



On a thin film model with insoluble surfactant

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Abstract

This paper studies the existence and asymptotic behavior of global weak solutions for a thin film equation with insoluble surfactant under the influence of gravitational, capillary, and van der Waals forces. We prove the existence of global weak solutions for *medium sized* initial data in *large function spaces*. Moreover, exponential decay towards the flat equilibrium state is established, where an estimate on the decay rate can be computed explicitly.

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1. Introduction

Surfactant is the short form for *surface active agent* and is a substance which – in contact with a fluid – reduces surface tension. The induced dynamic is twofold: On the one hand, the resulting surface tension gradients influence the evolution of the thin film; on the other hand, the surfactant spreads along the surface. The latter effect is known as *Marangoni effect*. Naturally, the surfactant induced dynamics are of particular interest in connection with *thin fluid films*, where surface

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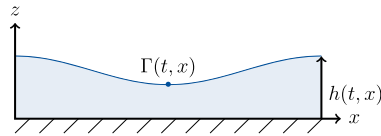


Fig. 1. Scheme of a thin film flow with insoluble surfactant.

tension forces have a very important impact. In particular, the interest in thin film equations with a layer of surfactant on the surface is motivated by various applications. For instance coating flow technology, film drainage in emulsions, foams and medical treatment of lungs of premature infants.

The present work studies the dynamics of a viscous, incompressible, Newtonian thin film over a flat bottom equipped with a layer of insoluble surfactant on the free surface. Thus, to study the full problem one has to consider a free boundary problem for the Navier-Stokes equations coupled with an advection-diffusion equation on the free surface. As this is a challenging issue, a common approach to simplify the problem is to consider the *lubrication approximation* to derive evolution equations for the film height and the surfactant concentration which capture the behavior and the main properties of the full free boundary problem. Pioneering works in this direction in absence of surfactant effects are due to Greenspan [32], Constantin, Dupont, Goldstein, Kadanoff, Shelley & Zhou [13], Bernis & Friedman [4], Beretta, Bertsch & Dal Passo [3] and Bertozzi & Pugh [5]. Also, Escher, Matioc & Matioc [23] considered the flow in porous media (see also Escher & Matioc [25], Matioc [39], Escher, Laurençot & Matioc [21], Laurençot & Matioc [35–38] and Bruell & Granero-Belinchón [10]) while the Stokes flow was considered by Escher, Matioc & Matioc [24] (see also Escher & Matioc [26] and Bruell & Granero-Belinchón [10]). A more recent reference is Pernas-Castaño & Velázquez [40], where the authors study the evolution of the interface between two different fluids in two concentric cylinders when the velocity is given by the Navier-Stokes equation and one of the fluids is thin.

Some of the main works on the evolution of a thin film with insoluble surfactant are the ones by Borgas & Grotberg [6], Gaver & Grotberg [29] and Jensen & Grotberg [34]. Under certain assumptions, Jensen & Grotberg [34] applied the lubrication approximation and cross-sectional averaging to derive the following system of evolution equations for the film height $h = h(t, x)$ and the surfactant concentration $\Gamma = \Gamma(t, x)$ (see Fig. 1):

$$\partial_t h = -\partial_x \left[\frac{h^2}{2} \partial_x \sigma(\Gamma) - \frac{\mathcal{G}}{3} h^3 \partial_x h + \frac{\mathcal{S}}{3} h^3 \partial_x^3 h + \mathcal{A} \frac{\partial_x h}{h} \right] \quad \text{in } \Omega_T \quad (1a)$$

$$\partial_t \Gamma = -\partial_x \left[\Gamma \left(h \partial_x \sigma(\Gamma) - \frac{\mathcal{G}}{2} h^2 \partial_x h + \frac{\mathcal{S}}{2} h^2 \partial_x^3 h + \frac{3\mathcal{A}}{2} \frac{\partial_x h}{h^2} \right) - \mathcal{D} \partial_x \Gamma \right] \quad \text{in } \Omega_T. \quad (1b)$$

Here, $\Omega_T := (0, T) \times \Omega$ denotes the time-space domain for the unknown functions h and Γ , with $\Omega \subset \mathbb{R}$ being an open, bounded interval. The system (1) is supplemented with initial conditions

$$h(0, x) = h_0(x) \quad \text{and} \quad \Gamma(0, x) = \Gamma_0(x) \quad \text{for all } x \in \Omega,$$

where h_0 and Γ_0 are given functions and boundary conditions

$$\partial_x h = \partial_x \Gamma = 0, \quad \mathcal{S} \partial_x^3 h = 0 \quad \text{for all } x \in \partial\Omega.$$

The appearing parameters represent a modified gravitational constant (\mathcal{G}), surface tension coefficient (\mathcal{S}), Hamaker constant (\mathcal{A}), which corresponds to the effects of van der Waals forces, and surface diffusion coefficient (\mathcal{D}). Moreover, σ is the constitutive equation of state relating the surface tension to the surfactant concentration. As the presence of surfactant reduces surface tension, σ is assumed to be positive and nonincreasing. A commonly used description for the dependence of the surface tension on the surfactant is given by (cf. [42,6,29])

$$\sigma_\beta(s) := (\beta + 1) \left[1 - s + \left(\frac{\beta + 1}{\beta} \right)^{\frac{1}{3}} s \right]^{-3} - \beta, \quad s \in [0, 1]$$

for $\beta \in (0, \infty)$. For simplicity reasons, in the present work, we assume that σ is given by the limit for $\beta \rightarrow \infty$ in σ_β , that is

$$\sigma(s) = 1 - s, \quad s \in [0, 1].$$

This assumption is also used in applications and numerical investigations; see for instance [33,1,2,44,45].

Let us mention that the scaling Jensen and Grotberg used for the surfactant concentration is given by

$$\Gamma(t, x) \mapsto \Gamma_m^{-1} \Gamma(t, x),$$

where $\Gamma_m > 0$ is the so-called *critical micelle concentration*. If the surfactant concentration Γ exceeds the value Γ_m , the molecules form micelles and thus there is no further decrease of surface tension to perceive. Consequently, it is natural to assume that the initial surfactant concentration Γ_0 satisfies $0 \leq \Gamma_0 \leq 1$.

From the analytical point of view, the system (1) includes many challenges. Notice that the evolution equations in (1) form a system of two strongly coupled, degenerate, parabolic partial differential equations. Under the assumption that all the appearing parameters \mathcal{G} , \mathcal{S} , \mathcal{A} , and \mathcal{D} are positive, the degeneracy occurs in the equation for the film height, when h approaches zero, i.e. when the surface touches the bottom. Moreover, (1) is a coupled system of mixed orders having cross diffusive terms. While the equation for the surfactant concentration Γ is an advection-diffusion equation of second order, the equation for the film height h is of fourth order. Notice that, if capillary effects are neglected ($\mathcal{S} = 0$), then the system is of second order in both equation. In our considerations, we are going to consider both cases: The gravity driven film ($\mathcal{S} = 0$) and the capillary driven film ($\mathcal{S} > 0$).

Even if during the last decades modeling as well as numerical investigations for the thin film equation with surfactant have attracted lots of attention (see for instance [29,34,33,1,2,44,16] and the references therein), the rigorous analytical studies have started recently. Existence of local solutions for a thin film equation with insoluble surfactant driven by Marangoni forces only ($\mathcal{G} = \mathcal{S} = \mathcal{A} = \mathcal{D} = 0$) has been studied by Renardy [41]. In absence of capillary and van der Waals forces ($\mathcal{S} = \mathcal{A} = 0$), the authors Escher, Hillairet, Laurençot & Walker [18] used lubrication approximation to derive a system of differential equations describing the evolution of a thin film with *soluble* surfactant under the influence of Marangoni and gravitational forces. Moreover, they proved local well-posedness in the space of square integrable functions L^2 by means of semigroup theory as well as asymptotic stability with exponential decay to equilibrium. We

would like to emphasize that their stability result is stated for positive initial data in the L^2 -based Sobolev space H^2 . The result shows in particular, that starting with H^2 data close to the flat steady state there exists a unique global strong solution. A similar result for a two-phase thin film equation with insoluble surfactant was shown by Bruell [7] for the gravity ($\mathcal{S} = \mathcal{A} = 0$) as well as for the capillary driven film ($\mathcal{G} = \mathcal{A} = 0$). Due to the degeneracy of the equation with respect to the film height, it is natural to expect that in general strong solutions exist only locally in time. The existence of nonnegative global weak solutions for the thin film equation with insoluble surfactant was proved by Escher, Hillairet, Laurençot & Walker [17] for the gravity driven thin film ($\mathcal{S} = \mathcal{A} = 0$), and [19] for the corresponding capillary driven film ($\mathcal{G} = \mathcal{A} = 0$) and by Bruell [8] for the two-phase thin film equation with insoluble surfactant under the influence of capillary forces (see also [28,12]). The main ingredient in all these works concerning the existence of global weak solutions is a regularization argument to overcome the degeneracy, followed by a two-step compactness argument based on a priori estimates provided by an energy functional for the system. Finally, we would like to mention that traveling wave solutions of a gravity thin film equation with insoluble surfactant ($\mathcal{S} = \mathcal{A} = 0$) were studied by Escher, Hillairet, Laurençot & Walker [20] (see also [22]).

1.1. Aim and outline of the present paper

The aim of the present work is to prove the existence of global weak solutions of (1) under fairly low regularity assumptions with respect to the initial data. Similarly as in the companion paper [10], we work in scales of Wiener spaces. Exploiting the algebra inequality verified by the norms of the underlying spaces, we show *a priori* energy estimates in Wiener algebra, which guarantee the existence of global weak solutions and imply the exponential decay towards the flat equilibrium state. Moreover, the decay rate can be bounded by explicit constants, which depend on the parameters of the system and the size of the initial data. In addition we prove uniqueness of the weak solutions provided that they belong to a (slightly) higher regularity class. A similar approach has been employed before for the Muskat problem [27,14,15] (and the references therein) for the doubly parabolic Keller-Segel system [11], PDEs modeling small steepness porous flow [31] and the evolution of crystal surfaces [30]. We consider two cases: The gravity driven film, where surface tension effects are neglected ($\mathcal{S} = 0$), which leads to a coupled system of second order equations; and the capillary driven film, where we take surface tension effects into account ($\mathcal{S} > 0$). In the latter case the evolution equations (1) build a coupled system of mixed orders. Let us emphasize that in our work we take *all* acting forces (gravity, surface tension, van der Waals) into account. To the best of our knowledge this is the first analytical existence result for the full system (1) where $\mathcal{G}, \mathcal{S}, \mathcal{A}, \mathcal{D} > 0$. The outline of the paper is as follows: We start in Section 2 with some preliminaries and auxiliary results concerning the scale of Wiener spaces. In Section 3 we reformulate the problems in terms of the distance to the (flat) equilibrium and state our main results. Eventually Sections 4 and 5 are devoted to the proofs of the main theorems for the gravity ($\mathcal{S} = 0$) and capillary ($\mathcal{S} > 0$) driven flow, respectively.

2. Preliminaries

We start by introducing the functional analytical framework. Let $\mathbb{T} := [-\pi, \pi)$. For $n \in \mathbb{N}$ we denote by

$$W^{n,p}(\mathbb{T}) = \left\{ u \in L^p(\mathbb{T}) \text{ such that } \|u\|_{W^{n,p}(\mathbb{T})}^p := \|u\|_{L^p}^p + \|\partial_x^n u\|_{L^p}^p < \infty \right\} \quad (2)$$

the standard L^p -based Sobolev spaces on \mathbb{T} . We recall the expression of the k -th Fourier coefficient and the Fourier series of a 2π -periodic integrable function u ,

$$\hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} u(x) e^{-ixk} dx, \quad u(x) = \sum_{k \in \mathbb{Z}} \hat{u}(k) e^{ixk}.$$

We introduce the Wiener spaces $\dot{A}^s(\mathbb{T})$ as

$$\dot{A}^s(\mathbb{T}) = \left\{ u \in L^1(\mathbb{T}) \text{ such that } \|u\|_{\dot{A}^s(\mathbb{T})} := \sum_{k \in \mathbb{Z}} |k|^s |\hat{u}(k)| < \infty \right\}. \quad (3)$$

We note that $\dot{A}^0(\mathbb{T}) = A(\mathbb{T})$ is a Banach algebra and $\{\dot{A}^s(\mathbb{T}) \mid s \geq 0\}$ form a Banach scale. Furthermore, the following inequalities hold true:

- Let $p \geq q \geq 0$, then

$$\|f\|_{\dot{A}^q} \leq \|f\|_{\dot{A}^p} \text{ for all } f \in \dot{A}^p(\mathbb{T}). \quad (4)$$

- Let $s \in \{0\} \cup [1, \infty)$ be a fixed parameter and $f, g \in \dot{A}^s(\mathbb{T})$, then, due to the convexity of x^s in this range,

$$\|fg\|_{\dot{A}^s} \leq 2^{s-1} (\|f\|_{\dot{A}^s} \|g\|_{\dot{A}^0} + \|f\|_{\dot{A}^0} \|g\|_{\dot{A}^s}) \leq 2^s \|f\|_{\dot{A}^s} \|g\|_{\dot{A}^s}, \quad (5)$$

while, as a consequence of subadditivity of x^s , for $s \in (0, 1)$, we have

$$\|fg\|_{\dot{A}^s} \leq 2 \|f\|_{\dot{A}^s} \|g\|_{\dot{A}^s}.$$

- Due to the Hölder inequality, we have the following interpolation inequality

$$\|f\|_{\dot{A}^{s\theta}} \leq \|f\|_{\dot{A}^0}^{1-\theta} \|f\|_{\dot{A}^s}^{\theta} \quad \text{for all } 0 < \theta < 1. \quad (6)$$

3. Reformulation of the problem and main results

3.1. Reformulation

In what follows we assume that $\sigma(s) = 1 - s$. Then, system (1) is given by

$$\partial_t h = -\partial_x \left[-\frac{h^2}{2} \partial_x \Gamma - \frac{\mathcal{G}}{3} h^3 \partial_x h + \frac{\mathcal{S}}{3} h^3 \partial_x^3 h + \mathcal{A} \frac{\partial_x h}{h} \right], \quad \text{in } \Omega_T \quad (7a)$$

$$\partial_t \Gamma = -\partial_x \left[\Gamma \left(-h \partial_x \Gamma - \frac{\mathcal{G}}{2} h^2 \partial_x h + \frac{\mathcal{S}}{2} h^2 \partial_x^3 h + \frac{3\mathcal{A}}{2} \frac{\partial_x h}{h^2} \right) - \mathcal{D} \partial_x \Gamma \right], \quad \text{in } \Omega_T, \quad (7b)$$

with initial conditions

$$h(0, x) = h_0(x) \quad \text{and} \quad \Gamma(0, x) = \Gamma_0(x) \quad \text{for all } x \in \Omega,$$

and boundary conditions

$$\partial_x h = \partial_x \Gamma = 0, \quad \mathcal{S} \partial_x^3 h = 0 \quad \text{for all } x \in \partial\Omega. \quad (8)$$

The problem is posed on a spatial interval $\Omega = (0, L)$ with the above lateral boundary conditions. However, without losing generality, instead of considering an interval and no-flux boundary conditions, we are going to consider periodic solutions h, Γ of (7) on a flat torus \mathbb{T} (which can be identified with $[-\pi, \pi)$). This generalization actually simplifies our approach and it was already used in [10] for similar problems. Let us explain why our formulation in the flat torus is actually equivalent to the original problem posed on the interval $(0, L)$. If (h_0, Γ_0) are the initial data on an interval $\bar{\Omega} = [0, L]$ satisfying the boundary conditions (8), we set

$$\bar{h}_0(x) := h_0(|x|), \quad \bar{\Gamma}_0(x) := \Gamma_0(|x|) \quad \text{for } x \in [-L, L].$$

In view of the symmetry of (7), the evenness of initial data is preserved and any solution of (7) on $[0, L]$ with initial data (h_0, Γ_0) satisfying the boundary conditions (8), can be identified with the corresponding solution to even initial data $(\bar{h}_0, \bar{\Gamma}_0)$ on the periodic cell $[-L, L]$ restricted to the half-domain $[0, L]$. In the sequel we drop the bar notation and consider periodic solutions of (7) defined on \mathbb{T} with initial conditions

$$h(0, x) = h_0(x) \quad \text{and} \quad \Gamma(0, x) = \Gamma_0(x) \quad \text{for all } x \in \mathbb{T},$$

where h_0, Γ_0 are given periodic functions. It follows immediately from the structure of the equations (7) that the initial mass is preserved in time:

Lemma 1 (Conservation of mass). *Let (h, Γ) be a solution of (7) on a time interval $[0, T)$, then*

$$\int_{\mathbb{T}} h(t, x) dx = \int_{\mathbb{T}} h_0(x) dx \quad \text{and} \quad \int_{\mathbb{T}} \Gamma(t, x) dx = \int_{\mathbb{T}} \Gamma_0(x) dx \quad \text{for all } t \in [0, T).$$

If (h_0, Γ_0) are nonnegative bounded initial data, we set

$$h_{\sharp} = \frac{1}{2\pi} \int_{\mathbb{T}} h_0(x) dx \quad \text{and} \quad \Gamma_{\sharp} = \frac{1}{2\pi} \int_{\mathbb{T}} \Gamma_0(x) dx.$$

The constants h_{\sharp} and Γ_{\sharp} represent the mean of the initial data and they are a steady state of the system (7). In our studies we consider the evolution of the distance of a solution (h, Γ) to the steady state $(h_{\sharp}, \Gamma_{\sharp})$. For this purpose, we define new unknowns

$$f = h - h_{\sharp}, \quad \Theta = \Gamma - \Gamma_{\sharp}, \quad (9)$$

which have zero mean. In the new variables (9), the system (7) can be rewritten as

$$\partial_t f - \frac{h_\#^2}{2} \partial_x^2 \Theta + \left(\frac{\mathcal{A}}{h_\#} - \frac{\mathcal{G}}{3} h_\#^3 \right) \partial_x^2 f + \frac{\mathcal{S}}{3} h_\#^3 \partial_x^4 f = \sum_{j=1}^4 N_j, \text{ in } (0, T) \times \mathbb{T} \quad (10a)$$

$$\partial_t \Theta - (h_\# \Gamma_\# + \mathcal{D}) \partial_x^2 \Theta + \left(\frac{3\mathcal{A}\Gamma_\#}{2h_\#^2} - \frac{\mathcal{G}}{2} \Gamma_\# h_\#^2 \right) \partial_x^2 f + \frac{\mathcal{S}}{2} \Gamma_\# h_\#^2 \partial_x^4 f = \sum_{j=5}^8 N_j, \text{ in } (0, T) \times \mathbb{T} \quad (10b)$$

with initial conditions

$$f(0, x) = f_0(x) \quad \text{and} \quad \Theta(0, x) = \Theta_0(x) \quad \text{for all } x \in \Omega,$$

where $f_0(x) = h_0(x) - h_\#$ and $\Theta_0(x) = \Gamma_0(x) - \Gamma_\#$ are the initial displacement functions from the flat states $h_\#$ and $\Gamma_\#$, respectively. The nonlinear terms N_i , $i = 1, \dots, 8$, on the right hand side of (10) are defined as

$$\begin{aligned} N_1 &= \partial_x \left[\left(\frac{f^2}{2} + f h_\# \right) \partial_x \Theta \right], \\ N_2 &= \partial_x \left[\frac{\mathcal{G}}{3} \left(3h_\#^2 f + 3f^2 h_\# + f^3 \right) \partial_x f \right], \\ N_3 &= -\partial_x \left[\frac{\mathcal{S}}{3} \left(3h_\#^2 f + 3f^2 h_\# + f^3 \right) \partial_x^3 f \right], \\ N_4 &= \partial_x \left[\mathcal{A} \frac{f}{h_\#^2 (1 + \frac{f}{h_\#})} \partial_x f \right], \\ N_5 &= \partial_x \left[(\Gamma_\# f + \Theta h_\# + \Theta f) \partial_x \Theta \right], \\ N_6 &= \partial_x \left[\frac{\mathcal{G}}{2} \left(\Gamma_\# f^2 + 2\Gamma_\# h_\# f + \Theta h_\#^2 + \Theta f^2 + 2\Theta h_\# f \right) \partial_x f \right], \\ N_7 &= -\partial_x \left[\frac{\mathcal{S}}{2} \left(\Gamma_\# f^2 + 2\Gamma_\# h_\# f + \Theta h_\#^2 + \Theta f^2 + 2\Theta h_\# f \right) \partial_x^3 f \right], \\ N_8 &= \partial_x \left[\Theta \frac{3\mathcal{A}}{2h_\#^2} \frac{(f^2 + 2h_\# f) \partial_x f}{(h_\# + f)^2} + \Gamma_\# \frac{3\mathcal{A}}{2h_\#^2} \frac{(f^2 + 2h_\# f) \partial_x f}{(h_\# + f)^2} - \Theta \frac{3\mathcal{A}}{2h_\#^2} \partial_x f \right] \\ &= \partial_x \left[\frac{3\mathcal{A}}{2} \frac{2f\Gamma_\# h_\# + f^2 \Gamma_\# - \Theta h_\#^2}{h_\#^2 (f + h_\#)^2} \partial_x f \right]. \end{aligned}$$

Note that, for $|r| < 1$, we have

$$\frac{1}{1+r} = \sum_{j=1}^{\infty} (-1)^{j+1} r^{j-1}, \quad (11)$$

$$\frac{1}{(1+r)^2} = \sum_{j=1}^{\infty} j(-1)^{j+1} r^{j-1}, \quad (12)$$

$$\frac{1}{(1+r)^3} = \frac{1}{2} \sum_{j=2}^{\infty} j(j-1)(-1)^j r^{j-2}. \quad (13)$$

Consequently, under the assumption that

$$\|f\|_{L^\infty} \leq \|f\|_{A^0} < h_\sharp, \quad (14)$$

we can use (11) and (12) and write

$$N_4 = \mathcal{A} \left[\left(\frac{f}{h_\sharp^2} \partial_x^2 f + \left(\frac{\partial_x f}{h_\sharp} \right)^2 \right) \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{f}{h_\sharp} \right)^{j-1} - \frac{f}{h_\sharp^3} (\partial_x f)^2 \sum_{j=1}^{\infty} j(-1)^{j+1} \left(\frac{f}{h_\sharp} \right)^{j-1} \right]. \quad (15)$$

Similarly, invoking (12) and (13) we have that

$$\begin{aligned} N_8 &= \partial_x \left[\frac{3\mathcal{A}}{2} \frac{2f\Gamma_\sharp h_\sharp + f^2\Gamma_\sharp - \Theta h_\sharp^2}{h_\sharp^2(f+h_\sharp)^2} \partial_x f \right] \\ &= \frac{3\mathcal{A}}{2h_\sharp^4} \left[\frac{2f\Gamma_\sharp h_\sharp + f^2\Gamma_\sharp - \Theta h_\sharp^2}{\left(\frac{f}{h_\sharp} + 1\right)^2} \partial_x^2 f + \frac{2\partial_x f \Gamma_\sharp h_\sharp + 2f\partial_x f \Gamma_\sharp - \partial_x \Theta h_\sharp^2}{\left(\frac{f}{h_\sharp} + 1\right)^2} \partial_x f \right] \\ &\quad - \frac{3\mathcal{A}}{h_\sharp^4} \left[\frac{2f\Gamma_\sharp h_\sharp + f^2\Gamma_\sharp - \Theta h_\sharp^2}{\left(\frac{f}{h_\sharp} + 1\right)^3} \frac{(\partial_x f)^2}{h_\sharp} \right] \\ &= \frac{3\mathcal{A}}{2h_\sharp^4} \left[\left(2f\Gamma_\sharp h_\sharp + f^2\Gamma_\sharp - \Theta h_\sharp^2 \right) \partial_x^2 f \right. \\ &\quad \left. + \left(2\partial_x f \Gamma_\sharp h_\sharp + 2f\partial_x f \Gamma_\sharp - \partial_x \Theta h_\sharp^2 \right) \partial_x f \right] \sum_{j=1}^{\infty} j(-1)^{j+1} \left(\frac{f}{h_\sharp} \right)^{j-1} \\ &\quad - \frac{3\mathcal{A}}{h_\sharp^4} \left[\left(2f\Gamma_\sharp h_\sharp + f^2\Gamma_\sharp - \Theta h_\sharp^2 \right) \frac{(\partial_x f)^2}{h_\sharp} \right] \frac{1}{2} \sum_{j=2}^{\infty} j(j-1)(-1)^j \left(\frac{f}{h_\sharp} \right)^{j-2}. \quad (16) \end{aligned}$$

We fix the initial data h_0, Γ_0 for problem (7). Thereby, the constants h_\sharp and Γ_\sharp are uniquely determined and we are going to state our results in terms of f, Θ for (10).

3.2. Main results

In what follows the constants \mathcal{G}, \mathcal{A} , representing the gravitational and van der Waals forces as well as the diffusion coefficient \mathcal{D} are assumed to be strictly positive.

Definition 1. Set $\zeta = 3$ for the capillary driven flow ($\mathcal{S} > 0$) and $\zeta = 1$ for the gravity driven flow ($\mathcal{S} = 0$). We say that $(f, \Theta) \in (L^1(0, T; W^{\zeta, 1}(\mathbb{T})))^2$ is a weak solution of (10) on $[0, T)$ corresponding to initial data (f_0, Θ_0) if and only if

$$\begin{aligned} & - \int_{\mathbb{T}} f_0 \phi(0) dx - \int_0^T \int_{\mathbb{T}} f \partial_t \phi dx dt + \int_0^T \int_{\mathbb{T}} \partial_x^2 \phi \left(-\frac{h_{\sharp}^2}{2} \Theta + \left(\frac{\mathcal{A}}{h_{\sharp}} - \frac{\mathcal{G}}{3} h_{\sharp}^3 \right) f \right) dx dt \\ & + \int_0^T \int_{\mathbb{T}} \partial_x^4 \phi \frac{\mathcal{S}}{3} h_{\sharp}^3 f dx dt + \sum_{j=1}^4 \int_0^T \int_{\mathbb{T}} \partial_x^{-1} N_j \partial_x \phi dx dt = 0 \end{aligned}$$

and

$$\begin{aligned} & - \int_{\mathbb{T}} \Theta_0 \psi(0) dx - \int_0^T \int_{\mathbb{T}} \Theta \partial_t \psi dx dt \\ & + \int_0^T \int_{\mathbb{T}} \partial_x^2 \psi \left(- (h_{\sharp} \Gamma_{\sharp} + \mathcal{D}) \Theta + \left(\frac{3\mathcal{A}\Gamma_{\sharp}}{2h_{\sharp}^2} - \frac{\mathcal{G}}{2} \Gamma_{\sharp} h_{\sharp}^2 \right) f \right) dx dt \\ & + \int_0^T \int_{\mathbb{T}} \partial_x^4 \psi \frac{\mathcal{S}}{2} \Gamma_{\sharp} h_{\sharp}^2 f dx dt + \sum_{j=5}^8 \int_0^T \int_{\mathbb{T}} \partial_x^{-1} N_j \partial_x \psi dx dt = 0 \end{aligned}$$

for all $(\phi, \psi) \in C_c^{\infty}([0, T) \times \mathbb{T})$, where ∂_x^{-1} denotes the operator given by $\widehat{\partial_x^{-1} u}(n) = -\frac{i}{n} \hat{u}(n)$.

Let $r, s \geq 0$. For $(f, \Theta) : [0, T) \rightarrow \dot{A}^r(\mathbb{T}) \times \dot{A}^s(\mathbb{T})$, we define the following functional:

$$\mathcal{E}_s^r(f, \Theta) : [0, T) \rightarrow [0, \infty), \quad \mathcal{E}_s^r(f, \Theta)(t) := \|f(t)\|_{\dot{A}^r} + \|\Theta(t)\|_{\dot{A}^s}. \quad (17)$$

We start by formulating our main result for the gravity driven flow, that is when $\mathcal{S} = 0$. To this end, let us define

$$\begin{aligned} \Lambda_1(t) &:= h_{\sharp} + \frac{19h_{\sharp}^2 \mathcal{G}}{3} + \frac{\mathcal{A}}{h_{\sharp}^2} \left(1 - \frac{\mathcal{E}_0^0(f, \Theta)(t)}{h_{\sharp}} \right)^{-1} \left[2 + \left(1 - \frac{\mathcal{E}_0^0(f, \Theta)(t)}{h_{\sharp}} \right)^{-1} \right] + \Gamma_{\sharp} \\ &+ \frac{3\mathcal{A}}{2h_{\sharp}^3} \left(1 - \frac{\mathcal{E}_0^0(f, \Theta)(t)}{h_{\sharp}} \right)^{-2} \left\{ 7\Gamma_{\sharp} + \frac{3}{2}h_{\sharp} + \left(1 - \frac{\mathcal{E}_0^0(f, \Theta)(t)}{h_{\sharp}} \right)^{-1} 4\Gamma_{\sharp} \right\} \\ &+ \mathcal{G} \left(4\Gamma_{\sharp} h_{\sharp} + 5h_{\sharp}^2 \right), \end{aligned} \quad (18)$$

$$\Lambda_2(t) := \frac{13}{2}h_{\sharp} + 2\Gamma_{\sharp} + \mathcal{G}h_{\sharp}^2 + \frac{3\mathcal{A}}{2h_{\sharp}^3} \left(1 - \frac{\mathcal{E}_0^0(f, \Theta)(t)}{h_{\sharp}} \right)^{-2} \frac{h_{\sharp}}{2}, \quad (19)$$

and the constants

$$\mathfrak{C}_1 := \frac{\mathcal{G}}{3} h_{\sharp}^3 - \frac{\mathcal{A}}{h_{\sharp}} - \left| \frac{3\mathcal{A}\Gamma_{\sharp}}{2h_{\sharp}^2} - \Gamma_{\sharp} h_{\sharp}^2 \frac{\mathcal{G}}{2} \right|, \quad (20)$$

$$\mathfrak{C}_2 := h_{\sharp} \Gamma_{\sharp} + \mathcal{D} - \frac{h_{\sharp}^2}{2}. \quad (21)$$

Then, we have the following result:

Theorem 1 (Global existence for $\mathcal{S} = 0$). Let $f_0 \in \dot{A}^0(\mathbb{T})$, $\Theta_0 \in \dot{A}^0(\mathbb{T})$ be the nonzero initial data for (10) such that

$$\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_{\sharp}, \Gamma_{\sharp}\}.$$

Assume that $\mathfrak{C}_1, \mathfrak{C}_2 > 0$. If

$$\gamma_1 := \mathfrak{C}_1 - \Lambda_1(0) \mathcal{E}_0^0(f_0, \Theta_0) > 0,$$

$$\gamma_2 := \mathfrak{C}_2 - \Lambda_2(0) \mathcal{E}_0^0(f_0, \Theta_0) > 0,$$

then there exists at least one global solution (f, Θ) of (10) in the sense of Definition 1 with regularity

$$(f, \Theta) \in \left(L^{\frac{2}{r}}(0, T; W^{r, \infty}(\mathbb{T})) \cap L^1(0, T; C^{1+\alpha}(\mathbb{T})) \cap L^2(0, T; H^2(\mathbb{T})) \right)^2,$$

for any $r \in [0, 2)$, $\alpha \in [0, \frac{1}{2})$ and $T > 0$. Moreover, the weak solution satisfies the following exponential decay:

$$\|f(t)\|_{L^\infty} + \|\Theta(t)\|_{L^\infty} \leq \mathcal{E}_0^0(f_0, \Theta_0) e^{-\delta t},$$

where

$$\delta = \min\{\gamma_1, \gamma_2\} > 0. \quad (22)$$

Furthermore, if the solution (f, Θ) satisfies the regularity

$$(f, \Theta) \in \left(L^1(0, T; \dot{A}^2(\mathbb{T})) \right)^2,$$

then the weak solution is unique.

Remark 1. The condition $\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_{\sharp}, \Gamma_{\sharp}\}$ in Theorem 1 implies in particular that

$$\|f_0\|_{L^\infty} < \|f_0\|_{\dot{A}^0} < h_{\sharp} \quad \text{and} \quad \|\Theta_0\|_{L^\infty} < \|\Theta_0\|_{\dot{A}^0} < \Gamma_{\sharp}.$$

This corresponds to a positivity condition of h_0 and Γ_0 and ensures that (14) is initially satisfied.

Remark 2. Note that the size restriction is explicitly computable and that the initial data do not need to be small in Sobolev spaces (one should compare the results in this paper with the global result in [9]). In particular, we prove the existence of global weak solutions and their decay towards the flat state for highly oscillating initial data. Let us further explain this with an explicit example. Consider the case where van der Waals forces are neglected, that is $\mathcal{A} = 0$ and set $\mathcal{G} = \mathcal{D} = 1$. We choose the initial data (h_0, Γ_0) to be

$$h_0(x) = 1 + \mu \sin(1000x) \quad \text{and} \quad \Gamma_0(x) = \frac{1}{2} + \mu \cos(1000x),$$

for $0 < \mu < \frac{1}{4}$. Then,

$$h_{\sharp} = 1, \quad \Gamma_{\sharp} = \frac{1}{2}, \quad \mathcal{E}_0^0(h_0 - h_{\sharp}, \Gamma_0 - \Gamma_{\sharp}) = 2\mu,$$

and the constants $\mathfrak{C}_1, \mathfrak{C}_2, \Lambda_1(0)$, and $\Lambda_2(0)$ are given by

$$\mathfrak{C}_1 = \frac{1}{12}, \quad \mathfrak{C}_2 = 1, \quad \Lambda_1(0) = \frac{89}{6}, \quad \Lambda_2(0) = \frac{17}{2}.$$

Then, for $\mu < \frac{1}{356} \sim 0.003$, this family of initial data satisfies the hypotheses and Theorem 1 guarantees the existence of a global weak solution (h, Γ) corresponding to the initial data (h_0, Γ_0) . Moreover the solution decays exponentially towards the flat equilibrium and

$$\|h(t) - h_{\sharp}\|_{L^\infty} + \|\Gamma - \Gamma_{\sharp}\|_{L^\infty} \leq 2\mu e^{-\delta t},$$

where $\delta = \frac{1}{12} - \frac{89}{3}\mu$. Furthermore, observe that

$$\|h_0\|_{\dot{C}^1} = \|\Gamma_0\|_{\dot{C}^1} = O(1) \quad \text{while} \quad \|h_0\|_{H^2} = \|\Gamma_0\|_{H^2} = O(10^3).$$

Next we formulate our main theorem for the capillary driven flow, that is for the full system (7) with $\mathcal{S} > 0$. In addition to (18)–(21), we define

$$\Lambda_3 := \frac{\mathcal{S}}{2}(14h_{\sharp}\Gamma_{\sharp} + 4h_{\sharp}^2) + \frac{19}{3}\mathcal{S}h_{\sharp}^2, \quad (23)$$

$$\mathfrak{C}_3 := \frac{\mathcal{S}}{3}h_{\sharp}^3 - \Gamma_{\sharp}h_{\sharp}^2\frac{\mathcal{S}}{2}. \quad (24)$$

Then, we have the following result:

Theorem 2 (Global existence for $\mathcal{S} > 0$). *Let $f_0 \in \dot{A}^0(\mathbb{T})$, $\Theta_0 \in \dot{A}^0(\mathbb{T})$ be the nonzero initial data for (10) such that*

$$\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_{\sharp}, \Gamma_{\sharp}\}.$$

Assume that $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 > 0$. If

$$\begin{aligned}\gamma_1 &:= \mathfrak{C}_1 - \Lambda_1(0) \mathcal{E}_0^0(f_0, \Theta_0) > 0, \\ \gamma_2 &:= \mathfrak{C}_2 - \left(\Lambda_2(0) + \frac{\mathcal{S}h_{\sharp}^2}{2} \right) \mathcal{E}_0^0(f_0, \Theta_0) > 0, \\ \gamma_3 &:= \mathfrak{C}_3 - \Lambda_3 \mathcal{E}_0^0(f_0, \Theta_0) > 0,\end{aligned}$$

then there exists at least one global solution (f, Θ) of (10) in the sense of Definition 1 with regularity

$$\begin{aligned}f &\in L^{\frac{4}{s}}(0, T; W^{s, \infty}(\mathbb{T})) \cap L^1(0, T; C^{3+\alpha}(\mathbb{T})) \cap L^2(0, T; H^2(\mathbb{T})), \quad s \in [0, 4), \\ \Theta &\in L^{\frac{4}{r}}(0, T; W^{r, \infty}(\mathbb{T})) \cap L^1(0, T; C^{1+\alpha}(\mathbb{T})) \cap L^2(0, T; H^1(\mathbb{T})), \quad r \in [0, 2)\end{aligned}$$

for any T and $0 \leq \alpha < \frac{1}{2}$. Moreover, the weak solutions satisfies the following exponential decay:

$$\|f(t)\|_{L^\infty} + \|\Theta(t)\|_{L^\infty} \leq \mathcal{E}_0^0(f_0, \Theta_0) e^{-\delta t},$$

where

$$\delta = \min\{\gamma_1, \gamma_2, \gamma_3\} > 0. \quad (25)$$

Furthermore, if the solution (f, Θ) satisfies the regularity

$$(f, \Theta) \in L^1(0, T; \dot{A}^4(\mathbb{T})) \times L^1(0, T; \dot{A}^2(\mathbb{T})),$$

then the weak solution is unique.

The remainder of this paper is devoted to the proof of the above theorems.

4. Proof of Theorem 1: global existence and decay when $\mathcal{S} = 0$

Let us start by outlining the steps of the proof. First, we provide in Subsection 4.1 some *a priori* estimates of a solution $(f, \Theta) \in (C^1(0, T; \dot{A}^0(\mathbb{T})))^2$ of (10). Under the assumptions on the initial data in Theorem 1, which in particular require that

$$\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_{\sharp}, \Gamma_{\sharp}\},$$

we will show that the solution (f, Θ) preserves this estimate, that is

$$\mathcal{E}_0^0(f, \Theta)(t) < \min\{h_{\sharp}, \Gamma_{\sharp}\} \quad \text{for all } 0 < t < T.$$

Notice that the above inequality implies that

$$\begin{aligned}\|f(t)\|_{\dot{A}^0} &\leq \mathcal{E}_0^0(f, \Theta)(t) < \min\{h_{\sharp}, \Gamma_{\sharp}\} < h_{\sharp} \quad \text{for all } 0 < t < T, \\ \|\Theta(t)\|_{\dot{A}^0} &\leq \mathcal{E}_0^0(f, \Theta)(t) < \min\{h_{\sharp}, \Gamma_{\sharp}\} < \Gamma_{\sharp} \quad \text{for all } 0 < t < T.\end{aligned}$$

Hence, the positivity conditions for h and Γ ,

$$\|f(t)\|_{L^\infty} \leq \|f(t)\|_{\dot{A}^0} < h_\sharp, \quad \|\Theta(t)\|_{L^\infty} \leq \|\Theta(t)\|_{\dot{A}^0} < \Gamma_\sharp$$

are preserved for all $t \in [0, T)$. Then we obtain that, for small enough initial energy $\mathcal{E}_0^0(f_0, \Theta_0)$, the following inequality holds:

$$\frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) + \delta \mathcal{E}_2^2(f, \Theta)(t) \leq 0$$

for some $\delta > 0$ dependent on the initial data. This inequality implies that a local solution (f, Θ) can not leave a ball in $(L^\infty((0, T); \dot{A}^0(\mathbb{T})) \cap L^1(0, T; \dot{A}^2(\mathbb{T})))^2$; thus, there is no finite time singularity in these functional spaces. In Subsection 4.2 we implement a standard Galerkin approximation argument to prove the existence of global weak solutions. The existence of a classical local solution of the Galerkin approximate system is guaranteed by the Picard–Lindelöf Theorem. These approximate solutions $(f_M, \Theta_M)_{M \in \mathbb{N}}$ are analytic. In particular, they satisfy

$$(f_M, \Theta_M) \in \left(C^1([0, T_M]; \dot{A}^0(\mathbb{T})) \right)^2,$$

where $T_M > 0$ denotes the maximal time of existence of (f_M, Θ_M) . The *a priori* estimates from before ensure the *global* existence of the approximate solutions and provide the necessary *a priori* bounds in order to use compactness arguments and pass to the limit $M \rightarrow \infty$, which yields a global weak solution of (10) in the sense of Definition 1. Now, let (f_M, Θ_M) be such approximate solution corresponding to the initial data (f_0, Θ_0) , which satisfy the hypothesis of Theorem 1, then, in view of the Poincaré-like inequality in (4), we see that $\mathcal{E}_0^0(f_M, \Theta_M)(t)$ is a Lyapunov functional, *i.e.*, that

$$\frac{d}{dt} \mathcal{E}_0^0(f_M, \Theta_M)(t) + \delta \mathcal{E}_0^0(f_M, \Theta_M)(0) \leq 0 \quad \text{for all } 0 < t < \infty.$$

The latter implies the exponential decay towards the equilibrium $(f^*, \Theta^*) = (0, 0)$ for the global weak solution (see Subsection 4.3). Eventually, in Subsection 4.4 it is shown that a global weak solution in

$$\left(L^1(0, T; \dot{A}^2(\mathbb{T})) \right)^2$$

is unique.

4.1. A priori estimates in $\dot{A}^0(\mathbb{T})$

Let $T \in (0, \infty]$ and

$$(f, \Theta) \in \left(C^1([0, T], \dot{A}^4(\mathbb{T})) \right)^2$$

be a local smooth solution of (10) with initial data

$$(f_0, \Theta_0) \in (\dot{A}(\mathbb{T}))^2$$

satisfying the condition

$$\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_\sharp, \Gamma_\sharp\}.$$

By continuity there exists $0 < t_* \leq T$ such that the above estimate is satisfied on $[0, t_*)$, that is

$$\mathcal{E}_0^0(f, \Theta)(t) < \min\{h_\sharp, \Gamma_\sharp\} \quad \text{for all } t \in [0, t_*). \quad (26)$$

Let us moreover assume that t_* is the maximal time such that (26) holds true on $[0, t_*)$. Hence, either

$$\mathcal{E}_0^0(f, \Theta)(t_*) = \min\{h_\sharp, \Gamma_\sharp\} \quad (27)$$

or $t_* = T$. Notice that the restriction (14) holds true on the time interval $[0, t_*)$. In the following let $t \in [0, t_*)$. We compute

$$\partial_t |\hat{f}(k)| = \frac{\operatorname{Re} \left(\bar{\hat{f}}(k) \partial_t \hat{f}(k) \right)}{|\hat{f}(k)|},$$

so, using that $\mathcal{S} = 0$,

$$\frac{d}{dt} \|f\|_{\dot{A}^0} \leq \frac{h_\sharp^2}{2} \|\Theta\|_{\dot{A}^2} + \left(\frac{\mathcal{A}}{h_\sharp} - \frac{\mathcal{G}}{3} h_\sharp^3 \right) \|f\|_{\dot{A}^2} + \|N_1\|_{\dot{A}^0} + \|N_2\|_{\dot{A}^0} + \|N_4\|_{\dot{A}^0}, \quad (28)$$

$$\frac{d}{dt} \|\Theta\|_{\dot{A}^0} \leq -(h_\sharp \Gamma_\sharp + \mathcal{D}) \|\Theta\|_{\dot{A}^2} + \left| \frac{3\mathcal{A}\Gamma_\sharp}{2h_\sharp^2} - \Gamma_\sharp h_\sharp^2 \frac{\mathcal{G}}{2} \right| \|f\|_{\dot{A}^2} + \|N_5\|_{\dot{A}^0} + \|N_6\|_{\dot{A}^0} + \|N_8\|_{\dot{A}^0} \quad (29)$$

We recall that

$$\widehat{uv}(k) = \sum_{j \in \mathbb{Z}} \hat{u}(j) \hat{v}(k - j)$$

and by the hypothesis of the theorem

$$\|f\|_{\dot{A}^0} \leq h_\sharp, \quad \|\Theta\|_{\dot{A}^0} \leq \Gamma_\sharp.$$

Using the algebra property of the Wiener space $A^s(\mathbb{T})$, i.e.

$$\|fg\|_{\dot{A}^s} \leq 2^s \|f\|_{\dot{A}^s(\mathbb{T})} \|g\|_{\dot{A}^s(\mathbb{T})}, \quad \text{for all } f, g \in \dot{A}^s(\mathbb{T}), \quad s \in \mathbb{N},$$

together with the interpolation inequality

$$\|f\|_{\dot{A}^{s\theta}(\mathbb{T})} \leq \|f\|_{\dot{A}^0(\mathbb{T})}^{1-\theta} \|f\|_{\dot{A}^s(\mathbb{T})}^\theta, \quad \text{for all } 0 < \theta < 1, \quad s \geq 0,$$

the contribution of the nonlinear terms N_k , $k = 1, 2$, and 4 , can be estimated as

$$\begin{aligned} \|N_1\|_{\dot{A}^0} &\leq \left(\frac{\|f\|_{\dot{A}^0}^2}{2} + \|f\|_{\dot{A}^0} h_{\sharp} \right) \|\Theta\|_{\dot{A}^2} + (\|f\|_{\dot{A}^1} \|f\|_{\dot{A}^0} + \|f\|_{\dot{A}^1} h_{\sharp}) \|\Theta\|_{\dot{A}^1} \\ &\leq \left(\frac{3}{2} \|f\|_{\dot{A}^0} h_{\sharp} + h_{\sharp} \|\Theta\|_{\dot{A}^0} \right) \|\Theta\|_{\dot{A}^2} + h_{\sharp} \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} \\ &\leq \mathcal{E}_0^0(f, \Theta) \frac{5}{2} h_{\sharp} \|\Theta\|_{\dot{A}^2} + h_{\sharp} \mathcal{E}_0^0(f, \Theta) \|f\|_{\dot{A}^2}, \\ \|N_2\|_{\dot{A}^0} &\leq \frac{\mathcal{G}}{3} \left[\left(3h_{\sharp}^2 \|f\|_{\dot{A}^0} + 3\|f\|_{\dot{A}^0}^2 h_{\sharp} + \|f\|_{\dot{A}^0}^3 \right) \|f\|_{\dot{A}^2} \right. \\ &\quad \left. + \left(3h_{\sharp}^2 \|f\|_{\dot{A}^1} + 6\|f\|_{\dot{A}^1} \|f\|_{\dot{A}^0} h_{\sharp} + 3\|f\|_{\dot{A}^0}^2 \|f\|_{\dot{A}^1} \right) \|f\|_{\dot{A}^1} \right] \\ &\leq \frac{19h_{\sharp}^2 \mathcal{G}}{3} \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} \\ &\leq \frac{19h_{\sharp}^2 \mathcal{G}}{3} \mathcal{E}_0^0(f, \Theta) \|f\|_{\dot{A}^2}, \end{aligned}$$

and

$$\begin{aligned} \|N_4\|_{\dot{A}^0} &\leq \mathcal{A} \left[\left(\frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \frac{\|f\|_{\dot{A}^2}}{h_{\sharp}} + \left(\frac{\|f\|_{\dot{A}^1}}{h_{\sharp}} \right)^2 \right) \sum_{j=1}^{\infty} \left(\frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{j-1} \right. \\ &\quad \left. + \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \left(\frac{\|f\|_{\dot{A}^1}}{h_{\sharp}} \right)^2 \sum_{j=1}^{\infty} j \left(\frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{j-1} \right] \\ &\leq \mathcal{A} \left[\left(\frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \frac{\|f\|_{\dot{A}^2}}{h_{\sharp}} + \left(\frac{\|f\|_{\dot{A}^1}}{h_{\sharp}} \right)^2 \right) \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-1} \right. \\ &\quad \left. + \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \left(\frac{\|f\|_{\dot{A}^1}}{h_{\sharp}} \right)^2 \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-2} \right] \\ &\leq \mathcal{A} \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \frac{\|f\|_{\dot{A}^2}}{h_{\sharp}} \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-1} \left[2 + \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-1} \right] \\ &\leq \mathcal{A} \frac{\mathcal{E}_0^0(f, \Theta)}{h_{\sharp}} \frac{\|f\|_{\dot{A}^2}}{h_{\sharp}} \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-1} \left[2 + \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_{\sharp}} \right)^{-1} \right], \end{aligned}$$

where we used the convergence of the geometric series $\sum_{j=1}^{\infty} r^{j-1}$ for $|r| < 1$, and

$$\sum_{j=1}^{\infty} j r^{j-1} = \partial_r \sum_{j=1}^{\infty} r^j = \frac{1}{(1-r)^2}, \quad |r| < 1.$$

Similarly, the nonlinear terms N_j , $j = 5, 6$, and 8 are bounded by

$$\begin{aligned}
\|N_5\|_{\dot{A}^0} &\leq (\Gamma_\sharp \|f\|_{\dot{A}^0} + (4h_\sharp + \Gamma_\sharp) \|\Theta\|_{\dot{A}^0}) \|\Theta\|_{\dot{A}^2} + \Gamma_\sharp \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} \\
&\leq \mathcal{E}_0^0(f, \Theta) (4h_\sharp + 2\Gamma_\sharp) \|\Theta\|_{\dot{A}^2} + \Gamma_\sharp \mathcal{E}_0^0(f, \Theta) \|f\|_{\dot{A}^2}, \\
\|N_6\|_{\dot{A}^0} &\leq \mathcal{G} \left[(4\Gamma_\sharp h_\sharp \|f\|_{\dot{A}^0} + 4\|\Theta\|_{\dot{A}^0} h_\sharp^2) \|f\|_{\dot{A}^2} + h_\sharp^2 (\|\Theta\|_{\dot{A}^0} \|\Theta\|_{\dot{A}^2} + \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2}) \right] \\
&\leq \mathcal{G} \mathcal{E}_0^0(f, \Theta) \left[(4\Gamma_\sharp h_\sharp + 4h_\sharp^2) \|f\|_{\dot{A}^2} + h_\sharp^2 (\|\Theta\|_{\dot{A}^2} + \|f\|_{\dot{A}^2}) \right], \\
\|N_8\|_{\dot{A}^0} &\leq \frac{3\mathcal{A}}{2h_\sharp^4} \left[(3\|f\|_{\dot{A}^0} \Gamma_\sharp h_\sharp + \|\Theta\|_{\dot{A}^0} h_\sharp^2) \|f\|_{\dot{A}^2} \right. \\
&\quad \left. + \left(4\|f\|_{\dot{A}^0} \Gamma_\sharp h_\sharp \|f\|_{\dot{A}^2} + \frac{h_\sharp^2}{2} [\|\Theta\|_{\dot{A}^0} \|\Theta\|_{\dot{A}^2} + \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2}] \right) \right] \sum_{j=1}^{\infty} j \left(\frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{j-1} \\
&\quad + \frac{3\mathcal{A}}{h_\sharp^3} [2\Gamma_\sharp \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2}] \sum_{j=2}^{\infty} j(j-1) \left(\frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{j-2} \\
&\leq \frac{3\mathcal{A}}{2h_\sharp^3} \left[(3\|f\|_{\dot{A}^0} \Gamma_\sharp + \|\Theta\|_{\dot{A}^0} h_\sharp) \|f\|_{\dot{A}^2} \right. \\
&\quad \left. + \left(4\|f\|_{\dot{A}^0} \Gamma_\sharp \|f\|_{\dot{A}^2} + \frac{h_\sharp}{2} [\|\Theta\|_{\dot{A}^0} \|\Theta\|_{\dot{A}^2} + \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2}] \right) \right] \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-2} \\
&\quad + \frac{3\mathcal{A}}{h_\sharp^3} [2\Gamma_\sharp \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2}] \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-3}.
\end{aligned}$$

Grouping terms, we find that

$$\begin{aligned}
\|N_8\|_{\dot{A}^0} &\leq \frac{3\mathcal{A}}{2h_\sharp^3} \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-2} \left\{ 7\Gamma_\sharp \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} + \frac{1}{2} h_\sharp [\|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} + \frac{h_\sharp}{2} \|\Theta\|_{\dot{A}^0} \|\Theta\|_{\dot{A}^2}] \right. \\
&\quad \left. + h_\sharp \|\Theta\|_{\dot{A}^0} \|f\|_{\dot{A}^2} + \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-1} 4\Gamma_\sharp \|f\|_{\dot{A}^0} \|f\|_{\dot{A}^2} \right\} \\
&\leq \frac{3\mathcal{A}}{2h_\sharp^3} \mathcal{E}_0^0(f, \Theta) \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-2} \left\{ 7\Gamma_\sharp \|f\|_{\dot{A}^2} + \frac{1}{2} h_\sharp [\|f\|_{\dot{A}^2} + \frac{h_\sharp}{2} \|\Theta\|_{\dot{A}^2}] \right. \\
&\quad \left. + h_\sharp \|f\|_{\dot{A}^2} + \left(1 - \frac{\|f\|_{\dot{A}^0}}{h_\sharp} \right)^{-1} 4\Gamma_\sharp \|f\|_{\dot{A}^2} \right\}.
\end{aligned}$$

We recall the definition (20) and (21), then we add equations (28) and (29), use the previous estimates for N_j , $j = 1, 2, 4, 5, 6$, and 8 and obtain that

$$\frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) \leq - \left(\mathfrak{C}_1 - \Lambda_1(t) \mathcal{E}_0^0(f, \Theta)(t) \right) \|f(t)\|_{\dot{A}^2} - \left(\mathfrak{C}_2 - \Lambda_2(t) \mathcal{E}_0^0(f, \Theta)(t) \right) \|\Theta(t)\|_{\dot{A}^2} \quad (30)$$

for $t \in [0, t_*)$, with Λ_1 and Λ_2 defined in (18) and (19), respectively. Using the hypothesis of the Theorem 1, we have that

$$\left. \frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) \right|_{t=0} < 0,$$

so, there exists a time $0 < t_0 \leq t_*$ such that

$$\mathcal{E}_0^0(f, \Theta)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0) \quad \text{for all } t \in [0, t_0]. \quad (31)$$

Let us assume that t_0 is the maximal time such that (31) holds true on $[0, t_0]$. We want to propagate this decay for all times, that is, we aim to show that $t_0 = t_*$, which in turn implies that $t_0 = t_* = T$, by (27). Let us emphasize that $\mathcal{E}_0^0(f, \Theta)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0)$ for $t \in [0, t_0]$ guarantees that

$$\Lambda_j(t) \leq \Lambda_j(0), \quad j = 1, 2 \quad \text{for all } t \in [0, t_0]. \quad (32)$$

Thereby, for any $t \in [0, t_0]$ we have that

$$\begin{aligned} \mathfrak{E}_1 - \Lambda_1(t) \mathcal{E}_0^0(f, \Theta)(t) &\geq \mathfrak{E}_1 - \Lambda_1(0) \mathcal{E}_0^0(f_0, \Theta_0) = \gamma_1 > 0, \\ \mathfrak{E}_2 - \Lambda_2(t) \mathcal{E}_0^0(f, \Theta)(t) &\geq \mathfrak{E}_2 - \Lambda_2(0) \mathcal{E}_0^0(f_0, \Theta_0) = \gamma_2 > 0. \end{aligned}$$

In particular, we obtain that

$$\frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) < 0 \quad \text{for all } t \in [0, t_0].$$

Assume that $t_0 < t_*$. By continuity, see (31), we deduce that $\mathcal{E}_0^0(f, \Theta)(t_0) = \mathcal{E}_0^0(f, \Theta)(0)$, but this implies that, again,

$$\left. \frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) \right|_{t=t_0} \leq 0,$$

and that contradicts the maximality assumption on t_0 . Thus, we have shown that in fact

$$\mathcal{E}_0^0(f, \Theta)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0) \quad \text{for all } t \in [0, t_*] \quad (33)$$

and thereby $t_* = T$, in view of (27). Then,

$$\frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) \leq -\gamma \mathcal{E}_2^2(f, \Theta)(t) \quad \text{for all } t \in [0, T], \quad (34)$$

where $\delta := \min\{\gamma_1, \gamma_2\}$. Eventually, the energy estimate for the gravity driven equation ($\mathcal{S} = 0$) reads

$$\mathcal{E}_0^0(f, \Theta)(t) + \delta \int_0^t \mathcal{E}_2^2(f, \Theta)(\tau) d\tau \leq \mathcal{E}_0^0(f_0, \Theta_0), \quad \forall t \in [0, T].$$

Similarly, using the equations for f and Θ together with the previous regularity, we find that

$$\int_0^t \|\partial_t f(\tau)\|_{\dot{A}^0} + \|\partial_t \Theta(\tau)\|_{\dot{A}^0} d\tau \leq c,$$

for all $t \in [0, T)$, where $c > 0$ is a constant depending on the initial data.

4.2. Existence of global weak solutions

We use a standard Galerkin approximation to obtain in the limit a global weak solution of (10) where surface tension effects are neglected, *i.e.* $\mathcal{S} = 0$. Let us fix $M \in \mathbb{Z}^+$. Set

$$f_M(t, x) := \sum_{|k| \leq M} \hat{f}(t, k) e^{ikx} \quad \text{and} \quad \Theta_M(t, x) := \sum_{|k| \leq M} \hat{\theta}(t, k) e^{ikx}$$

and the initial data

$$f_M(0, x) := \sum_{|k| \leq M} \hat{f}_0(k) e^{ikx} \quad \text{and} \quad \Theta_M(0, x) := \sum_{|k| \leq M} \hat{\theta}_0(k) e^{ikx}$$

to coincide with the Fourier truncation of the f_0 and Θ_0 , respectively. Recall that the convergence of the Fourier series of the initial data is guaranteed by the assumption that $f_0, \Theta_0 \in \dot{A}^0(\mathbb{T})$. We consider the Galerkin approximate problems:

$$\begin{aligned} \partial_t f_M - \frac{h_\#^2}{2} \partial_x^2 \Theta_M + \left(\frac{\mathcal{A}}{h_\#} - \frac{\mathcal{G}}{3} h_\#^3 \right) \partial_x^2 f_M + \frac{\mathcal{S}}{3} h_\#^3 \partial_x^4 f_M &= \sum_{j=1,2,4} N_j^M, \quad \text{in } (0, T) \times \mathbb{T} \\ \partial_t \Theta_M - (h_\# \Gamma_\# + \mathcal{D}) \partial_x^2 \Theta_M + \left(\frac{3\mathcal{A}\Gamma_\#}{2h_\#^2} - \frac{\mathcal{G}}{2} \Gamma_\# h_\#^2 \right) \partial_x^2 f_M + \frac{\mathcal{S}}{2} \Gamma_\# h_\#^2 \partial_x^4 f_M \\ &= \sum_{j=5,6,8} N_j^M, \quad \text{in } (0, T) \times \mathbb{T} \end{aligned}$$

where the nonlinearities N_j^M are given by

$$\begin{aligned} N_1^M &= \partial_x P_M \left[\left(\frac{f^2}{2} + f h_\# \right) \partial_x \Theta \right], \\ N_2^M &= \partial_x P_M \left[\frac{\mathcal{G}}{3} \left(3h_\#^2 f + 3f^2 h_\# + f^3 \right) \partial_x f \right], \\ N_4^M &= \mathcal{A} P_M \left[\left(\frac{f}{h_\#^2} \partial_x^2 f + \left(\frac{\partial_x f}{h_\#} \right)^2 \right) \sum_{j=1}^M (-1)^{j+1} \left(\frac{f}{h_\#} \right)^{j-1} \right. \\ &\quad \left. - \frac{f}{h_\#^3} (\partial_x f)^2 \sum_{j=1}^M j (-1)^{j+1} \left(\frac{f}{h_\#} \right)^{j-1} \right], \\ N_5^M &= \partial_x P_M \left[(\Gamma_\# f + \Theta h_\# + \Theta f) \partial_x \Theta \right], \end{aligned}$$

$$\begin{aligned} N_6^M &= \partial_x P_M \left[\frac{\mathcal{G}}{2} \left(\Gamma_{\sharp} f^2 + 2\Gamma_{\sharp} h_{\sharp} f + \Theta h_{\sharp}^2 + \Theta f^2 + 2\Theta h_{\sharp} f \right) \partial_x f \right], \\ N_8^M &= \frac{3\mathcal{A}}{2h_{\sharp}^4} P_M \left[\left(2f\Gamma_{\sharp} h_{\sharp} + f^2\Gamma_{\sharp} - \Theta h_{\sharp}^2 \right) \partial_x^2 f \right. \\ &\quad \left. + \left(2\partial_x f\Gamma_{\sharp} h_{\sharp} + 2f\partial_x f\Gamma_{\sharp} - \partial_x \Theta h_{\sharp}^2 \right) \partial_x f \right] \sum_{j=1}^M j(-1)^{j+1} \left(\frac{f}{h_{\sharp}} \right)^{j-1} \\ &\quad - \frac{3\mathcal{A}}{h_{\sharp}^4} P_M \left[\left(2f\Gamma_{\sharp} h_{\sharp} + f^2\Gamma_{\sharp} - \Theta h_{\sharp}^2 \right) \frac{(\partial_x f)^2}{h_{\sharp}} \right] \frac{1}{2} \sum_{j=2}^M j(j-1)(-1)^j \left(\frac{f}{h_{\sharp}} \right)^{j-2}. \end{aligned}$$

Here, the operator P_M denotes the Fourier truncation operator

$$P_M g(x) = \sum_{|k| \leq M} \hat{g}(k) e^{ikx} \quad \text{for any } g \in \dot{A}^0(\mathbb{T}).$$

The Picard–Lindelöf Theorem ensures the existence of classical solutions

$$f_M, \Theta_M \in C^1([0, T_M]; C^\infty(\mathbb{T})),$$

where $T_M > 0$ is the maximal existence time. Furthermore, the approximate problems provide the same *a priori* bounds as in the previous sections. Consequently the Galerkin solutions $(f_M, \Theta_M)_{M \in \mathbb{N}}$ exist globally and for any $T > 0$ we have the bounds:

$$(f_M, \Theta_M)_{M \in \mathbb{N}} \text{ is uniformly bounded in } \left(L^\infty(0, T; \dot{A}^0(\mathbb{T})) \cap L^1(0, T; \dot{A}^2(\mathbb{T})) \right)^2 \quad (35)$$

and

$$(\partial_t f_M, \partial_t \Theta_M)_{M \in \mathbb{N}} \text{ is uniformly bounded in } \left(L^1(0, T; \dot{A}^0(\mathbb{T})) \right)^2. \quad (36)$$

Following the lines in [10], the above uniform regularities of the Galerkin approximations $(f_M, \Theta_M)_{M \in \mathbb{N}}$ guarantee the existence of a weakly convergent subsequences (not relabeled) such that

$$(f_M, \Theta_M) \overset{*}{\rightharpoonup} (f, \Theta) \text{ in } (L^\infty(0, T; L^\infty(\mathbb{T})))^2.$$

Similarly, using interpolation in Wiener spaces, the finite measure of the spatial domain and the fact that $\dot{A}^r(\mathbb{T}) \subset \dot{W}^{r, \infty}(\mathbb{T})$ for any $r \geq 0$, we obtain the existence of a subsequence (not relabeled), such that

$$(f_M, \Theta_M) \rightharpoonup (f, \Theta) \text{ in } \left(L^{\frac{2}{r}}(0, T; \dot{W}^{r, p}(\mathbb{T})) \right)^2, \quad 0 \leq r < 2, \quad 1 \leq p < \infty. \quad (37)$$

In particular, using that the norm of a Banach space is weakly lower semicontinuous, we have the following bound

$$\|(f, \Theta)\|_{L^{\frac{2}{r}}(0, T; \dot{W}^{r, p}(\mathbb{T}))^2} \leq \liminf_{M \rightarrow \infty} \|(f_M, \Theta_M)\|_{L^{\frac{2}{r}}(0, T; \dot{W}^{r, p}(\mathbb{T}))^2} \leq C(f_0, \Theta_0).$$

From the previous fact, using the finiteness of the domain and taking the limit in p , we can infer that actually

$$\limsup_{p \rightarrow \infty} \|(f, \Theta)\|_{L^{\frac{2}{r}}(0, T; \dot{W}^{r, p}(\mathbb{T}))^2} \leq C(f_0, \Theta_0),$$

so

$$(f, \Theta) \in \left(L^{\frac{2}{r}}(0, T; \dot{W}^{r, \infty}(\mathbb{T})) \right)^2, \quad 0 \leq r < 2.$$

Furthermore (35), (36) imply that

$$(f_M, \Theta_M)_{M \in \mathbb{N}} \text{ is uniformly bounded in } \left(L^2(0, T; H^1(\mathbb{T})) \right)^2,$$

which is due to $\dot{A}^s(\mathbb{T}) \subset \dot{H}^s(\mathbb{T})$ for any $s \geq 0$ and an interpolation inequality for fractional Sobolev spaces. Eventually, as a consequence of (35), (36) and the Aubin-Lions compactness result [43, Corollary 4] with

$$X = H^2, \quad B = C^{1+s}, \quad Y = L^2,$$

we obtain (up to a subsequence) that

$$(f_M, \Theta_M) \rightarrow (f, \Theta) \quad \text{in } \left(L^1(0, T; C^{1+s}(\mathbb{T})) \right)^2, \quad 0 \leq s < \frac{1}{2}.$$

Passing to the limit in the weak formulation of the Galerkin approximation yields the existence of a global weak solution of (10) in the sense of Definition 1.

4.3. Exponential trend to equilibrium

Using (30) and the definition of $\delta > 0$ in (22), we have that

$$\frac{d}{dt} \mathcal{E}_0^0(f_M, \Theta_M)(t) \leq -\delta \mathcal{E}_2^0(f_M, \Theta_M)(t) \quad \text{for all } t \geq 0. \quad (38)$$

Using the Poincaré-like inequality (4), we also conclude that

$$\frac{d}{dt} \mathcal{E}_0^0(f_M, \Theta_M)(t) \leq -\delta \mathcal{E}_0^0(f_M, \Theta_M)(t) \quad \text{for all } t \geq 0,$$

which in turn implies the exponential decay towards the equilibrium:

$$\mathcal{E}_0^0(f_M, \Theta_M)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0)e^{-\delta t} \quad \text{for all } t \geq 0.$$

Using (37) and the lower semi-continuity of the weak-* convergence, we have that

$$\|f(t)\|_{L^\infty(\mathbb{T})} + \|\Theta(t)\|_{L^\infty(\mathbb{T})} \leq \liminf_{k \rightarrow \infty} \mathcal{E}_0^0(f_M, \Theta_M)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0)e^{-\delta t}.$$

4.4. Uniqueness

The proof follows a standard contradiction argument. For the sake of brevity, we only sketch the idea. Assume that there exist two different solutions (f_1, Θ_1) and (f_2, Θ_2) starting from the same initial data

$$(f_0, \Theta_0) \in \left(\dot{A}^0(\mathbb{T})\right)^2.$$

Assume also that these solutions satisfy

$$(f_i, \Theta_i) \in \left(L^1(0, T; \dot{A}^2(\mathbb{T}))\right)^2.$$

Using the smallness of the initial data, the same estimates as in Subsection 4.1 yield that

$$\begin{aligned} & \frac{d}{dt} (\|f_1 - f_2\|_{\dot{A}^0} + \|\Theta_1 - \Theta_2\|_{\dot{A}^0}) \\ & \leq C [\|f_1 - f_2\|_{\dot{A}^0} + \|\Theta_1 - \Theta_2\|_{\dot{A}^0}] (\|f_1\|_{\dot{A}^2} + \|f_2\|_{\dot{A}^2} + \|\Theta_1\|_{\dot{A}^2} + \|\Theta_2\|_{\dot{A}^2} + 1). \end{aligned}$$

Now the statement is a consequence of Gronwall's inequality and the fact that $(f_1, \Theta_1)(0) = (f_2, \Theta_2)(0)$.

5. Proof of Theorem 2: global existence and decay when $\mathcal{S} > 0$

The proof essentially follows the arguments in the previous section, the main difference relying in the fact that for $\mathcal{S} > 0$ we have that (10) is a system of mixed orders. Thereby the energy estimates require some additional investigation. The existence of local solutions of the approximate Galerkin systems are straightforward due to Picard–Lindelöf's theorem and follows as in Subsection 4.2. The energy estimates then ensure that the approximate solutions exist globally. Furthermore, the energy estimates provide the necessary a priori bounds to pass to the limit in the Galerkin approximation; thereby guaranteeing the existence of a global weak solution in the sense of Definition 1. In view of the weak lower semicontinuity of the norm, the global weak solution inherits the energy estimates for the approximate solutions and we can conclude the regularity and exponential decay of the solution.

5.1. A priori estimates in $\dot{A}^0(\mathbb{T})$

To perform our energy estimates let us assume that there exists a local smooth solution

$$(f, \Theta) \in \left(C^1([0, T), \dot{A}^4(\mathbb{T}))\right)^2$$

of the fourth-order system (10) where $\mathcal{S} > 0$, corresponding to initial data (f_0, Θ_0) , satisfying

$$\mathcal{E}_0^0(f_0, \Theta_0) < \min\{h_\sharp, \Gamma_\sharp\}.$$

Similarly as before, we have that

$$\frac{d}{dt} \|f\|_{\dot{A}^0} \leq \frac{h_{\sharp}^2}{2} \|\Theta\|_{\dot{A}^2} + \left(\frac{\mathcal{A}}{h_{\sharp}} - \frac{\mathcal{G}}{3} h_{\sharp}^3 \right) \|f\|_{\dot{A}^2} - \frac{\mathcal{S}}{3} h_{\sharp}^3 \|f\|_{\dot{A}^4} + \sum_{j=1}^4 \|N_j\|_{\dot{A}^0}, \quad (39)$$

$$\frac{d}{dt} \|\Theta\|_{\dot{A}^0} \leq -(h_{\sharp} \Gamma_{\sharp} + \mathcal{D}) \|\Theta\|_{\dot{A}^2} + \left| \frac{3\mathcal{A}\Gamma_{\sharp}}{2h_{\sharp}^2} - \Gamma_{\sharp} h_{\sharp}^2 \frac{\mathcal{G}}{2} \right| \|f\|_{\dot{A}^2} + \frac{\mathcal{S}}{2} h_{\sharp}^2 \Gamma_{\sharp} \|f\|_{\dot{A}^4} + \sum_{j=5}^8 \|N_j\|_{\dot{A}^0}. \quad (40)$$

Keeping in mind the definitions of \mathfrak{C}_i , $i = 1, 2, 3$, in (20), (21), and (24), we take the sum of the two inequalities above and obtain that

$$\frac{d}{dt} \mathcal{E}_0^0(f, \Theta) + \mathfrak{C}_1 \|f\|_{\dot{A}^2} + \mathfrak{C}_2 \|\Theta\|_{\dot{A}^2} + \mathfrak{C}_3 \|f\|_{\dot{A}^4} \leq \sum_{j=1}^8 \|N_j\|_{\dot{A}^0}.$$

Recalling (30), we have that

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_0^0(f, \Theta) + \mathfrak{C}_3 \|f\|_{\dot{A}^4} \\ & \leq \|N_3\|_{\dot{A}^0} + \|N_7\|_{\dot{A}^0} - \left(\mathfrak{C}_1 - \Lambda_1(t) \mathcal{E}_0^0(t) \right) \|f(t)\|_{\dot{A}^2} - \left(\mathfrak{C}_2 - \Lambda_2(t) \mathcal{E}_0^0(t) \right) \|\Theta(t)\|_{\dot{A}^2}, \end{aligned}$$

where Λ_1 and Λ_2 are defined in (18) and (19), respectively. Thus we are left to estimate the remaining terms $\|N_3\|_{\dot{A}^0}$ and $\|N_7\|_{\dot{A}^0}$. Similarly as before, using the interpolation inequality in Wiener spaces, we estimate

$$\|N_3\|_{\dot{A}^0} \leq \frac{19}{3} \mathcal{S} h_{\sharp}^2 \mathcal{E}_0^0(f, \Theta) \|f\|_{\dot{A}^4},$$

and

$$\begin{aligned} \|N_7\|_{\dot{A}^0} & \leq \frac{\mathcal{S}}{2} \left(6\Gamma_{\sharp} h_{\sharp} \|f\|_{\dot{A}^0} + h_{\sharp}^2 \|\Theta\|_{\dot{A}^0} \right) \|f\|_{\dot{A}^4} \\ & \quad + \frac{\mathcal{S}}{2} \left(8\Gamma_{\sharp} h_{\sharp} \|f\|_{\dot{A}^1} + 4\|\Theta\|_{\dot{A}^1} h_{\sharp}^2 \right) \|f\|_{\dot{A}^3}. \end{aligned}$$

Note that by Young's inequality, we have that

$$\|\Theta\|_{\dot{A}^1} \|f\|_{\dot{A}^3} \leq \|\Theta\|_{\dot{A}^1} \|f\|_{\dot{A}^0}^{\frac{1}{4}} \|f\|_{\dot{A}^4}^{\frac{3}{4}} \leq \|\Theta\|_{\dot{A}^1} \left(\frac{1}{4\varepsilon} \|f\|_{\dot{A}^0} + \frac{3\varepsilon}{4} \|f\|_{\dot{A}^4} \right),$$

for any $\varepsilon > 0$. In particular, as Θ and f are nonzero (provided the initial data are nonzero), we can take

$$\varepsilon = \frac{\mathcal{E}_0^0(f, \Theta)}{\|\Theta\|_{\dot{A}^1}}.$$

Thus,

$$\|\Theta\|_{\dot{A}^1} \|f\|_{\dot{A}^3} \leq \frac{\|\Theta\|_{\dot{A}^1}^2}{4\mathcal{E}_0^0(f, \Theta)} \|f\|_{\dot{A}^0} + \frac{3\mathcal{E}_0^0}{4} \|f\|_{\dot{A}^4} \leq \frac{\|\Theta\|_{\dot{A}^2} \mathcal{E}_0^0(f, \Theta)}{4} + \frac{3\mathcal{E}_0^0(f, \Theta)}{4} \|f\|_{\dot{A}^4}.$$

Then, we conclude that

$$\|N_7\|_{\dot{A}^0} \leq \frac{\mathcal{S}}{2} \left((14h_{\sharp} \Gamma_{\sharp} + 4h_{\sharp}^2) \mathcal{E}_0^0(f, \Theta) \|f\|_{\dot{A}^4} + h_{\sharp}^2 \mathcal{E}_0^0(f, \Theta) \|\Theta\|_{\dot{A}^2} \right).$$

Using (23), we can group terms as follows:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0^0(f, \Theta)(t) &\leq - \left(\mathfrak{C}_1 - \Lambda_1(t) \mathcal{E}_0^0(f, \Theta)(t) \right) \|f(t)\|_{\dot{A}^2} \\ &\quad - \left(\mathfrak{C}_2 - \left(\Lambda_2(t) + \frac{\mathcal{S}h_{\sharp}^2}{2} \right) \mathcal{E}_0^0(f, \Theta)(t) \right) \|\Theta(t)\|_{\dot{A}^2} \\ &\quad - \left(\mathfrak{C}_3 - \Lambda_3 \mathcal{E}_0^0(f, \Theta)(t) \right) \|f(t)\|_{\dot{A}^4}. \end{aligned}$$

Repeating the argument from Section 4, we conclude that

$$\mathcal{E}_0^0(f, \Theta)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0) e^{-\delta t} \quad (41)$$

where δ is defined in (25). The energy estimate for the capillary driven thin film ($\mathcal{S} > 0$) reads

$$\mathcal{E}_0^0(f, \Theta)(t) + \int_0^t \delta \mathcal{E}_2^4(f, \Theta)(\tau) d\tau \leq \mathcal{E}_0^0(f_0, \Theta_0)$$

and

$$\int_0^t \|\partial_t f(\tau)\|_{\dot{A}^0} + \|\partial_t \Theta(\tau)\|_{\dot{A}^0} d\tau \leq c,$$

for all $t \in [0, T)$, where $c > 0$ is a constant depending on the initial data.

5.2. Existence of global weak solutions

In a similar way as in Subsection 4.2, we obtain the existence of a sequence of global Galerkin approximations $(f_M, \Theta_M)_{M \in \mathbb{N}}$. Then, the energy estimates from above guarantee that for any $T > 0$:

$$(f_M)_{M \in \mathbb{N}} \text{ is uniformly bounded in } L^\infty(0, T; \dot{A}^0(\mathbb{T})) \cap L^1(0, T; \dot{A}^4(\mathbb{T})) \subset L^2(0, T; \dot{H}^2(\mathbb{T}))$$

and

$(\Theta_M)_{M \in \mathbb{N}}$ is uniformly bounded in $L^\infty(0, T; \dot{A}^0(\mathbb{T})) \cap L^1(0, T; \dot{A}^2(\mathbb{T})) \subset L^2(0, T; \dot{H}^1(\mathbb{T}))$.

Moreover, the time derivatives satisfy

$$(\partial_t f_M, \partial_t \Theta_M)_{M \in \mathbb{N}} \text{ is uniformly bounded in } \left(L^1(0, T; \dot{A}^0(\mathbb{T})) \right)^2$$

for any $T > 0$. Consequently, we obtain the existence of a subsequence (not relabeled) such that

$$\begin{aligned} f_M &\rightharpoonup f \text{ in } L^{\frac{4}{s}}(0, T; \dot{W}^{s,p}(\mathbb{T})), \quad 0 \leq s < 4, \quad 1 \leq p < \infty \\ \Theta_M &\rightharpoonup \Theta \text{ in } L^{\frac{2}{r}}(0, T; \dot{W}^{r,p}(\mathbb{T})), \quad 0 \leq r < 2, \quad 1 \leq p < \infty. \end{aligned} \quad (42)$$

Moreover,

$$\begin{aligned} f_M &\rightarrow f \quad \text{in } L^1(0, T; C^{3+\alpha}(\mathbb{T})), \quad 0 \leq \alpha < \frac{1}{2}, \\ f_M &\rightharpoonup f \quad \text{in } L^2(0, T; \dot{H}^2(\mathbb{T})) \end{aligned}$$

and

$$\begin{aligned} \Theta_M &\rightarrow \Theta \quad \text{in } L^1(0, T; C^{1+\alpha}(\mathbb{T})), \quad 0 \leq \alpha < \frac{1}{2}, \\ \Theta_M &\rightharpoonup \Theta \quad \text{in } L^2(0, T; \dot{H}^1(\mathbb{T})). \end{aligned}$$

Equipped with these convergences we can pass to the limit in the weak formulation and conclude the global existence of a weak solution of (10) for $\mathcal{S} > 0$ in the sense of Definition 1.

5.3. Exponential trend to equilibrium

The proof follows the same ideas as in Subsection 4.3. The sequence of Galerkin approximation $(f_M, \Theta_M)_{M \in \mathbb{N}}$ satisfies the energy estimates. In particular, we have the exponential decay towards the equilibrium in (41):

$$\mathcal{E}_0^0(f_M, \Theta_M)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0)e^{-\delta t} \quad \text{for all } t \geq 0,$$

where $\delta > 0$ is defined in (25). Using (42) and the lower semi-continuity of the weak-* convergence, we have that

$$\|f(t)\|_{L^\infty(\mathbb{T})} + \|\Theta(t)\|_{L^\infty(\mathbb{T})} \leq \liminf_{k \rightarrow \infty} \mathcal{E}_0^0(f_M, \Theta_M)(t) \leq \mathcal{E}_0^0(f_0, \Theta_0)e^{-\delta t}$$

for all $t \geq 0$, which proves the claim.

5.4. Uniqueness

Also the proof for the conditional uniqueness is similar to the one in Subsection 4.4. Assume that there exist two different solutions (f_1, Θ_1) and (f_2, Θ_2) starting from the same initial data

$$(f_0, \Theta_0) \in (\dot{A}(\mathbb{T}))^2.$$

Moreover, we suppose that the solutions satisfy the additional regularity

$$(f_i, \Theta_i) \in \left(L^1(0, T; \dot{A}^4(\mathbb{T})) \times L^1(0, T; \dot{A}^2(\mathbb{T})) \right)^2$$

for $i = 1, 2$. Similarly as for the energy estimate we compute that

$$\begin{aligned} \frac{d}{dt} (\|f_1 - f_2\|_{\dot{A}^0} + \|\Theta_1 - \Theta_2\|_{\dot{A}^0}) \\ \leq C [\|f_1 - f_2\|_{\dot{A}^0} + \|\Theta_1 - \Theta_2\|_{\dot{A}^0}] \\ \times (\|f_1\|_{\dot{A}^2} + \|f_2\|_{\dot{A}^2} + \|f_1\|_{\dot{A}^4} + \|f_2\|_{\dot{A}^4} + \|\Theta_1\|_{\dot{A}^2} + \|\Theta_2\|_{\dot{A}^2}). \end{aligned}$$

Now the assertion follows by applying Gronwall's inequality and recalling that the solutions (f_i, Θ_i) , $i = 1, 2$ share the same initial data.

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