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GROWTH IN THE MUSKAT PROBLEM*

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Abstract. We review some recent results on the Muskat problem modelling multiphase flow in porous media. Furthermore, we prove a new regularity criteria in terms of some norms of the initial data in critical spaces ($\dot{W}^{1,\infty}$ and $\dot{H}^{3/2}$).

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1. Introduction

The mathematical study of multiphase flow in porous media is a very active research area [7, 79, 82]. Besides being mathematically challenging, this problem is also physically interesting as it models oil extraction [11, 65, 80, 81], tumor growth [56], beach evolution [95] or a geothermal reservoir [18]. The problem of studying the free boundary flow in porous media is also known as the Muskat problem [79]. The purpose of this paper is to review some recent results on the Muskat problem and also to prove a new regularity criteria following the same approach as the one developed in [33].

Flow (at relatively slow velocities) in porous media evolves according to Darcy's Law

$$\frac{\mu}{\kappa}u(x,y,t)+\nabla p(x,y,t)=-\rho(x,y,t)G(0,1)^T,\ \text{ for } (x,y,t)\in\mathbb{R}^2\times[0,T] \tag{1.1a}$$

$$\nabla\cdot u(x,y,t)=0,\ \text{ for } (x,y,t)\in\mathbb{R}^2\times[0,T] \tag{1.1b}$$

$$\partial_t\rho(x,y,t)+\nabla\cdot(u(x,y,t)\rho(x,y,t))=0 \text{ for } (x,y,t)\in\mathbb{R}^2\times[0,T], \tag{1.1c}$$

where p,u,ρ and μ are the pressure, velocity, density and viscosity of the incompressible fluids while κ denotes the permeability of the medium and G denotes the acceleration due to gravity. In what follows we will assume that, in an appropriate choice of units, $G=1$. Unless otherwise stated, we will also fix $\kappa=1$. Darcy's Law was derived heuristically by Henry Darcy in 1856 [42] (although it can be derived rigorously using homogenization techniques [68, 94]). Remarkably, (1.1a) was derived independently by Hele-Shaw [66, 67] when he was studying viscid flow between two parallel flat plates separated by a narrow distance. The mathematical literature on the

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(horizontal, *i.e.* where gravity is neglected) Hele-Shaw cell problem is also large. The interested reader can refer to [20–22, 44, 46, 47, 92] and the references therein.

Equation (1.1) is a system of hyperbolic active scalar equations. Some other equation in this family are the famous surface quasi-geostrophic equation [8, 19, 23, 24, 28, 69, 70], the magnetogeostrophic equation [52–55, 78], the Stokes system [5, 71] or the 2D Euler equation in vorticity formulation [72, 73]. The Muskat problem studies the particular type of solution where there are two different immiscible fluids, a fluid on top with label $+$ and a fluid below with label $-$, with properties given by (ρ^+,μ^+) and (ρ^-,μ^-) (or a fluid with (ρ^-,μ^-) and a dry zone with $\rho^+=\mu^+=0$) separated by a moving interface, parametrized as

$$\Gamma(t)=\{(x,y)\in\mathbb{R}^2,\ \ (x,y)=(z_1(\alpha,t),z_2(\alpha,t)),\ \alpha\in\mathbb{R}\},\tag{1.2}$$

for certain functions $z_i:\mathbb{R}\times\mathbb{R}^+\mapsto\mathbb{R}$. We observe that, in this paper, unless otherwise stated we will assume $\mu^+=\mu^-$.

Thus, the goal is that, by getting a smooth enough solution for the interface equation, we obtain a weak solution of the conservation law (1.1) on the whole plane (of course, at the same time, one each phase, the restriction of the weak solution is a strong solution). Besides obtaining a solution, we would like to understand whether the solution exists for all time and the dynamical properties of this solution or, at the contrary, if the solution presents finite time singularities. Similar problems have been studied for other active scalars as the surface quasi-geostrophic equation by Rodrigo, Gancedo and Gancedo & Strain [57, 58, 89].

Another motivation to study the problem (1.1) comes from the fact that, in a certain sense, the Muskat problem is a sort of *parabolic* version of the water waves problem. To observe that, we have to use Lagrangian coordinates. Indeed, in Lagrangian variables, system (1.1) reads

$$\frac{\mu}{\kappa}\frac{\mathrm{d}}{\mathrm{dt}}\eta+A^T\nabla q=-\rho^-G(0,1)^T,\tag{1.3}$$

where $A=(\nabla\eta)^{-1}$, $q=p\circ\eta$ and η is the Lagrangian coordinates. At the same time, the water waves problem can be written as

$$\rho^-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta+A^T\nabla q=-\rho^-G(0,1)^T.\tag{1.4}$$

Thus, we see that the connection between the Muskat and the water waves problems resembles the link between the heat and the wave equations. Similarly, one can compare the asymptotic models for the Muskat problem in [63] and for the water waves problem in [4].

Using the previous parametrization for $\Gamma(t)$ (1.2), the two-phase Muskat problem with parameters $(\rho^+,1)$, $(\rho^-,1)$ is equivalent to the following nonlinear and nonlocal evolution system for the unknowns z_i

$$\partial_t z(\alpha)=\frac{\bar\rho}{\pi}\int_{\mathbb{R}}\frac{z_1(\alpha)-z_1(\beta)}{|z(\alpha)-z(\beta)|^2}(\partial_\alpha z(\alpha)-\partial_\alpha z(\beta))\mathrm{d}\beta,\tag{1.5}$$

where the integral is understood in Cauchy principal value sense, *i.e.*

$$\int_{\mathbb{R}}=\lim_{\epsilon\to 0}\int_{B(0,\epsilon)^c\cap B(0,\epsilon^{-1})},$$

and $\bar\rho=\frac{\kappa(\rho^--\rho^+)}{2}$. We observe that every integral is taken in principal value sense from this point onwards.

Equivalently, when the viscosities satisfy $\mu^+=\mu^-$ and the interface is assumed to be the graph of the function $f(x,t)$, the Muskat system (1.1) (or analogously (1.5)) can be written as a single nonlocal, nonlinear equation for the interface (1.2):

$$\begin{aligned} \partial_t f &= \frac{\bar{\rho}}{\pi}\, \partial_x \int_{\mathbb{R}} \arctan\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right)\, \mathrm{d}\alpha \\ f(0,x) &= f_0(x). \end{aligned} \tag{1.6}$$

The previous formulation as a conservation law with a nonlocal and nonlinear flux was obtained in [25]. Remarkably, the Muskat problem can also be written in terms of oscillatory integrals as

$$\begin{aligned} \partial_t f(t,x) &= \frac{\bar{\rho}}{\pi} \int_{\mathbb{R}} \partial_x \left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right) \int_0^\infty e^{-\delta} \cos\left(\delta\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right)\right)\, \mathrm{d}\delta\, \mathrm{d}\alpha \\ f(0,x) &= f_0(x). \end{aligned} \tag{1.7}$$

This latter formulation was observed in [33].

2. Notation and functional setting

We denote

$$\Delta_\alpha f \equiv \frac{f(x,t)-f(x-\alpha,t)}{\alpha}.$$

Similarly,

$$\delta_y f(x) = f(x) - f(x-y) \text{ and } \bar{\delta}_y f(x) = f(x) - f(x+y).$$

We define the Calderón operator $\Lambda = \sqrt{-\Delta}$. On the Fourier side, this operator is given as the action of the multiplier $|\xi|$, *i.e.*

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi).$$

In an analogous manner, we consider the Hilbert transform $\mathcal{H}$ given as the action of the multiplier $-i\mathrm{sgn}(\xi)$, *i.e.*

$$\widehat{\mathcal{H} f}(\xi) = -i\mathrm{sgn}(\xi) \hat{f}(\xi).$$

We shall use the homogeneous L^2-based Sobolev space $\dot{H}^s$, $s \in \mathbb{R}^+$, which is endowed with the (semi)-norm

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}.$$

We will also use the L^p-based Sobolev spaces, $W^{n,p}(\mathbb{R})$, which are defined as

$$W^{n,p} = \{u \in L^p(\mathbb{R}), \partial_x^n u \in L^p(\mathbb{R})\},$$

with (semi-)norm

$$\|u\|_{\dot{W}^{n,p}} = \|\partial_x^n u\|_{L^p}.$$

Similarly, we define the homogeneous Wiener spaces $\dot{A}^\alpha(\mathbb{R})$ as

$$\dot{A}^\alpha(\mathbb{R}) = \left\{ u(x) \in L^1(\mathbb{R}), \text{ such that } \|u\|_{\dot{A}^\alpha(\mathbb{R})} = \int_{\mathbb{R}} |\xi|^\alpha |\hat{u}(\xi)| \mathrm{d}\xi \right\}. \tag{2.1}$$

Let us recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R})$ (see [6, 10, 90]). Let $(p,q,s) \in [1,\infty]^2 \times \mathbb{R}$. Let f be a tempered distribution (which is such that its Fourier transform is integrable near 0), then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ is the space endowed with the following (semi)-norm

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \frac{\|\mathbb{1}_{]0,1[}(s)\delta_y f + \mathbb{1}_{[1,2[}(s)(\delta_y + \bar{\delta}_y f)\|_{L^p}}{|y|^s} \right\|_{L^q(\mathbb{R},|y|^{-1}\mathrm{dy})}$$

We shall use the following classical embeddings. Let $(p_1,p_2,r_1,r_2) \in [1,\infty]^4$, then

$$\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s_2}_{p_2,r_2}(\mathbb{R}),$$

where $s_1 + \frac{1}{p_2} = s_2 + \frac{1}{p_1}$ and $r_1 \le r_2$. We also have for all $(p_1,s_1) \in [2,\infty] \times \mathbb{R}$,

$$\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s_1}_{p_1,r_2}(\mathbb{R}),$$

for all $(r_1,r_2) \in \,]1,\infty]$ such that $r_1 \le r_2$. Let $(s_1,s_2) \in \mathbb{R}^2$ so that $s_1 < s_2$, then for all $\theta \in]0,1[$ and $(p,r) \in [1,\infty]^2$, we have the following interpolation inequality

$$\|f\|_{\dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|f\|^\theta_{\dot{B}^{s_1}_{p,r}} \|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}. \tag{2.2}$$

We shall use the following useful generalized Calderón commutator type estimate (see *e.g.* Dawson, McGahagan, and Ponce [43] for a proof). Let $\Phi \in \dot{W}^{k+l,\infty}$ and let us consider the commutator

$$[\mathcal{H}, \Phi] f = \mathcal{H}(\Phi f) - \Phi \mathcal{H} f.$$

Then, for all $p \in]1,\infty[$ and $(k,l) \in \mathbb{N}$

$$\left\| [\mathcal{H}, \Phi]\, \partial_x^k f \right\|_{\dot{W}^{l,p}} \le C_{k,l} \|\Phi\|_{\dot{W}^{k+l,\infty}} \|f\|_{L^p}, \tag{2.3}$$

for all $f \in L^p$.

Throughout the article, $A \le B$ means that there exists a constant $C > 0$ depending only on controlled quantities such that $A \le CB$.

3. Well-posedness

Linearizing (1.5), we obtain that the linear problem is

$$\partial_t f = -\bar{\rho} \partial_x \mathcal{H} f.$$

We note that the sign of $\bar{\rho}$ is crucial in the evolution. When $\bar{\rho} > 0$, the linear problem reduces to a (fractional) heat equation, and it is therefore trivially well-posed in Sobolev spaces. However, when $\bar{\rho} < 0$, the linear problem has an anti-diffusive character that makes the problem ill-posed in Sobolev spaces but well posed for analytic functions. The condition on the sign of $\bar{\rho}$ states that the fluids are in the stable regime if the lighter fluid is above the heavier fluid. Hence, it is a condition on the stratification of the fluids.

Due to (1.1a), the condition on the sign of $\bar{\rho}$ is equivalent to

$$RT(t) = -(\nabla p^-(\Gamma(t)) - \nabla p^+(\Gamma(t))) \cdot n > 0,$$

where n denotes the (upward) normal to $\Gamma(t)$. This latter condition is the well-known Rayleigh–Taylor stability condition [88, 91]. This stability condition is ubiquitous in free boundary problems and it appears also when studying the water waves problem or the free-surface Euler equation [34, 40].

This stability condition that appears when studying the linear problem has to be taken into account when dealing with the full nonlinear problem (1.6) (unless surface tension effects are considered).

Recalling (1.5) for $(x, f(x,t))$, we observe that the equation is invariant by the scaling

$$f_\lambda(x,t) = \lambda^{-1} f(\lambda x, \lambda t). \tag{3.1}$$

We observe that there are several spaces whose norm is also left invariant by this scaling. These spaces are called *critical* for this equation. Three examples of critical spaces are

$$L^\infty(0,T;\dot{W}^{1,\infty}), \;\; L^\infty(0,T;\dot{H}^{3/2}), \text{ and } L^\infty(0,T;\dot{B}^1_{\infty,\infty}).$$

Spaces with more regularity (as for instance $L^\infty(0,T;\dot{H}^2)$) are called subcritical while spaces with less regularity (as for instance $L^\infty(0,T;\dot{H}^1)$) are called supercritical. The heuristic idea is that it is *easy* to construct solutions in subcritical spaces and *very difficult* to construct solutions in supercritical spaces.

Due to all this, we see that there are three main ingredients we have to take into account when proving a well-posedness result:

1. the fluids need to have the good stratification (*i.e.* the heavy fluid has to lie below the lighter fluid),
2. we should be able to parametrize the interface $\Gamma(t)$ as the graph of certain function $f(x,t)$ (otherwise, there exists some region where the fluid have the bad stratification),
3. the function f needs to have subcritical (or, at most, critical) regularity.

3.1. Local existence

Following the previous discussion, the *basic* local existence result reads as follows

Theorem 3.1. *Let $\rho^+ < \rho^-$ be two fixed parameters. Fix $s = 3$. Assume that $f_0 \in \dot{H}^s(\mathbb{R}) \cap L^2$. Then, there exists $0 < T = T(\|f_0\|_{L^2}, \|f_0\|_{\dot{H}^s})$ and a unique solution to (1.6)*

$$f \in C([0,T], L^2 \cap \dot{H}^s).$$

Furthermore, if $T = T_{\max} < \infty$, then

$$\limsup_{t\to T_{\max}} \|f(t)\|_{C^{2+\delta}} = \infty.$$

Physically, the previous hypotheses mean that the internal wave has no turning points (it is given as a graph) and separates fluids having the good stratification. More geometrically, the required smoothness on the data precludes consideration of initial data with a cusp. Actually, as noted in [2], (at the time) *it is(was) an open problem as to whether the problem is well-posed for initial data with a cusp.*

Theorem 3.1 was proved by Córdoba & Gancedo [29] using energy methods (a similar result for the case where the spatial domain is a strip, for the case of a porous medium with two different permeabilities and for the case of three fluids was proved by Córdoba, Granero-Belinchón and Orive [37] and Berselli, Córdoba and Granero-Belinchón [9] and Córdoba & Gancedo [31], respectively). A different proof (using a formulation for (1.6) based on the tangent angle and arclength) was given by Ambrose [1, 2] (see also [96]). Another approach is the one by Escher, Matioc & Walker [49] where the authors used semigroup theory to obtain the similar result when the initial data is in the *little Hölder* spaces $h^{2+\delta} \subset C^{2+\delta}$ (see also the papers by Escher & Matioc [45] and Escher, Matioc & Matioc [48] where the case with small initial data is studied).

In order the initial regularity $\dot{H}^s$ can be relaxed for $3/2 < s \leq 5/2$ (so it allows for interfaces whose curvature is not bounded pointwise) some new ideas were required. Mathematically, an H^s well-posedness result is challenging because the *standard* energy estimates suggest $\|h\|_{C^{2+\delta}}$ as the quantity one needs to control. In the case of a fluid and a dry zone, *i.e.* where the upper fluid is replaced by a dry zone, Cheng, Granero-Belinchón & Shkoller [3] introduced a new method to analyze (1.1) and prove the local existence of an $\dot{H}^2$ solution. As the domain in (1.1) $\Omega(t)$ is unknown, these authors first pull-back (1.1) onto a fixed-in-time reference domain. By doing this, (1.1) is transformed into a system of equations set on a fixed reference domain Ω, but having time-dependent coefficients. Then, this new method is based on the analysis of the resulting quasilinear system of partial differential equations and combines new energy estimates in the bulk of the fluid with estimates for the interface (see also Shkoller & Granero-Belinchón [62] for the case of two permeabilities). Due to the fact that this approach does not rely on the explicit structure of the singular integral equation (1.5), it can be applied to study general domain geometries and permeability functions.

In the case of two fluids with same viscosity, Constantin, Gancedo, Shvydkoy & Vicol [27] proved the local existence of solution for $W^{2,p}$ $p \in (1, \infty]$ initial data. The proofs exploit the nonlocal nonlinear parabolic nature of (1.6) through a series of nonlinear lower bounds for the nonlocal operators involved. Furthermore, these authors also prove that, as long as the slope of the interface remains uniformly bounded, the curvature remains bounded. A related result is the one by Prüss and Simmonet [87] where the authors prove the local existence for small initial data in $W^{2+\frac{1}{p},p}$.

Matioc [74], by rewritting the Muskat problem as an abstract evolution equation in an appropriate functional setting, was able to prove the local existence for arbitrary H^s, $3/2 < s < 2$ initial data (see also [75–77]).

For the case of two different viscosities in the RT stable regime the results are more scarce: Córdoba, Córdoba & Gancedo proved the local existence for H^3 curves [35] and H^4 surfaces [36] (see also [86] for the case of two different permeabilities) while Cheng, Granero-Belinchón & Shkoller prove the results for a H^2 graph.

3.2. Global existence

Before we can go over global existence results, we need to identify quantities that can be bounded for all positive times. The first of such results appeared in [30] and establishes the decay of $\|f(t)\|_{L^\infty}$:

Theorem 3.2. *Let $\rho^+ < \rho^-$ be two fixed parameters. Then the solution to the Muskat problem (1.6) satisfies*

$$\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}$$

To prove this result, Córdoba & Gancedo used a pointwise estimate to compute the evolution of

$$M(t) = \max_x f(x,t).$$

For a similar result for the case of a bounded porous medium we refer to [37]. Physically, this theorem means that the amplitude of the internal wave in a porous medium decays.

Similarly, one can also prove an L^2 energy balance

Theorem 3.3. *Let $\rho^+ < \rho^-$ be two fixed parameters. Then the solution to the Muskat problem (1.6) satisfies*

$$\|f(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u(s)\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}s \leq \|f_0\|^2_{L^2(\mathbb{R})},$$

or, equivalently,

$$\|f(t)\|_{L^2(\mathbb{R})}^2+\frac{\rho^- - \rho^+}{2\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\log\left(1+\left(\frac{f(x,s)-f(y,s)}{x-y}\right)^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}s\leq\|f_0\|_{L^2(\mathbb{R})}^2.$$

The previous result was given by Constantin, Córdoba, Gancedo & Strain [25] (see also [3, 9]).

Although the solution enjoys this decay of the *relatively strong* L^∞ norm and a energy balance that controls the velocity in $L^2(0,\infty;L^2(\mathbb{R}^2))$, this is not enough to obtain a global existence result of any kind. Then we have to turn our attention to other *stronger* norms.

In that regards, the first result was given by Córdoba & Gancedo [30], where these authors proved that

Theorem 3.4. *Let $\rho^+<\rho^-$ be two fixed parameters and assume that*

$$\|f_0\|_{\dot{W}^{1,\infty}}<1.$$

Then the solution to the Muskat problem (1.6) *satisfies*

$$\|f(t)\|_{\dot{W}^{1,\infty}}\leq\|f_0\|_{\dot{W}^{1,\infty}}.$$

This result gives conditions ensuring the decay of a *critical* norm. In the same spirit one has the analog result in terms of Wiener spaces (see [25, 26] and also the related work [85])

Theorem 3.5. *Let $\rho^+<\rho^-$ be two fixed parameters and assume that*

$$\|f_0\|_{\dot{A}^1}<1/3.$$

Then the solution to the Muskat problem (1.6) *satisfies*

$$\|f(t)\|_{\dot{A}^1}\leq\|f_0\|_{\dot{A}^1}.$$

Theorems 3.4 and 3.5 imply that the internal wave will not break if the initial slope satisfies certain size restrictions (when measured in appropriate critical norms).

We have to distinguish two different global existence results:

1. global *weak* solutions (when the equation (1.6) is satisfied in distributional sense)
2. global *classical* solutions (when the equation (1.6) holds pointwise).

Using Theorem (3.4), Constantin, Córdoba, Gancedo & Strain [25] proved the global existence of weak Lipschitz solutions. These result was later extended to the case where the porous medium is bounded by one sof the authors [61]. Remarkably, in order for the wave to not break-down in the case where the domain is bounded, not only the slope has to be suitably small, but also the amplitude, the depth and the slope of the internal wave have to satisfy appropriate (explicit) conditions. Roughly speaking, these extra conditions linking the depth, the amplitude and the slope of the wave mean that the amplitude can not be such that the wave is *close* to the bottom and that, the bigger the amplitude is, the smaller the slope has to be.

Using Theorem (3.5), Constantin, Córdoba, Gancedo & Strain [25] and Constantin, Córdoba, Gancedo, Rodríguez-Piazza & Strain [26] proved the global existence of classical solutions for initial data satisfying $\|f_0\|_{\dot{A}^1}<0.2$. The early work by Córdoba & Gancedo [29] has already a global existence result for (1.6) in the spirit of Theorem 3.5, but with a non-explicit size restriction $\|f_0\|_{\dot{A}^1}<\epsilon$ for *certain* ϵ.

With an initial data with suitably small Lipschitz norm, Constantin, Gancedo, Shvydkoy & Vicol [27] proved the global existence of classical solution. This result, by further restricting the size of $\|f_0\|_{\dot{W}^{1,\infty}}$ establishes that the weak solution (which exists due to [25]) is, indeed, a classical solution. This result was later extended to the full range $\|f_0\|_{\dot{W}^{1,\infty}} < 1$ by Cameron [12] (actually, the result by Cameron is more general as the criteria is given in terms of the product of the supremum and infimum of the slope).

Another global existence result in a critical space (which allows the slope to be arbitrarily large) is the one by Córdoba & Lazar [33]:

Theorem 3.6. *Let $\rho^+ < \rho^-$ be two fixed parameters and assume that $f_0 \in L^2 \cap \dot{H}^{3/2} \cap \dot{H}^{5/2}$ such that $\|f_0\|_{\dot{H}^{3/2}}$ is suitably small. Then, there exists a unique global strong solution of* (1.7)

$$f \in L^\infty(0,T;H^{5/2}) \cap L^2(0,T;H^3),\ \forall T > 0.$$

Some other global existence results are those in [3] (smallness of the initial data in H^2, see also [76, 77]), [59] (smallness in $\dot{A}^1$ but allows for two different viscosities) and [48] (smallness in the little Hölder space $h^{2+\delta}$, see also [45]).

4. Finite time singularities

The first type of singularity for the Muskat problem is the *turning singularity* and was proved by Castro, Córdoba, Fefferman, Gancedo & López-Fernández [13] (see also [9, 37]). In these singularities, the initial interface is assumed to be a smooth graph and then, in a finite time $T_{breaking}$,

$$\limsup_{t \to T_{breaking}} \|f(t)\|_{\dot{W}^{1,\infty}} = \infty.$$

This singularity implies that the internal wave breaks, *i.e.* after time $T_{breaking}$, the internal wave cannot be parametrized as a graph. Equivalently, this singularity means that the Muskat problem leaves the RT stable regime in finite time. However, a turning singularity do not imply a loss in derivatives for the solution.

The precise statement is

Theorem 4.1. *Let $\rho^+ < \rho^-$ be two fixed parameters. Then, there exist smooth initial data such that the unique strong solution of* (1.6) *satisfies*

$$\limsup_{t \to T_{breaking}} \|f(t)\|_{\dot{W}^{1,\infty}} = \infty,$$

for $0 < T_{breaking} < \infty$.

The proof of this result has several steps:

1. First, one considers the Muskat problem in its formulation for arbitrary curves (1.5) and proves the local existence of solution (forward and backward in time) for initial data who are analytic *via* a Cauchy–Kovalevsky theorem.
2. Then, one identifies $\partial_\alpha v_1(z(\alpha,t),t)$ as the quantity to track. Indeed, for initial data who are *'about to break'* such that

$$\partial_\alpha z(\alpha,t)\Big|_{\alpha=0} = (0,1),$$

i.e. whose tangent vector is vertical, one has that, the sign condition

$$\partial_\alpha v_1(z(\alpha,0),0)\Big|_{\alpha=0} < 0$$

is equivalent to breaking.

3. Now one constructs initial data such that $\partial_\alpha v_1(z(0,0),0) < 0$ and $\partial_\alpha z(0,0) = (0,1)$.
4. Finally, one takes an analytic initial data such that the previous condition holds (using some mollification argument) and invokes the Cauchy–Kovalevsky Theorem in step 1. Using the forward and backward existence, we conclude the existence of internal waves (that can be parametrized as a graph for $-\delta < t < 0$ that reach the initial data constructed in step 3 at time $t = 0$ and that cannot be parametrized as a graph for $0 < t < \delta$.

We would like to remark that the analytic curves remain valid classical solutions to the Muskat problem (1.5) for $0 < t < \delta$.

It is also interesting to note that the initial data leading to breaking waves can be taken with arbitrary small amplitude [64].

Gómez-Serrano and Granero-Belinchón [60] (see also [38, 39]), by using a computer assisted proof together with a variation of the previous ideas, were able to study the effect of finite depth and varying permeability. Among other results, these authors showed that the existence of top and bottom for the porous medium can enhance the formation of turning singularities in the sense that there exists initial curves such that, when the depth is finite, they break in finite time, but, if the depth is infinite, they become smooth graphs.

In terms of loss of derivatives, Castro, Córdoba, Fefferman and Gancedo [14] proved that there exist analytic initial data in the RT stable regime for the Muskat problem such that the solution turns to the RT unstable regime and later breaks down *i.e.* no longer belongs to C^4.

Finally, another possible singularity is the self-intersection of the interface or the intersection of two different interfaces (for the problem with three different fluids or two fluids and a dry zone) while the curve remain itself smooth. These singularities are known as *splash singularities* (when the self intersection happens at a single point) or *splat singularities* (when the self-intersection happens along an interval). In this case, it was proved by Castro, Córdoba, Fefferman and Gancedo [16] that splash singularities can indeed occur for the one-phase Muskat problem, while the case of splat singularities was disregarded by Córdoba and Pernas-Castaño [32]. When the case of several fluids is considered, Gancedo and Strain [58] proved that splash/splat singularities cannot occur in finite time (see for the analog result for the case of the Euler equations [41, 50]).

5. Wild solutions and mixing

The Muskat problem (1.6) is *ill-posed* in the RT unstable regime [29, 37]. Thus, the construction of weak solutions to (1.1) *via* the construction of classical solutions to (1.6) fails in the RT unstable regime. This leaves open the existence of weak solutions in the RT unstable regime and its dynamical properties.

Another topic that has attracted a lot of interest recently in the mathematical community is the construction of *wild solutions* to different fluid dynamical problems. These wild solutions are weak solutions that have compact support in space and time and, thus, they break the uniqueness. In the Muskat problem these solutions are particularly interesting since they can be related to mixing of the fluids in the RT unstable regime.

Thus, the existence of (possibly infinitely many) weak solutions in the RT unstable regime and their link to the mixing of the fluids appears as a very interesting research topic.

In this regards, it was proved by Székelyhidi Jr. [93] the existence of weak solutions in the RT unstable regime (see also the works by Förster and Székelyhidi Jr. [51] and Otto [83, 84]). Also, Castro, Córdoba and Faraco [15] proved that, starting with a smooth interface in the RT unstable regime, there exists a weak solution such that a *mixing strip-like* region opens around the interface (see also Castro, Faraco and Mengual [17]). In other words, the free boundary assumption is replaced by the opening of this mixing zone where the fluids begin to mix.

6. A new result

In this section, we shall prove the following theorem

Theorem 6.1. *Let $T>0$, assume that $f_0 \in \dot{H}^{1/2} \cap \dot{W}^{1,\infty}$ then, if*

- *the $K = L^\infty([0,T], \dot{W}^{1,\infty})$ norm of the corresponding solution remains bounded*
- *$\|f_0\|_{\dot{H}^{3/2}} < C(K)$ is preserved*

Then, the solution is global in time and we have

$$\|f\|^2_{\dot{H}^{1/2}}(T) + \frac{\pi}{1+K^2}\int_0^T \|f\|^2_{\dot{H}^1}\, \mathrm{d}s \lesssim \|f_0\|^2_{\dot{H}^{1/2}} + P\left(\|f\|_{L^\infty([0,T],\dot{H}^{3/2})}\right)\int_0^T \|f\|^2_{\dot{H}^1}\, \mathrm{d}s$$

where $P(X) = X + X^2$

Proof of Theorem 6.1 For the sake of notational simplicity, we take $\bar{\rho} = \pi$. We do $\dot{H}^{1/2}$ estimates. Using (1.7), we have that

$$\begin{aligned}
\frac{1}{2}\partial_t \|f\|^2_{\dot{H}^{1/2}} &= \int \Lambda^{1/2} f \int_0^\infty e^{-\delta}\Lambda^{1/2}\left(\partial_x \Delta_\alpha f \cos(\delta \Delta_\alpha f(x))\right)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&= \int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \equiv L_1
\end{aligned}$$

Let us set $\bar{\Delta}_\alpha f = \frac{f(x,t)-f(x+\alpha,t)}{\alpha}$. By denoting $S = \Delta_\alpha f + \bar{\Delta}_\alpha f$ and $D = \Delta_\alpha f - \bar{\Delta}_\alpha f$, one observes that by doing $\alpha \to -\alpha$ if necessary, one may write

$$\begin{aligned}
\frac{1}{2}\partial_t \|f\|^2_{\dot{H}^{1/2}} &= \int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&\quad - \int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\bar{\Delta}_\alpha f)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&= \int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&\quad + \int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\left(\cos(\delta\Delta_\alpha f) - \cos(\delta\bar{\Delta}_\alpha f)\right)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&\quad - \int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&= \frac{1}{4}\int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}(\cos(\delta\Delta_\alpha f(x)) + \cos(\delta\bar{\Delta}_\alpha f(x))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&\quad + \frac{1}{2}\int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\left(\cos(\delta\Delta_\alpha f) - \cos(\delta\bar{\Delta}_\alpha f)\right)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&= \frac{1}{2}\int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\cos(\frac{\delta}{2}S)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
&\quad - \int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{1/2}} &= -\int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\sin^2(\frac{\delta}{4}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad +\frac{1}{2}\int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= L_{1,1}+L_{1,2}+L_{1,3}
\end{aligned}$$

We need to further decompose the last term, more precisely we write

$$\begin{aligned}
L_{1,3} &= -\int \Lambda f \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= -\int \Lambda f \int \frac{f_x(x)-f_x(x-\alpha)}{\alpha} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= \int \Lambda f \int \frac{f_x(x-\alpha)-f_x(x)}{\alpha} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= \int \Lambda f \int \frac{\partial_\alpha\left(f(x)-f(x-\alpha)\right)}{\alpha} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\int \Lambda f \int \frac{f_x(x)}{\alpha} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x
\end{aligned}$$

We obtain,

$$\begin{aligned}
L_{1,3} &= \int \Lambda f \int \frac{f(x)-f(x-\alpha)}{\alpha^2} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{2}\int \Lambda f \int \frac{f(x)-f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta}\partial_\alpha D\cos(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{2}\int \Lambda f \int \frac{f(x)-f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta}\partial_\alpha S\sin(\frac{\delta}{2}D)\cos(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\int \Lambda f \int \frac{f_x(x)}{\alpha} \int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= \sum_{i=1}^4 L_{1,3,i}
\end{aligned}$$

6.1.1. Estimates of $L_{1,1}$

In order to control $L_{1,1}$, we use Holder inequality $L^2-L^2-L^\infty$,

$$\begin{aligned}
L_{1,1} &= -\int \Lambda f \int \partial_x D \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\sin^2(\frac{\delta}{4}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\le \frac{\Gamma(3)}{4}\|f\|^2_{\dot{H}^1}\int \frac{\|\delta_\alpha f+\bar{\delta}_\alpha f\|^2_{L^\infty}}{|\alpha|^3}\ \mathrm{d}\alpha \\
&\le \frac{1}{2}\|f\|^2_{\dot{H}^1}\|f\|^2_{\dot{B}^1_{\infty,2}} \\
&\le \frac{1}{2}\|f\|^2_{\dot{H}^1}\|f\|^2_{\dot{H}^{3/2}}
\end{aligned}$$

where we used the embedding $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,2}$ along with the fact that (since $\dot{H}^1$ and L^2 are shift invariant spaces)

$$\|\partial_x D\|_{L^2} \leq \frac{2}{|\alpha|}\|f_x(x) - f_x(x-\alpha)\|_{L^2} \leq \frac{4}{|\alpha|}\|\partial_x D\|_{L^2} \leq \frac{4}{|\alpha|}\|f\|_{\dot{H}^1}.$$

6.1.2. Estimates of $L_{1,3}$

In the next subsection, we shall estimate the $L_{1,3,i}$ for $i = 1, \ldots, 4$.

6.1.2.1. Estimates of $L_{1,3,1}$

By observing that $\dot{H}^1 \hookrightarrow \dot{B}^{1/2}_{\infty,2}$, we have that

$$\begin{aligned}
|L_{1,3,1}| &\leq \|f\|_{\dot{H}^1} \int_0^\infty \delta e^{-\delta} \frac{\|f(x) - f(x-\alpha)\|_{L^\infty}\|f(x-\alpha) + f(x+\alpha) - 2f(x)\|_{L^2}}{|\alpha|^3} \, \mathrm{d}\delta \, \mathrm{d}\alpha \\
&\leq \Gamma(2)\|f\|_{\dot{H}^1}\left(\int \frac{\|f(x) - f(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2} \, \mathrm{d}\alpha \int \frac{\|f(x-\alpha) + f(x+\alpha) - 2f(x)\|^2_{L^2}}{|\alpha|^4} \, \mathrm{d}\alpha\right)^{1/2} \\
&\leq \|f\|_{\dot{H}^1}\|f\|_{\dot{H}^{1/2}}\|f\|_{\dot{B}^{1/2}_{\infty,2}} \\
&\leq \|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}
\end{aligned}$$

6.1.2.2. Estimates of $L_{1,3,2}$

We may rewrite (just by a direct integration in x) D and S as follows:

$$D = \frac{f(x+\alpha) - f(x-\alpha)}{\alpha} = \frac{1}{\alpha}\int_0^\alpha (f_x(x+s) + f_x(x-s) - 2f_x(x)) \, \mathrm{d}s + 2f_x(x) \tag{6.1}$$

and $S = \Delta_\alpha f + \bar{\Delta}_\alpha f = -\frac{(f(x+\alpha)+f(x-\alpha)-2f(x))}{\alpha}$,

We also need to give a suitable expression of their derivatives with respect to α. In that regards, it is not difficult to check that

$$\partial_\alpha D = \frac{f_x(x+\alpha) + f_x(x-\alpha) - 2f_x(x)}{\alpha} - \frac{\int_0^\alpha (f_x(x-s) + f_x(x+s) - 2f_x(x)) \, \mathrm{d}s}{\alpha^2}$$

and

$$\partial_\alpha S = \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x + \frac{f(x+\alpha) + f(x-\alpha) - 2f(x)}{\alpha^2},$$

we then rewrite $L_{1,3,2}$ as

$$\begin{aligned}
L_{1,3,2} = &-\frac{1}{2}\int \Lambda f \int \frac{f(x) - f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{f_x(x+\alpha) + f_x(x-\alpha) - 2f_x(x)}{\alpha} \\
&\times \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right)\sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x \\
&+\frac{1}{2}\int \Lambda f \int \frac{f(x) - f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{\int_0^\alpha (f_x(x-s) + f_x(x+s) - 2f_x(x)) \, \mathrm{d}s}{\alpha^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x \\
= \; & L_{1,3,2,1} + L_{1,3,2,2}.
\end{aligned}
$$

By using that $\dot{H}^1 \hookrightarrow \dot{B}^{1/2}_{\infty,2}$, we find

$$
\begin{aligned}
|L_{1,3,2,1}| &\leq \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^1} \int \frac{\|f(x) - f(x-\alpha)\|_{L^\infty}}{\alpha} \; \frac{\|f_x(x+\alpha) + f_x(x-\alpha) - 2f_x(x)\|_{L^2}}{\alpha} \, \mathrm{d}\alpha \\
&\leq \|f\|_{\dot{H}^1} \|f\|_{\dot{B}^{1/2}_{\infty,2}} \|f_x\|_{\dot{B}^{1/2}_{2,2}} \\
&\leq \|f\|^2_{\dot{H}^1} \|f\|_{\dot{H}^{3/2}}
\end{aligned}
$$

In order to estimate $L_{1,3,2,2}$, we consider q, r and $\bar{r}$ (so that $1/r + 1/\bar{r} = 1$) that will be chosen latter, and we write

$$
\begin{aligned}
|L_{1,3,2,2}| &\leq \frac{1}{2} \|f\|_{\dot{H}^1} \int \frac{\|f(x) - f(x-\alpha)\|_{L^\infty}}{|\alpha|^3} \int_0^\infty \delta e^{-\delta} \\
&\quad \times |\alpha|^{q+\frac{1}{\bar{r}}} \left(\int_0^\alpha \frac{\|f_x(x-s) + f_x(x+s) - 2f_x(x)\|^r_{L^2}}{s^{qr}} \, \mathrm{d}s \right)^{1/r} \\
&\quad \times \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|_{L^\infty}}{\alpha} \, \mathrm{d}\delta \, \mathrm{d}\alpha \\
&\leq \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^1} \|f_x\|_{\dot{B}^{q-\frac{1}{r}}_{2,r}} \int \frac{\|f(x) - f(x-\alpha)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar{r}}}} \; \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|_{L^\infty}}{\alpha} \, \mathrm{d}\alpha \\
&\leq \frac{1}{2} \|f\|_{\dot{H}^1} \|f\|_{\dot{B}^{1+q-\frac{1}{r}}_{2,r}} \left(\int \frac{\|f(x) - f(x-\alpha)\|^2_{L^\infty}}{|\alpha|^{5-2q-\frac{2}{\bar{r}}}} \, \mathrm{d}\alpha \int \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|^2_{L^\infty}}{\alpha^3} \, \mathrm{d}\alpha \right)^{1/2} \\
&\leq \frac{1}{2} \|f\|_{\dot{H}^1} \|f\|_{\dot{B}^{1+q-\frac{1}{r}}_{2,r}} \|f\|_{\dot{B}^{2-q-\frac{1}{\bar{r}}}_{\infty,2}} \|f\|_{\dot{B}^1_{\infty,2}}
\end{aligned}
$$

Then, for $\bar{r} = r = 2$ and $q = 1$, we obtain since $\dot{H}^1 \hookrightarrow \dot{B}^{1/2}_{\infty,2}$ and $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,2}$

$$
\begin{aligned}
|L_{1,3,2,2}| &\leq \frac{1}{2} \|f\|_{\dot{H}^1} \|f\|_{\dot{H}^{3/2}} \|f\|_{\dot{B}^{1/2}_{\infty,2}} \|f\|_{\dot{B}^1_{\infty,2}} \\
&\leq \frac{1}{2} \|f\|^2_{\dot{H}^1} \|f\|^2_{\dot{H}^{3/2}},
\end{aligned}
$$

hence,

$$
|L_{1,3,2}| \leq \|f\|^2_{\dot{H}^1} \|f\|^2_{\dot{H}^{3/2}}
$$

6.1.2.3. Estimates of $L_{1,3,3}$

We now estimate $L_{1,3,3}$. We need to decompose this term as follows

$$
\begin{aligned}
L_{1,3,3} \; &= \; -\frac{1}{2} \int \Lambda f \int \frac{f(x) - f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2} D_\alpha f) \partial_\alpha S \cos(\frac{\delta}{2} S) \, \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x \\
&= \; -\frac{1}{2} \int \Lambda f \int \frac{f(x) - f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \; \sin(\frac{\delta}{2} D) \bar{\Delta}_\alpha f_x \; \cos(\frac{\delta}{2} S) \, \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int \Lambda f \int \frac{f(x) - f(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \; \sin(\frac{\delta}{2} D) \Delta_\alpha f_x \; \cos(\frac{\delta}{2} S) \, \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int \Lambda f \int \frac{f(x)-f(x-\alpha)}{\alpha}\int_0^\infty \delta e^{-\delta}\sin(\frac{\delta}{2}D)\frac{f(x+\alpha)+f(x-\alpha)-2f(x)}{\alpha^2}\\
&\times\cos(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x\\
=&\sum_{i=1}^{3} L_{1,3,3,i}
\end{aligned}
$$

To estimate $L_{1,3,3,1}$ we see that

$$
\begin{aligned}
|L_{1,3,3,1}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^1}\left(\int \frac{\|f(x)-f(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2}\ \mathrm{d}\alpha\int\frac{\|f_x(x)-f_x(x+\alpha)\|^2_{L^2}}{|\alpha|^2}\ \mathrm{d}\alpha\right)^{1/2}\\
&\leq \|f\|_{\dot{H}^1}\|f\|_{\dot{B}^{1/2}_{\infty,2}}\|f_x\|_{\dot{B}^{1/2}_{2,2}}\\
&\leq \|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}
\end{aligned}
$$

As well, one may easily estimate $L_{1,3,3,2}$ and we find that

$$
|L_{1,3,3,2}|\leq\frac{1}{2}\|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}
$$

For $L_{1,3,3,3}$, it suffices to write that

$$
\begin{aligned}
|L_{1,3,3,3}| &\leq \frac{1}{2}\|f\|_{\dot{H}^1}\left(\int \frac{\|f(x)-f(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2}\ \mathrm{d}\alpha\int\frac{\|f(x+\alpha)+f(x-\alpha)-2f(x)\|^2_{L^2}}{|\alpha|^4}\ \mathrm{d}\alpha\right)^{1/2}\\
&\leq \frac{1}{2}\|f\|_{\dot{H}^1}\|f\|_{\dot{B}^{1/2}_{\infty,2}}\|f\|_{\dot{B}^{3/2}_{2,2}}\\
&\leq \frac{1}{2}\|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}
\end{aligned}
$$

So that,

$$
|L_{1,3,3}|\|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}} \tag{6.2}
$$

6.1.2.4. Estimates of $L_{1,3,4}$

As for $L_{1,3,4}$, we use the antisymmetry of $\mathcal{H}$ together with the commutator estimates (2.3), as follows

$$
\begin{aligned}
L_{1,3,4} &= -\int\Lambda f\int\frac{f_x(x)}{\alpha}\int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x\\
&= \frac{1}{2}\int f_x\int_0^\infty\int e^{-\delta}\frac{1}{\alpha}\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right]f_x\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x.
\end{aligned}
$$

and we find that

$$
\begin{aligned}
L_{1,3,4} =& \frac{1}{2}\int\int\frac{f_x(x)-f_x(x-\alpha)}{\alpha}\int_0^\infty e^{-\delta}\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right]f_x\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x\\
&+\frac{1}{2}\int\int\frac{f_x(x-\alpha)}{\alpha}\int_0^\infty e^{-\delta}\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right]f_x\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x.
\end{aligned}
$$

By integrating by parts we find

$$
\begin{aligned}
L_{1,3,4} &= \frac{1}{2}\int\int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\int_0^\infty e^{-\delta}\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right] f_x \ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{2}\int\int \frac{f(x-\alpha)-f(x)}{\alpha^2}\int_0^\infty e^{-\delta}\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right] f_x \ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad +\frac{1}{2}\int\int \frac{f(x-\alpha)-f(x)}{\alpha}\int_0^\infty e^{-\delta}\partial_\alpha\left[\mathcal{H},\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\right] f_x \ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= L_{1,3,4,1}+L_{1,3,4,2}+L_{1,3,4,3}
\end{aligned}
$$

The commutator estimate (2.3) (in the case $l=1$ and $k=0$) and the embeddings $\dot{H}^{3/2}\hookrightarrow \dot{B}^1_{\infty,4}$ and $\dot{H}^1\hookrightarrow \dot{B}^{1/2}_{\infty,4}$ allows us to find

$$
\begin{aligned}
|L_{1,3,4,1}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^1}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}}{\alpha}\frac{\|f(x)-f(x-\alpha)\|_{L^\infty}}{\alpha} \\
&\quad\times \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{\alpha}\ \mathrm{d}\alpha \\
&\leq \|f\|_{\dot{H}^1}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|^2_{L^2}}{\alpha^2}\ \mathrm{d}\alpha\right)^{1/2}\left(\int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|^4_{L^\infty}}{\alpha^5}\ \mathrm{d}\alpha\right)^{1/4} \\
&\quad\times\left(\int \frac{\|f(x)-f(x-\alpha)\|^4_{L^\infty}}{\alpha^3}\ \mathrm{d}\alpha\right)^{1/4} \\
&\leq \|f\|_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}\|f\|_{\dot{B}^1_{\infty,4}}\|f\|_{\dot{B}^{1/2}_{\infty,4}} \\
&\leq \|f\|^2_{\dot{H}^1}\|f\|^2_{\dot{H}^{3/2}}
\end{aligned}
$$

Then, we estimate $L_{1,3,4,2}$. We first see that

$$
\begin{aligned}
L_{1,3,4,2} &= -\frac{1}{2}\int \Lambda f\frac{f(x-\alpha)-f(x)}{\alpha^2}\int_0^\infty\int e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{2}\int f_x\frac{\mathcal{H}f(x-\alpha)-\mathcal{H}f(x)}{\alpha^2}\int_0^\infty\int e^{-\delta}\sin(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= L_{1,3,4,2,1}+L_{1,3,4,2,2}
\end{aligned}
$$

Then, we have by using Holder inequality $L^2-L^\infty-L^2$

$$
\begin{aligned}
|L_{1,3,4,2,1}| &\leq \frac{1}{2}\|f\|_{\dot{H}^1}\int \frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{\alpha^2}\int_0^\infty e^{-\delta}\frac{\|\delta_\alpha f+\bar{\delta}_\alpha f\|_{L^2}}{\alpha}\ \mathrm{d}\delta\ \mathrm{d}\alpha \\
&\leq \frac{1}{2}\|f\|_{\dot{H}^1}\left(\int \frac{\|f(x-\alpha)-f(x)\|^2_{L^\infty}}{\alpha^3}\ \mathrm{d}\alpha\right)^{1/2}\left(\int \frac{\|\delta_\alpha f+\bar{\delta}_\alpha f\|^2_{L^2}}{\alpha^3}\ \mathrm{d}\alpha\right)^{1/2} \\
&\leq \frac{1}{2}\|f\|_{\dot{H}^1}\|f\|_{\dot{B}^1_{\infty,2}}\|f\|_{\dot{H}^1} \\
&\leq \frac{1}{2}\|f\|^2_{\dot{H}^1}\|f\|_{\dot{H}^{3/2}}
\end{aligned}
$$

Analogously (since $\mathcal{H}$ maps L^2 onto L^2), we also find

$$|L_{1,3,4,2}| \leq \frac{1}{2}\|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}}$$

The control of the term $L_{1,3,4,3}$ is challenging and one needs to use the following decomposition

$$\begin{aligned}
L_{1,3,4,3} &= -\frac{1}{4}\int f_x \frac{\mathcal{H}f(x-\alpha)-\mathcal{H}f(x)}{\alpha} \int_0^\infty \int \delta e^{-\delta}\partial_\alpha D \ \cos(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{4}\int \Lambda f \frac{f(x-\alpha)-f(x)}{\alpha} \int_0^\infty \int \delta e^{-\delta}\partial_\alpha D \ \cos(\frac{\delta}{2}D)\sin(\frac{\delta}{2}S)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{4}\int f_x \int \frac{\mathcal{H}f(x-\alpha)-\mathcal{H}f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\partial_\alpha S \ \cos(\frac{\delta}{2}S)\sin(\frac{\delta}{2}D)\ \ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad -\frac{1}{4}\int \Lambda f \int \frac{f(x-\alpha)-f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\partial_\alpha S \ \cos(\frac{\delta}{2}S)\sin(\frac{\delta}{2}D)\ \ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= \sum_{i=1}^4 L_{1,3,4,3,i}
\end{aligned}$$

All those previous term are as regular as $L_{1,3,j}$ for $j=2,3$ (up to some Hilbert transform, we shall do L^p estimate for safe values of p, that is $p\neq 1,\infty$), it is therefore an easy task to get that

$$|L_{1,3,4,3,1}+L_{1,3,4,3,2}| \leq 2\|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}}^2$$

and

$$|L_{1,3,4,3,3}+L_{1,3,4,3,4}| \leq 3\|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}}$$

Therefore, we have obtained

$$|L_{1,3}| \leq \|f\|_{\dot{H}^1}^2\left(\|f\|_{\dot{H}^{3/2}}^2+\|f\|_{\dot{H}^{3/2}}\right). \tag{6.3}$$

6.1.3. Estimates of $L_{1,2}$

We have

$$\begin{aligned}
L_{1,2} &= \frac{1}{2}\int \Lambda f \int (\partial_x \Delta_\alpha f - \partial_x \bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&= -\frac{1}{2}\int \Lambda f \int \frac{1}{\alpha}\partial_\alpha(\delta_\alpha f+\bar{\delta}_\alpha f) \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x,
\end{aligned}$$

By integrating by parts one finds

$$\begin{aligned}
L_{1,2} &= \frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\ \mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x \\
&\quad +\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\partial_\alpha D\sin(\frac{\delta}{2}D)\mathrm{d}\delta\ \mathrm{d}\alpha\ \mathrm{d}x.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
L_{1,2} = & \frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& +\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha} \\
& \times \sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& +\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f_x(x)}{\alpha}\sin(\frac{\delta}{2}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& -\frac{1}{2}\int \Lambda f \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f(x+\alpha)-f(x-\alpha)}{\alpha^2} \\
& \times \sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x
\end{aligned}
$$

The first term gives the paraboliticity, since we may write

$$
\begin{aligned}
& \frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& = -\pi\int |\Lambda f|^2\, \mathrm{d}x - \int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x, \\
& = -\pi\|f\|^2_{\dot{H}^1} - \int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x,
\end{aligned}
$$

Then, using formula (6.1) we may rewrite this term as follows

$$
\begin{aligned}
L_{1,2} = & -\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& +\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha} \\
& \times \sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& -\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta}\int_0^\alpha f_x(x-s)+f_x(x+s)-2f_x(x)\, \mathrm{d}s \\
& \times \sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
& -\pi\|f\|^2_{\dot{H}^1} \qquad (6.4) \\
= & L_{1,2,1}+L_{1,2,2}+L_{1,2,3}+L_{1,2,4}.
\end{aligned}
$$

We need to further decompose $L_{1,2,1}$ as follows

$$
\begin{aligned}
L_{1,2,1} = & -\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}D)\, \mathrm{d}\delta\, \mathrm{d}\alpha\, \mathrm{d}x \\
= & -\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sin(\frac{\delta}{4}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s-\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}D)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
=&-\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \sin(\frac{\delta}{4}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s)\cos(\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}D)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
&-\int \mathcal{H}f_x\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \cos\left(\frac{\delta}{4}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s\right)\sin(\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}D)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x
\end{aligned}
$$

In the last integral, with add and substract 1 in the cosine term, and we shall repeat this process after having used the trigonometry formula we develop the $\sin(\frac{\delta}{4}D)$, we obtain that

$$
\begin{aligned}
L_{1,2,1}=&-\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \sin(\frac{\delta}{4}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s)\cos(\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}D)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
&+2\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \sin^2\left(\frac{\delta}{8}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s\right)\sin(\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}D)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
&-\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \sin(\frac{\delta}{2}f_x(x))\;\sin(\frac{\delta}{4}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s)\cos(\frac{\delta}{2}f_x(x))\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
&-2\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\\
&\times \sin^2(\frac{\delta}{2}f_x(x))\;\sin^2\left(\frac{\delta}{8}\frac{1}{\alpha}\int_0^\alpha f_x(x+s)+f_x(x-s)-2f_x(x)\;\mathrm{d}s\right)\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
&-\int \Lambda f\int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\quad\int_0^\infty e^{-\delta}\;\sin^2(\frac{\delta}{2}f_x(x))\;\mathrm{d}\delta\;\mathrm{d}\alpha\;\mathrm{d}x\\
=&\sum_{j=1}^5 L_{1,2,1,j}
\end{aligned}
$$

For $L_{1,2,1,1}$ we write

$$
\begin{aligned}
|L_{1,2,1,1}|\leq \|f\|_{\dot{H}^1}&\int_0^\infty \delta e^{-\delta}\frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|^3}\\
&\times\int_0^\alpha \|f_x(x-s)+f_x(x+s)-2f_x(x)\|_{L^2}\;\mathrm{d}s\;\mathrm{d}\alpha\;\mathrm{d}\delta
\end{aligned}
$$

By Minkowski's inequality, we find

$$
\begin{aligned}
|L_{1,2,1,1}| \leq\ & \|f\|_{\dot{H}^1} \int_0^\infty \delta e^{-\delta} \int \frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2} \, \mathrm{d}s \, \mathrm{d}\alpha \, \mathrm{d}\delta \\
&+\|f\|_{\dot{H}^1} \int_0^\infty \delta e^{-\delta} \int \frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x+s)-f_x(x)\|_{L^2} \, \mathrm{d}s \, \mathrm{d}\alpha \, \mathrm{d}\delta \\
&+\|f\|_{\dot{H}^1} \int_0^\infty \delta e^{-\delta} \int \frac{\|f(x+\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2} \, \mathrm{d}s \, \mathrm{d}\alpha \, \mathrm{d}\delta \\
&+\|f\|_{\dot{H}^1} \int_0^\infty \delta e^{-\delta} \int \frac{\|f(x+\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x+s)-f_x(x)\|_{L^2} \, \mathrm{d}s \, \mathrm{d}\alpha \, \mathrm{d}\delta
\end{aligned}
$$

Since those terms have the same regularity, it is easy to conclude that

$$
\begin{aligned}
|L_{1,2,1,1}| &\leq 4\Gamma(2)\|f\|_{\dot{H}^1} \int \frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2} \, \mathrm{d}s \, \mathrm{d}\alpha \\
&\leq 4\|f\|_{\dot{H}^1} \int \frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^3} |\alpha|^{q+\frac{1}{\bar{r}}} \left(\int \frac{\|f_x(x-s)-f_x(x)\|_{L^2}^r}{|s|^{qr}} \, \mathrm{d}s \right)^{1/r} \mathrm{d}\alpha \\
&\leq 4\|f\|_{\dot{H}^1} \|f\|_{\dot{B}^{2-q-\frac{1}{\bar{r}}}_{\infty,1}} \|f_x\|_{\dot{B}^{q-\frac{1}{r}}_{2,r}}
\end{aligned}
$$

Then, choosing $q = 3/4$, $r = \bar{r} = 2$, using interpolation and the embedding $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,\infty}$,

$$
\begin{aligned}
|L_{1,2,1,1}| &\leq 4\|f\|_{\dot{H}^1} \|f\|_{\dot{B}^{\frac{3}{4}}_{\infty,1}} \|f_x\|_{\dot{B}^{1/4}_{2,2}} \\
&\leq 4\|f\|_{\dot{H}^1} \|f\|^{1/2}_{\dot{B}^{1/2}_{\infty,\infty}} \|f\|^{1/2}_{\dot{B}^1_{\infty,\infty}} \|f\|_{\dot{H}^{5/4}} \\
&\leq 4\|f\|_{\dot{H}^1} \|f\|^{1/2}_{\dot{H}^1} \|f\|^{1/2}_{\dot{B}^1_{\infty,\infty}} \|f\|^{1/2}_{\dot{B}^1_{2,2}} \|f\|^{1/2}_{\dot{B}^{3/2}_{2,2}} \\
&\leq 4\|f\|^2_{\dot{H}^1} \|f\|_{\dot{H}^{3/2}}
\end{aligned}
$$

Thus,

$$
|L_{1,2,1,1}| \leq \|f\|^2_{\dot{H}^1} \|f\|_{\dot{H}^{3/2}}
$$

Analogously, we find that

$$
\left| \sum_{i=1}^4 L_{1,2,1,i} \right| \leq \|f\|^2_{\dot{H}^1} \|f\|_{\dot{H}^{3/2}} \tag{6.5}
$$

As for $L_{1,2,1,5}$, we observe that since

$$
-2\pi\Lambda f = \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \, \mathrm{d}\alpha \quad \text{and} \quad \int_0^\infty e^{-\delta} \, \sin^2(\frac{\delta}{2}A) \, \mathrm{d}\delta = \frac{1}{2}\frac{1}{1+A^2}
$$

$$
\begin{aligned}
L_{1,2,1,5} &= -\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2} \int_0^\infty e^{-\delta} \, \sin^2(\frac{\delta}{2} f_x(x)) \, \mathrm{d}\delta \, \mathrm{d}\alpha \, \mathrm{d}x \\
&\leq \pi \frac{K^2}{1+K^2} \|f\|^2_{\dot{H}^1},
\end{aligned} \tag{6.6}
$$

where

$$K = \|f_x\|_{L^\infty L^\infty}.$$

Therefore, we find that

$$\left|\sum_{i=1}^{5} L_{1,2,1,i}\right| \leq \|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}} + \pi\frac{K^2}{1+K^2}\|f\|_{\dot{H}^1}^2 \tag{6.7}$$

For $L_{1,2,2}$, we write that

$$\begin{aligned}
L_{1,2,2} &= \frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha} \\
&\quad\times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\,\mathrm{d}\delta\,\mathrm{d}\alpha\,\mathrm{d}x \\
&\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^1}\left(\int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}^2}{\alpha^2}\,\mathrm{d}\alpha \int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^2}^2}{\alpha^2}\,\mathrm{d}\alpha\right)^{1/2} \\
&\leq \|f\|_{\dot{H}^1}\|f\|_{\dot{B}^{1/2}_{2,2}}\|f\|_{\dot{H}^{3/2}} \\
&\leq \|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}}
\end{aligned}$$

as well, for $L_{1,2,3}$ we observe that

$$\begin{aligned}
L_{1,2,3} &= -\frac{1}{2}\int \Lambda f \int \frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta}\int_0^\alpha f_x(x-s)+f_x(x+s)-2f_x(x)\,\mathrm{d}s \\
&\quad\times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\,\mathrm{d}\delta\,\mathrm{d}\alpha\,\mathrm{d}x \\
&\leq \frac{1}{4}\|f\|_{\dot{H}^1}\int_0^\infty \delta e^{-\delta}\frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|^3} \\
&\quad\times \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2}+\|f_x(x+s)-f_x(x)\|_{L^2}\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\delta \\
&\leq \frac{1}{4}\|f\|_{\dot{H}^1}\int_0^\infty \delta e^{-\delta}\left(\frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar{r}}}}+\frac{\|f_x(x+\alpha)-f_x(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar{r}}}}\right) \\
&\quad\times \left(\int \frac{\|f_x(x+s)-f_x(x)\|_{L^2}^r}{|s|^{qr}}\,\mathrm{d}s\right)^{1/r}\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\delta \\
&\leq \frac{1}{4}\|f\|_{\dot{H}^1}\int_0^\infty \delta e^{-\delta}\left(\frac{\|f(x-\alpha)-f(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar{r}}}}\right)\times\left(\int \frac{\|f_x(x+s)-f_x(x)\|_{L^2}^r}{|s|^{qr}}\,\mathrm{d}s\right)^{1/r}\,\mathrm{d}\alpha\,\mathrm{d}\delta \\
&\leq \frac{\Gamma(2)}{4}\|f\|_{H^1}\|f\|_{\dot{B}^{2-q-\frac{1}{\bar{r}}}_{\infty,1}}\|f_x\|_{\dot{B}^{q-\frac{1}{r}}_{2,r}}
\end{aligned}$$

Then, by choosing $q=3/4$, $r=\bar{r}=2$, one gets

$$\begin{aligned}
|L_{1,2,3}| &\leq \|f\|_{H^1}\|f\|_{\dot{B}^{3/4}_{\infty,1}}\|f\|_{\dot{B}^{5/4}_{2,2}} \\
&\leq \frac{1}{2}\|f\|_{\dot{H}^1}\|f\|_{\dot{B}^{1/2}_{\infty,\infty}}^{1/2}\|f\|_{\dot{B}^{1}_{\infty,\infty}}^{1/2}\|f\|_{\dot{B}^{3/2}_{2,2}}^{1/2}\|f\|_{\dot{B}^{1}_{2,2}}^{1/2} \\
&\leq \frac{1}{2}\|f\|_{\dot{H}^1}^2\|f\|_{\dot{H}^{3/2}}
\end{aligned}$$

Since (6.4) is a dissipative term and by (6.6), we have obtained that

$$|L_1| \leq \|f\|^2_{H^1} P(\|f\|_{\dot{H}^{3/2}}) - \pi \|f\|^2_{\dot{H}^1} + \pi \frac{K^2}{1+K^2} \|f\|^2_{\dot{H}^1}$$

Finally,

$$|L_1| \leq \|f\|^2_{H^1} P(\|f\|_{\dot{H}^{3/2}}) - \frac{\pi}{1+K^2} \|f\|^2_{\dot{H}^1} \tag{6.8}$$

where $P(X) = X + X^2$.

And then integrating in time $s \in [0,T]$ one gets the desired energy inequality. Therefore, if $\|f_0\|_{\dot{H}^{3/2}}$ is smaller than some $C(K)$ that depends only on K, then the solution is in $L^\infty([0,T], \dot{H}^{1/2}) \cap L^2([0,T], \dot{H}^1)$. This concludes the $\dot{H}^{1/2}$-estimates.

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