijj} u_{,i} + u_{,ij} u_{,ij}}{1+u} \\
 &\quad - \frac{u_{,ii} u_{,j} u_{,j} + 3u_{,ij} u_{,i} u_{,j} + u_{,jj} u_{,i} u_{,i}}{(1+u)^2} + 2 \frac{u_{,j} u_{,j} u_{,i} u_{,i}}{(1+u)^3}.
 \end{aligned}$$

Thus, we can rewrite the equation as

$$u_t + \Delta^2 u = I_1 + I_2 + I_3, \quad (14)$$

where

$$\begin{aligned}
 I_1 &= \frac{u_{,jj} u_{,ii} + u_{,j} u_{,iij} + u_{,ijj} u_{,i} + u_{,ij} u_{,ij}}{1+u}, \\
 I_2 &= -\frac{u_{,ii} u_{,j} u_{,j} + 3u_{,ij} u_{,i} u_{,j} + u_{,jj} u_{,i} u_{,i}}{(1+u)^2}, \\
 I_3 &= \frac{2u_{,j} u_{,j} u_{,i} u_{,i}}{(1+u)^3}.
 \end{aligned}$$

By the hypothesis $|u_0|_0 < 1$, we take the Taylor expansion of the rational functions in I_1 , I_2 , and I_3 :

$$\begin{aligned}
 I_1 &= (u_{,jj} u_{,ii} + u_{,j} u_{,iij} + u_{,ijj} u_{,i} + u_{,ij} u_{,ij}) \sum_{n=0}^{\infty} (-1)^n u^n \\
 I_2 &= (u_{,ii} u_{,j} u_{,j} + 3u_{,ij} u_{,i} u_{,j} + u_{,jj} u_{,i} u_{,i}) \sum_{n=1}^{\infty} (-1)^n n u^{n-1} \\
 I_3 &= 2(u_{,j} u_{,j} u_{,i} u_{,i}) \sum_{n=2}^{\infty} (-1)^n n(n-1) u^{n-2}.
 \end{aligned}$$

Writing

$$\begin{aligned}
 S_1 &= \sum_{n=0}^{\infty} (-1)^n u^n, \\
 S_2 &= \sum_{n=1}^{\infty} (-1)^n n u^{n-1}, \\
 S_3 &= \sum_{n=2}^{\infty} (-1)^n n(n-1) u^{n-2},
 \end{aligned}$$

and using the convolution theorem for the Fourier series together with Tonelli's theorem for exchanging the order of summation, we can obtain the following estimates:

$$\begin{aligned}
 |S_1|_0 &= \sum_{k \in \mathbb{Z}^d} \left| \sum_{n=0}^{\infty} (-1)^n \widehat{u^n}(k) \right| \leq \sum_{n=0}^{\infty} |u|_0^n = \frac{1}{1 - |u|_0} \\
 |S_2|_0 &= \sum_{k \in \mathbb{Z}^d} \left| \sum_{n=1}^{\infty} n (-1)^n \widehat{u^{n-1}}(k) \right| \leq \sum_{n=1}^{\infty} n |u|_0^{n-1} = \frac{1}{(1 - |u|_0)^2} \\
 |S_3|_0 &= \sum_{k \in \mathbb{Z}^d} \left| \sum_{n=2}^{\infty} n(n-1) (-1)^n \widehat{u^{n-2}}(k) \right| \leq \sum_{n=2}^{\infty} n(n-1) |u|_0^{n-2} = \frac{2}{(1 - |u|_0)^3}.
 \end{aligned} \quad (15)$$

Moreover, for fixed i, j , we apply (7) to derive

$$\begin{aligned} |u_{,i} u_{,ijj}|_0 &\leq |u|_1 |u|_3 \leq |u|_0 |u|_4, \\ |u_{,ij} u_{,ij}|_0 &\leq |u|_2^2 \leq |u|_0 |u|_4, \\ |u_{,j} u_{,j} u_{,ii}|_0 &\leq |u|_1^2 |u|_2 \leq |u|_0^2 |u|_4, \\ |u_{,i} u_{,j} u_{,ij}|_0 &\leq |u|_1^2 |u|_2 \leq |u|_0^2 |u|_4, \\ |u_{,i} u_{,j} u_{,i} u_{,j}|_0 &\leq |u|_1^4 \leq |u|_0^3 |u|_4. \end{aligned} \quad (16)$$

By (14), (15) and (16) and summing up in i, j , we have

$$\frac{d}{dt} |u(t)|_0 + |u(t)|_4 \leq d^2 P_1(|u(t)|_0) |u(t)|_4, \quad (17)$$

where P_1 is defined in (10). Suppose u_0 satisfies

$$|u_0|_0 < 1 \quad \text{and} \quad d^2 P_1(|u_0|_0) < 1. \quad (18)$$

Then, using the monotonicity of P_1 , we obtain the following a priori bound

$$|u(t)|_0 + (1 - d^2 P_1(|u_0|_0)) \int_0^t |u(s)|_4 ds \leq |u_0|_0 \quad \text{for all } t \geq 0. \quad (19)$$

3.2 Approximate Sequence of Solutions

We consider the following approximate equation:

$$\partial_t u + \Delta^2 u = -\mathcal{P}_N \left(\left(u_{,i} u_{,j} \sum_{k=0}^N (-1)^k u^k \right)_{,ij} \right), \quad (20a)$$

$$u(0, x) = \mathcal{P}_N u_0, \quad (20b)$$

where $\mathcal{P}_N g$ is defined as

$$\mathcal{P}_N g = \sum_{k=-N}^N \widehat{g}(k) e^{ikx}.$$

Since the right-hand side of (20a) is a polynomial of u , we can solve (20) using Picard's iteration for the ODE's to obtain a solution u^N for each N . Following the computations used to derive (19), we can show that u^N satisfies the same estimates as above:

$$\begin{aligned} \frac{d}{dt} |u^N(t)|_0 + |u^N(t)|_4 &\leq d^2 P_1(|u^N(t)|_0) |u^N(t)|_4, \\ |\mathcal{P}_N u_0|_0 &\leq |u_0|_0. \end{aligned} \quad (21)$$

We note that $P_1(z)$ is increasing for $0 < z < 1$. Using this fact with u_0 satisfying (18), we obtain

$$|u^N(t)|_0 + (1 - d^2 P_1(|u_0|_0)) \int_0^t |u^N(s)|_4 ds \leq |u_0|_0 \quad (22)$$

for all $t \geq 0$ uniformly in $N \in \mathbb{N}$. In particular we find that

$$\sup_N |u^N(t)|_0 < 1.$$

3.3 Passing to the Limit in the Weak Formulation

The inequality (22) implies that $\{u^N\}$ is uniformly bounded in $L_T^\infty \mathcal{A}_0 \cap L_T^1 \mathcal{A}_4$. By interpolating $L_T^\infty \mathcal{A}_0$ and $L_T^1 \mathcal{A}_4$, $\{u^N\}$ is also uniformly bounded in

$$\{u^N\} \in L_T^p \mathcal{A}_{\frac{4}{p}}, \quad 1 < p < \infty. \quad (23)$$

We observe that (see [12]), for $1 < p \leq \infty$

$$L_T^p \ell^1 = (L_T^q c_0)^*, \quad q^{-1} + p^{-1} = 1, \quad (24)$$

where c_0 is the space formed by the sequences whose limit is zero and ℓ^1 is the space formed by the summable sequences. Due to this, the space

$$L_T^p \mathcal{A}_0$$

is a dual space.

Furthermore, since

$$\|u^N\|_{H^2}^2 = \sum_{k \in \mathbb{Z}^d} |\widehat{u^N}(k)|^2 + \sum_{k \in \mathbb{Z}^d} |k|^4 |\widehat{u^N}(k)|^2 \leq \|u^N\|_0^2 + \|u^N\|_0 \|u^N\|_4,$$

$\{u^N\}$ is uniformly bounded in

$$\{u^N\} \in L_T^2 H^2.$$

We also note that $\{\partial_t u^N\}$ is uniformly bounded in $L_T^2 H^{-2}$. Indeed, since $u^N, i \in L^4 \mathcal{A}_0$ by (23), we have

$$u_{,i} u_{,j} \sum_{k=0}^N (-1)^k u^k \in L_T^2 \mathcal{A}_0 \subset L_T^2 L^2$$

and thus

$$\partial_t u^N = -\Delta^2 u - \mathcal{P}_N \left(\left(u_{,i} u_{,j} \sum_{k=0}^N (-1)^k u^k \right)_{,ij} \right) \in L_T^2 H^{-2}.$$

We now recall Simon's compactness lemma.

Lemma 1 [35] *Let X_0 , X_1 , and X_2 be Banach spaces such that X_0 is compactly embedded in X_1 and X_1 is a subset of X_2 . Then, for $1 \leq p < \infty$, any bounded subset of $\{v \in L_T^p X_0 : \frac{\partial v}{\partial t} \in L_T^1 X_2\}$ is precompact in $L_T^p X_1$.*

Then, by Lemma 1 and Banach-Alaoglu theorem, we have that up to subsequences, still denoted by $\{u^N\}$,

$$u^N \rightharpoonup^* u \in L^\infty([0, T] \times \mathbb{T}^d), \quad (25a)$$

$$u^N \rightharpoonup^* u \in L_T^p \mathcal{A}_{\frac{4}{p}}, \quad 1 < p < \infty, \quad (25b)$$

$$u^N \rightarrow u \in L_T^2 H^r, \quad 0 \leq r < 2, \quad (25c)$$

$$u^N \rightharpoonup^* u \in L^\infty([0, T], \mathcal{A}_0), \quad (25d)$$

We now rewrite (20) as

$$\partial_t u^N = \mathcal{P}_N \left(u_{,ij}^N - \frac{u_{,i}^N u_{,j}^N}{1 + u^N} + \frac{u_{,i}^N u_{,j}^N}{1 + u^N} - u_{,i} u_{,j} \sum_{k=0}^N (-1)^k u^k \right)_{,ij}.$$

We multiply this equation with ϕ . After integrating by parts and using the Taylor series for the nonlinear term, we obtain

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^d} \partial_t \phi(t, x) u^N(t, x) dx dt - \int_{\mathbb{T}^d} \phi(0, x) \mathcal{P}_N u_0(x) dx \\ & + \int_0^T \int_{\mathbb{T}^d} \mathcal{P}_N \left(u_{,ij}^N - \frac{u_{,i}^N u_{,j}^N}{1 + u^N} \right) \phi_{,ij}(t, x) dx dt \\ & - \int_0^T \int_{\mathbb{T}^d} \mathcal{P}_N \left(u_{,i}^N u_{,j}^N \sum_{k=N+1}^{\infty} (-u^N)^k \right) \phi_{,ij}(t, x) dx dt = 0. \end{aligned} \quad (26)$$

Due to the previous convergences, we have that

- By (25b) with $\tilde{p} = 2$, $u_{,ij}^N \rightharpoonup u_{,ij}$ in $L_T^2 L^2$.
- By (25c) with $r = \frac{3}{2}$, we have $u_{,i}^N \rightarrow u_{,i} \in L_T^2 H^{r-1} \subset L_T^2 L^{q_1}$ with $q_1 = \frac{2d}{d-\frac{1}{2}}$.
- By (25b) with $p = 4$, we also have $u_{,j}^N \rightharpoonup u_{,j} \in L_T^4 \mathcal{A}_0 \subset L_T^4 L^{q_2}$ with $1 \leq q_2 \leq \infty$.
- We finally have $u \in L_T^4 H^{\frac{1}{4}} \subset L_T^4 L^{q_3}$ with $q_3 = \frac{4d}{2d-1}$ which can be obtained by (25c) with $r = \frac{1}{2}$ and then interpolated with $L_T^\infty \mathcal{A}_0 \subset L_T^\infty L^2$. This implies

$$\frac{1}{1 + u^N} - \frac{1}{1 + u} = \frac{u - u^N}{(1 + u^N)(1 + u)} \rightarrow 0 \quad \text{in } L_T^4 L^{q_3}$$

because the denominator $(1 + u^N)(1 + u)$ does not vanish.

We now choose q_2 such that

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1.$$

Then, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \mathcal{P}_N \left(u_{,ij}^N - \frac{u_{,i}^N u_{,j}^N}{1 + u^N} \right) \phi_{,ij}(t, x) dx dt \\ & \rightarrow \int_0^T \int_{\mathbb{T}^d} \left(u_{,ij} - \frac{u_{,i} u_{,j}}{1 + u} \right) \phi_{,ij}(t, x) dx dt \\ & = \int_0^T \int_{\mathbb{T}^d} \log(u(t, x) + 1)_{,ij} (u(t, x) + 1) \phi_{,ij}(t, x) dx dt. \end{aligned}$$

We finally note that

$$\left\| u_{,i}^N u_{,j}^N \sum_{k=N+1}^{\infty} (u^N)^k \right\|_{L_T^2 L^2} \leq \|u^N\|_{L_T^4 \mathcal{A}_1}^2 \sum_{k=N+1}^{\infty} \|u^N\|_{L_T^\infty \mathcal{A}_0}^k \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

We now can pass to the limit to (26) to derive (9) for any $\phi \in C^\infty([0, \infty) \times \mathbb{T}^d)$.

3.4 Decay to Equilibrium

Since $\langle u^N(t) \rangle = 0$, we have $|u^N(t)|_0 \leq |u^N(t)|_4$. Using (21), we obtain

$$|u^N(t)|_0 \leq |u_0|_0 e^{-(1-d^2 P_1(|u_0^N|_0))t} \leq |u_0|_0 e^{-(1-d^2 P_1(|u_0|_0))t}$$

uniformly in N . In particular,

$$\|u(t)\|_{L^\infty} \leq |u_0|_0 e^{-(1-d^2 P_1(|u_0|_0))t}.$$

3.5 Uniqueness

Due to the regularity of the solution constructed, the uniqueness follows from a standard contradiction argument. As this part is classical, we only sketch its proof. We consider two different solutions emanating from the same initial data, $u^{(1)}$ and $u^{(2)}$, and define

$$v = u^{(1)} - u^{(2)}.$$

Then

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_{,ij}\|_{L^2}^2 = \int_{\mathbb{T}^d} \left(\frac{u_{,i}^{(1)} u_{,j}^{(1)}}{1 + u^{(1)}} - \frac{u_{,i}^{(2)} u_{,j}^{(2)}}{1 + u^{(2)}} \right) v_{,ij} dx.$$

Integrating by parts and using Hölder inequality together with interpolation in Sobolev spaces we find the inequality

$$\frac{d}{dt} \|v\|_{L^2}^2 \leq c(\|u^{(1)}\|_{C^2} + \|u^{(2)}\|_{C^2}) \|v\|_{L^2}^2.$$

As the solution that we constructed is of class $L_T^2 C^2$, we conclude the uniqueness of the solution.

3.6 Spatial Analyticity

Let $\sigma \in (0, 1)$ be a sufficiently small constant which will be fixed later. We define the function

$$V^N(t, x) = e^{\sigma t \sqrt{-\Delta}} u^N(t, x) \quad \text{or equivalently} \quad \widehat{V^N}(t, k) = e^{\sigma t |k|} \widehat{u^N}(t, k).$$

Let

$$G(u) = \mathcal{P}_N \left(\left(u_{,i} u_{,j} \sum_{k=0}^N (-1)^k u^k \right)_{,ij} \right) \quad (27)$$

Then,

$$\partial_t \widehat{V^N}(t, k) = -|k|^4 \widehat{V^N}(t, k) - e^{\sigma t |k|} \widehat{G(u)}(t, k) + \sigma |k| \widehat{V^N}(t, k) \quad (28)$$

with $\widehat{V^N}(0, k) = \widehat{P_N u_0}(k)$. We note that

$$\begin{aligned} \left| e^{\sigma s \sqrt{-\Delta}} (fg) \right|_0 &\leq \sum_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} e^{\sigma s |k-l|} |\widehat{f}(k-l)| e^{\sigma s |l|} |\widehat{g}(l)| \\ &\leq \left| e^{\sigma s \sqrt{-\Delta}} f \right|_0 \left| e^{\sigma s \sqrt{-\Delta}} g \right|_0. \end{aligned} \quad (29)$$

In particular,

$$\left| e^{\sigma s \sqrt{-\Delta}}(f^2) \right|_0 \leq \sum_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} e^{\sigma s |k-l|} |\widehat{f}(k-l)| e^{\sigma s |l|} |\widehat{f}(l)| \leq \left| e^{\sigma s \sqrt{-\Delta}} f \right|_0^2.$$

Furthermore,

$$\begin{aligned} \left| e^{\sigma s \sqrt{-\Delta}}(f^3) \right|_0 &\leq \sum_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} e^{\sigma s |k-l|} |\widehat{f^2}(k-l)| e^{\sigma s |l|} |\widehat{f}(l)| \\ &\leq \left| e^{\sigma s \sqrt{-\Delta}}(f^2) \right|_0 \left| e^{\sigma s \sqrt{-\Delta}} f \right|_0 \\ &\leq \left| e^{\sigma s \sqrt{-\Delta}} f \right|_0^3. \end{aligned}$$

Similarly,

$$\left| e^{\sigma s \sqrt{-\Delta}}(f^n) \right|_0 \leq \left| e^{\sigma s \sqrt{-\Delta}} f \right|_0^n.$$

We apply the previous estimates to the nonlinear terms in (28). Following the computations in Sect. 3.1 used to obtain (17), we derive

$$\left| e^{\sigma s \sqrt{-\Delta}}(u_{,ij} u_{,j} u_{,ij}) \right|_0 \leq \left| e^{\sigma s \sqrt{-\Delta}} u \right|_1^2 \left| e^{\sigma s \sqrt{-\Delta}} u \right|_2 \leq \left| e^{\sigma s \sqrt{-\Delta}} u \right|_0^2 \left| e^{\sigma s \sqrt{-\Delta}} u \right|_4.$$

Performing the same estimates to the rest of nonlinear terms, we find the following inequality:

$$\frac{d}{dt} |V^N(t)|_0 + |V^N(t)|_4 \leq d^2 P_1(|V^N(t)|_0) |V^N(t)|_4 + \sigma |V^N(t)|_1.$$

Since $\langle V^N \rangle = \widehat{u^N}(0) = 0$, we have $|V^N(t)|_1 \leq |V^N(t)|_4$ and thus

$$\frac{d}{dt} |V^N(t)|_0 + (1 - \sigma) |V^N(t)|_4 \leq d^2 P_1(|V^N(t)|_0) |V^N(t)|_4.$$

We now take $\sigma > 0$ sufficiently small to satisfy $d^2 P_1(|u_0|_0) < 1 - \sigma$. Then, we obtain

$$|V^N(t)|_0 \leq |u_0|_0 \quad \text{for all } t > 0. \quad (30)$$

Using (24), we conclude that

$$V^N \xrightarrow{*} V \in L_T^\infty \mathcal{A}_0.$$

Now we observe that we can pass to the weak-* limit in N and conclude the desired bound for the function V . The convergence of u^N implies that

$$V(t, x) = e^{\sigma t \sqrt{-\Delta}} u(t, x).$$

4 Proof of Theorem 2

For the sake of notational simplicity, we write $\mu \Gamma = \gamma > 0$ and $\epsilon(2\Gamma^2 - 2\Gamma) = -\nu < 0$ in (11). We first rewrite (11) as

$$u_t - \frac{\gamma}{3} u_{xxx} - \gamma((u+1)(\log(u+1))_{xx})_x = -\nu((u+1)(\log(u+1))_{xx})_{xx}.$$

4.1 Existence of Solution

Since $((u + 1)(\log(u + 1))_{xx})_x$ is a lower order term, the first part of Theorem 2 is very similar to the one in Theorem 1 and so we only provide a priori estimates for the existence of solutions. Since

$$\begin{aligned} ((1 + u)(\log(u + 1))_{xx})_x &= u_{xxx} - \frac{2u_x u_{xx}}{1 + u} + \frac{u_x^3}{(1 + u)^2}, \\ ((1 + u)(\log(u + 1))_{xx})_{xx} &= u_{xxxx} - \frac{2(u_x u_{xxx} + u_{xx}^2)}{1 + u} \\ &\quad + \frac{2u_x^2 u_{xx} + 3u_x^2 u_{xx}}{(1 + u)^2} - \frac{2u_x^4}{(1 + u)^3}, \end{aligned}$$

we take the Taylor expansion of the rational functions to derive

$$u_t - \frac{4\gamma}{3} u_{xxx} + \nu u_{xxxx} = v(I_1 + I_2 + I_3) + \gamma(I_4 + I_5),$$

with

$$\begin{aligned} I_1 &= \frac{2u_x u_{xxx} + 2u_{xx}^2}{1 + u} = (2u_x u_{xxx} + 2u_{xx}^2) \sum_{n=0}^{\infty} (-1)^n u^n, \\ I_2 &= -\frac{5u_x^2 u_{xx}}{(1 + u)^2} = 5u_x^2 u_{xx} \sum_{n=1}^{\infty} (-1)^n n u^{n-1}, \\ I_3 &= 2\frac{u_x^4}{(1 + u)^3} = 2u_x^4 \sum_{n=2}^{\infty} (-1)^n n(n-1) u^{n-2}, \\ I_4 &= -\frac{2u_x u_{xx}}{1 + u} = -2u_x u_{xx} \sum_{n=0}^{\infty} (-1)^n u^n, \\ I_5 &= \frac{u_x^3}{(1 + u)^2} = -u_x^3 \sum_{n=1}^{\infty} (-1)^n n u^{n-1}. \end{aligned}$$

Since

$$\frac{\operatorname{Re}(\widehat{\bar{u}}(t, k) \widehat{u_{xxx}}(t, k))}{|\widehat{u}(t, k)|} = 0, \quad (31)$$

we have

$$\frac{d}{dt} |u(t)|_0 + \nu |u(t)|_4 \leq \nu \sum_{j=1}^3 |I_j|_0 + \gamma \sum_{j=4}^5 |I_j|_0.$$

We recall P_1 and P_2 in (10) and (13):

$$P_1(z) = \frac{4z}{1-z} + \frac{5z^2}{(1-z)^2} + \frac{4z^3}{(1-z)^3}, \quad P_2(z) = \frac{z}{1-z} + \frac{z^2}{(1-z)^2}.$$

By applying (7) and (15), we obtain

$$\sum_{i=1}^3 |I_i|_0 \leq \left(\frac{4|u(t)|_0}{1-|u|_0} + \frac{5|u(t)|_0^2}{(1-|u|_0)^2} + \frac{4|u(t)|_0^3}{(1-|u|_0)^3} \right) |u|_4 = P_1(|u|_0)|u|_4,$$

$$\sum_{i=4}^5 |I_i|_0 \leq \left(\frac{|u(t)|_0}{1-|u|_0} + \frac{|u(t)|_0^2}{(1-|u|_0)^2} \right) |u|_3 \leq P_2(|u|_0)|u|_4$$

which imply that

$$\frac{d}{dt} |u(t)|_0 + v|u(t)|_4 \leq vP_1(|u|_0)|u|_4 + \gamma P_2(|u|_0)|u|_4. \quad (32)$$

Suppose u_0 satisfies

$$|u_0|_0 < 1 \quad \text{and} \quad \gamma P_2(|u_0|_0) + vP_1(|u_0|_0) < v.$$

Then, we have

$$|u(t)|_0 + (v - \gamma P_2(|u_0|_0) - vP_1(|u_0|_0)) \int_0^t |u(s)|_4 ds \leq |u_0|_0 \quad \text{for all } t \geq 0.$$

4.2 Decay to Equilibrium

As before, since $\langle u(t) \rangle = 0$, we have $|u(t)|_0 \leq |u(t)|_4$. By (32),

$$\|u(t)\|_{L^\infty} \leq |u_0|_0 e^{-(v - \gamma P_2(|u_0|_0) - vP_1(|u_0|_0))t}.$$

4.3 Uniqueness

The uniqueness follows similarly as before.

4.4 Spatial Analyticity

Let $\sigma \in (0, 1)$ be a small parameter that will be fixed later. As the approximation process mimics the method of constructing approximate solutions in Theorem 1, we skip the approximation process. Thus, we continue directly to obtain appropriate estimates for

$$V(t, x) = e^{\sigma t \sqrt{-\Delta}} u(t, x).$$

We observe that V satisfies

$$\begin{aligned} \partial_t \widehat{V}(t, k) &= -|k|^4 \widehat{V}(t, k) - \frac{4\gamma}{3} \widehat{V_{xxx}}(t, k) + v e^{\sigma t |k|} (I_1 + I_2 + I_3) \\ &\quad + \gamma e^{\sigma t |k|} (I_4 + I_5) + \sigma |k| \widehat{V}(t, k) \end{aligned}$$

with $\widehat{V}(0, k) = \widehat{u_0}(k)$. Using (31) applied to V , $|V(t)|_1 \leq |V(t)|_4$, and (29), we obtain

$$\frac{d}{dt} |V(t)|_0 + v|V(t)|_4 \leq (\gamma P_3(|V(t)|_0) + vP_4(|V(t)|_0)) |V(t)|_4 + \sigma |V(t)|_4.$$

Suppose u_0 is such that

$$v - \sigma - \gamma P_2(|u_0|_0) - vP_1(|u_0|_0) > 0.$$

Then, we have

$$|V(t)|_0 \leq |u_0|_0 \quad \text{for all } t > 0. \quad (33)$$

This completes the proof of Theorem 2.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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