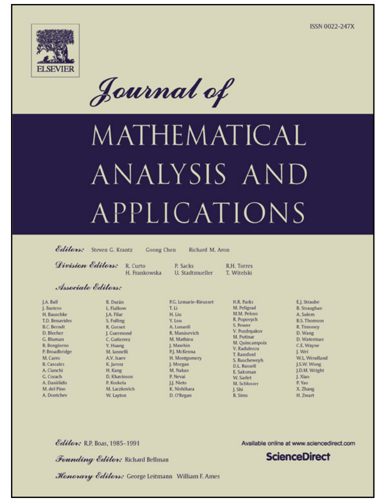
Accepted Manuscript



Convolution operators on group algebras which are tauberian or cotauberian

Liliana Cely, Elói M. Galego, Manuel González

PII: S0022-247X(18)30400-1
DOI: https://doi.org/10.1016/j.jmaa.2018.05.007
Reference: YJMAA 22239

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 27 November 2017

CONVOLUTION OPERATORS ON GROUP ALGEBRAS WHICH ARE TAUBERIAN OR COTAUBERIAN

LILIANA CELY, ELÓI M. GALEGO, AND MANUEL GONZÁLEZ

Abstract. We study the convolution operators T_μ acting on the group algebras $L_1(G)$ and $M(G)$, where G is a locally compact abelian group and μ is a complex Borel measure on G. We show that a cotauberian convolution operator T_μ acting on $L_1(G)$ is Fredholm of index zero, and that T_μ is tauberian if and only if so is the corresponding convolution operator acting on the algebra of measures $M(G)$, and we give some applications of these results.

1. Introduction

Tauberian and cotauberian operators were introduced in [14] and [19] respectively, and they have been useful in Banach space theory (see [9]). Recently, some examples of non-trivial (i.e. with non-closed range) tauberian operators $T : L_1(\mu) \to L_1(\mu)$ have been found in [13], answering a question in [9]. In [3] we considered the case in which T is a convolution operator T_μ acting on the group algebra $L_1(G)$, where G be a locally compact abelian group. We proved that T_μ tauberian implies T_μ invertible when G is non-compact, and that T_μ tauberian implies T_μ Fredholm when G is compact and the singular continuous part μ_{sc} of μ with respect to the Haar measure on G is zero. In particular, tauberian operators T_μ are trivial in those cases. The case $\mu_{sc} \neq 0$ remains open.

Our main result in this paper shows that the cotauberian convolution operators T_μ acting on $L_1(G)$ are always Fredholm, hence trivial. This fact is surprising, because it is not difficult to obtain examples of non-trivial cotauberian operators $T : L_1(\mu) \to L_1(\mu)$. And it is useful because it gives a new characterization of the Fredholm multipliers of $L_1(G)$ considered in [1, Theorem 5.97] as those that satisfy the definition of cotauberian operator.

In [3], our main tool was a result in [8] characterizing the tauberian operators $T : L_1(\mu) \to Y$ in terms of the action of T over sequences of normalized disjoint functions in $L_1(\mu)$. Here we follow a more algebraic approach, based in the characterization of the convolution operators T_μ as the multipliers of the group algebra $L_1(G)$ [15, Chapter 0]. We show first that T_μ cotauberian implies T_μ tauberian. In the case G non-compact, T_μ tauberian implies T_μ

Supported in part by MINECO (Spain), Grant MTM2016-76958.
2010 Mathematics Subject Classification. Primary: 47A53, 43A22.
Keywords: multiplier; tauberian operator; convolution operator.

invertible by a result in [3]. In the case G compact, $L_1(G)^{**}$ is a Banach algebra in which $L_1(G)$ is a closed ideal, hence $L_1(G)^{**}/L_1(G)$ is a Banach algebra, and we prove that T_μ cotauberian implies that the operator induced by T_μ^{**} on the quotient algebra $L_1(G)^{**}/L_1(G)$ is bijective. From the inverse of the induced operator we obtain an inverse of T_μ modulo the compact operators, hence T_μ is Fredholm.

We also show that T_μ is tauberian if and only if its natural extension to the algebra of measures $M(G)$ is tauberian, we derive some results for convolution operators acting on $C_0(G)$ and $L_\infty(G)$, and we answer a question raised in [7] about the measures $\mu \in M(G)$ such that $\nu \in M(G)$ and $\mu \star \nu \in L_1(G)$ imply $\nu \in L_1(G)$.

Along the paper X and Y are complex Banach spaces and $T : X \to Y$ is a (continuous linear) operator from X to Y. We denote by $R(T)$ and $N(T)$ the range and the kernel of T respectively. An operator $T : X \to Y$ is called *tauberian* if the second conjugate $T^{**} : X^{**} \to Y^{**}$ satisfies $T^{**-1}(Y) = X$, and T is called *cotauberian* if its conjugate T^* is tauberian.

We denote by X^{co} the quotient space X^{**}/X, and $T^{co} : X^{co} \to Y^{co}$ is the operator induced by T^{**} and defined by $T^{co}(m+X) := T^{**}m+Y$, $m \in X^{**}$, which is called the *residuum operator* of T. Note that T is tauberian if and only if T^{co} is injective, and T is cotauberian if and only if T^{co} has dense range [9, Proposition 3.1.8 and Corollary 3.1.12]. The set of operators of the form T^{co} for some T was studied in [11].

An operator $T : X \to Y$ is *Fredholm* if it has closed finite codimensional range and finite dimensional kernel. In this case the *index* of T is defined by $\operatorname{ind}(T) = \dim N(T) - \dim X/R(T)$. Fredholm operators are tauberian and cotauberian [9, Thm. 2.1.5 and Prop. 3.1.5].

2. Preliminaries

We denote by G a locally compact abelian group (a *LCA group*, for short), and by m the Haar measure on G. Moreover $L_1(G)$ is the space of m-integrable complex functions on G endowed with the L_1-norm $\|\cdot\|_1$, and $M(G)$ denotes the space of complex Borel measures on G endowed with the variation norm. Also $C_0(G)$ and $L_\infty(G)$ denote the space of complex continuous functions on G which vanish at infinity and the space of m-essentially bounded measurable complex valued functions on G, both endowed with the supremum norm. Note that $M(G)$ and $L_\infty(G)$ can be identified with the dual spaces of $C_0(G)$ and $L_1(G)$ respectively, and $L_1(G)$ can be identified with the subspace of those $\mu \in M(G)$ that are absolutely continuous with respect to m.

The space $L_1(G)$ with the convolution $(f \star g)(x) = \int_G f(x-y)g(y)dm(y)$ is a commutative Banach algebra. Moreover, if G is a compact group $L_\infty(G) \subseteq L_1(G)$, thus by Young's inequality, $C(G)$ and $L_\infty(G)$ are commutative Banach algebras with the convolution product. Given $\mu \in M(G)$ and $f \in L_1(G)$, the expression $(\mu\star f)(x) = \int_G f(x-y)d\mu(y)$ defines $\mu\star f \in L_1(G)$

satisfying $\|\mu \star f\|_1 \leq \|\mu\| \cdot \|f\|_1$. Thus we obtain a *convolution operator* T_μ on $L_1(G)$ defined by $T_\mu f = \mu \star f$, and satisfying $\|T_\mu\| = \|\mu\|$. Moreover, given $\mu, \nu \in M(G)$, the convolution of measures $\mu \star \nu \in M(G)$ is commutative [18]. Therefore $T_{\mu\star\nu} = T_\mu T_\nu = T_\nu T_\mu$, and we have an operator $M_\mu : M(G) \longrightarrow M(G)$ defined by $M_\mu \lambda = \lambda \star \mu$ which is an extension of T_μ. We can also define the convolution operator $S_\mu : C_0(G) \to C_0(G)$ and its extension $L_\mu : L_\infty(G) \to L_\infty(G)$ by $L_\mu g = \mu \star g$.

For $\mu \in M(G)$, we denote by $\widetilde{\mu}$ the measure in $M(G)$ such that $\widetilde{\mu}(E) = \mu(-E)$ for each Borel set E. Given $\mu, \eta \in M(G)$ and $f \in C_0(G)$, it is not difficult to show that

$$\langle \eta, \mu \star f \rangle = \langle \widetilde{\mu} \star \eta, f \rangle.$$

Thus $S_\mu^* = M_{\widetilde{\mu}}$, and similarly we get $T_\mu^* = L_{\widetilde{\mu}}$.

Following [15], we say that a Banach algebra A is *without order* if for all $x \in A$, $xA = \{0\}$ implies $x = 0$, or, for all $x \in A$, $Ax = \{0\}$ implies $x = 0$. A map $T : A \to A$ is a *multiplier of* A if $x(Ty) = (Tx)y$ for all $x, y \in A$. If G is a LCA group, then $L_1(G)$ is without order and the multipliers of $L_1(G)$ are precisely the convolution operators T_μ, $\mu \in M(G)$ [15, Chapter 0].

A net $(e_\alpha)_{\alpha \in I}$ in a commutative Banach algebra A is an *approximate identity* if, for each $a \in A$, $\lim_\alpha \|ae_\alpha - a\| = 0$. If G is a LCA group, then $L_1(G)$ admits a bounded approximate identity [15, Appendix F].

Given a Banach algebra A, the second dual space A^{**} of A is also a Banach algebra endowed with the (first) Arens product [2]. Specifically, given $M, N \in A^{**}$, $f \in A^*$ and $a, b \in A$, we define the product $M \cdot N$ in three steps as follows:

$$\begin{aligned} f \cdot a \in A^* : &\qquad \langle f \cdot a, b \rangle := \langle f, ab \rangle \\ N \cdot f \in A^* : &\qquad \langle N \cdot f, a \rangle := \langle N, f \cdot a \rangle \\ M \cdot N \in A^{**} : &\qquad \langle M \cdot N, f \rangle := \langle M, N \cdot f \rangle. \end{aligned}$$

Thus, for G a LCA group, $L_1(G)^{**}$ is a Banach algebra. Given $f \in L_\infty(G)$ and $\phi \in L_1(G)$, it is not difficult to show that

$$f \cdot \phi = f \star \widetilde{\phi}, \tag{1}$$

where $\widetilde{\phi}(x) = \phi(-x)$. Moreover, since G is commutative, the centre of $L_1(G)^{**}$ is $L_1(G)$ [16, Corollary 3] i.e.,

$$L_1(G) = \{m \in L_1(G)^{**} : \; m \cdot n = n \cdot m \text{ for each } n \in L_1(G)^{**}\}.$$

When G is compact, $L_1(G)$ is a (closed) ideal of $L_1(G)^{**}$ [20, Proposition 4.2]. Thus $L_1(G)^{co}$ is a Banach algebra.

We will need later the following result proved in [3].

Theorem 2.1. [3, Theorem 3.2] *Let G be a non-compact LCA group. Then every tauberian convolution operator $T_\mu : L_1(G) \to L_1(G)$ is invertible.*

For basic results on Fredholm theory, tauberian operators and multipliers of Banach algebras without order we refer to [1], [9] and [15].

3. Main results

First we show that the Banach algebras involved in our arguments are without order.

Proposition 3.1. *Let G be a LCA group. Then the algebra $L_1(G)^{**}$ admits a norm-one right identity; hence it is a Banach algebra without order. Moreover, when G is compact, the quotient algebra $L_1(G)^{co}$ also admits a norm-one right identity and it is a Banach algebra without order.*

Proof. The first part of our first result was proved in [6, Proposition 2.1]. For the second part, note that if E is a norm-one right identity of $L_1(G)^{**}$ and G is compact, then $L_1(G)^{co}$ is a Banach algebra and $E + L_1(G)$ is a norm-one right identity in $L_1(G)^{co}$, hence $L_1(G)^{co}$ is a Banach algebra without order. □

Let us see that the multipliers of algebras without order have a good behavior under duality.

Proposition 3.2. *Let T be a multiplier of a Banach algebra without order A. Then the second conjugate $T^{**} : A^{**} \to A^{**}$ is a multiplier of A^{**}.*

Proof. The proof has three steps. Let $M, N \in A^{**}$, $f \in A^*$ and $a, b \in A$. Then $T(ab) = (Ta)b = a(Tb)$, because A is without order, and from this equality we derive $T^*(f \cdot a) = f \cdot Ta$ as follows:

$$\begin{aligned} \langle T^*(f \cdot a), b\rangle &= \langle f \cdot a, Tb\rangle \\ &= \langle f, a(Tb)\rangle \\ &= \langle f, (Ta)b\rangle \\ &= \langle f \cdot Ta, b\rangle . \end{aligned}$$

In a similar way, from $T^*(f \cdot a) = f \cdot Ta$ we derive $T^{**}N \cdot f = T^*(N \cdot f)$, and from the latter equality we get that $M \cdot T^{**}N = T^{**}M \cdot N$, showing that T^{**} is a multiplier. □

Corollary 3.3. *Let G be a compact LCA group. Then the residuum operator T_μ^{co} is a multiplier of $L_1(G)^{co}$.*

Proof. Let $n + L_1(G)$ and $m + L_1(G)$ in $L_1(G)^{co}$. Since T_μ^{**} is a multiplier of $L_1(G)^{**}$,

$$\begin{aligned} (n + L_1(G)) \cdot T_\mu^{co}(m + L_1(G)) &= (n + L_1(G)) \cdot (T_\mu^{**}m + L_1(G)) \\ &= n \cdot T_\mu^{**}m + L_1(G) \\ &= T_\mu^{**}n \cdot m + L_1(G) \\ &= (T_\mu^{**}n + L_1(G)) \cdot (m + L_1(G)) \\ &= T_\mu^{co}(n + L_1(G)) \cdot (m + L_1(G)). \end{aligned}$$

Hence T_μ^{co} is a multiplier of $L_1(G)^{co}$. □

Now we are ready to show that cotauberian convolution operators on $L_1(G)$ are tauberian. This result contrasts with the fact that it is easy to find non-trivial cotauberian operators $T : L_1(0,1) \to L_1(0,1)$: just consider a surjective operator S with infinite dimensional kernel from $L_1(0,1)$ onto a closed infinite dimensional subspace M of $L_1(0,1)$ such that the quotient $L_1(0,1)/M$ is reflexive, and add a compact operator K so that $T = S + K$ has non-closed range. However it is difficult to obtain a non-trivial tauberian operator $T : L_1(0,1) \to L_1(0,1)$ (see [13]).

Proposition 3.4. *Let T be a multiplier of a Banach algebra without order A. Suppose that A coincides with the centre of A^{**} and T is cotauberian. Then T is tauberian.*

Proof. Let $M \in A^{**}$ such that $T^{**}M \in A$. Then for each $K \in A^{**}$,

$$M \cdot T^{**}K = T^{**}M \cdot K = K \cdot T^{**}M = T^{**}K \cdot M$$

because T^{**} is a multiplier of A^{**}. In particular $M \cdot T^{**}K = T^{**}K \cdot M$. Since T is cotauberian, $R(T^{**}) + A$ is dense in A^{**} [9, Corollary 3.1.12]. Consequently, given $S \in A^{**}$, we can find sequences $(W_n) \subset A^{**}$ and $(f_n) \subset A$ with $(T^{**}W_n + f_n)$ norm-convergent to S. Therefore

$$\begin{aligned} M \cdot S &= \lim_n (M \cdot (T^{**}W_n + f_n)) \\ &= \lim_n (M \cdot T^{**}W_n + M \cdot f_n) \\ &= \lim_n (T^{**}W_n \cdot M + f_n \cdot M) \\ &= \lim_n ((T^{**}W_n + f_n) \cdot M) \\ &= S \cdot M. \end{aligned}$$

Then $M \in A$, the centre of A^{**}. Hence T is tauberian. □

Corollary 3.5. *Let G be a LCA group.*

(1) *Every cotauberian convolution operator T_μ is tauberian.*
(2) *Suppose that G is non-compact. Then T_μ is cotauberian if and only if it is invertible.*

Proof. (1) It follows from the facts that $L_1(G)$ is a Banach algebra without order and $L_1(G)$ is the centre of $L_1(G)^{**}$.

(2) It is enough to observe that, since G is non-compact, every tauberian convolution operator on $L_1(G)$ is invertible (Theorem 2.1). □

Let E be a right identity in the algebra $L_1(G)^{**}$ provided by Proposition 3.1. We consider the map $\Gamma_E : M(G) \to L_1(G)^{**}$ defined by

$$\Gamma_E(\mu) := T_\mu^{**}(E), \quad \mu \in M(G).$$

The map Γ_E is an isometric algebra homomorphism of $M(G)$ into $L_1(G)^{**}$ extending the natural embedding of $L_1(G)$ into $L_1(G)^{**}$ [6, Proposition 2.3].

Since T_μ^{**} is a multiplier of $L_1(G)^{**}$, for each $m \in L_1(G)^{**}$ we have

$$T_\mu^{**}m = (T_\mu^{**}m) \cdot E = m \cdot T_\mu^{**}E = m \cdot \Gamma_E(\mu). \tag{2}$$

Thus T_μ^{**} is a right multiplication operator (by $\Gamma_E(\mu)$). Moreover

$$E \cdot \Gamma_E(\mu) = T_\mu^{**}(E) = \Gamma_E(\mu). \tag{3}$$

Next we give our main result.

Theorem 3.6. *Let G be a LCA group and let $T_\mu : L_1(G) \to L_1(G)$ be a convolution operator. Then T_μ is cotauberian if and only if it is Fredholm of index zero.*

Proof. For G non-compact, it is a consequence of Corollary 3.5.

Suppose that G is compact and T_μ cotauberian. Then T_μ is also tauberian by Proposition 3.4, hence T_μ^{co} is injective with dense range by [9, Prop. 3.1.8 and Cor. 3.1.12], and has closed range by [9, Theorem 4.4.2]. Thus T_μ^{co} is invertible.

Since T_μ^{co} is a multiplier of the Banach algebra without order $L_1(G)^{co}$ (Proposition 3.1), its inverse $S := (T_\mu^{co})^{-1}$ is a multiplier of $L_1(G)^{co}$ [15, Theorem 1.1.3].

Let E be a right identity in $L_1(G)^{**}$. Then

$$T_\mu^{co}(E + L_1(G)) = T_\mu^{**}E + L_1(G) = \Gamma_E(\mu) + L_1(G).$$

Set $\beta + L_1(G) := S(E + L_1(G))$. Then

$$\begin{aligned}(\beta + L_1(G)) \cdot (\Gamma_E(\mu) + L_1(G)) &= S(E + L_1(G)) \cdot T_\mu^{co}(E + L_1(G)) \\ &= T_\mu^{co}(S(E + L_1(G))) \cdot (E + L_1(G)) \\ &= E + L_1(G).\end{aligned}$$

Thus $(\beta \cdot \Gamma_E(\mu) - E) \in L_1(G)$. On the other hand, we have

$$\begin{aligned}(E + L_1(G)) \cdot (\beta + L_1(G)) &= (E + L_1(G)) \cdot S(E + L_1(G)) \\ &= S(E + L_1(G)) \cdot (E + L_1(G)) = \beta + L_1(G),\end{aligned}$$

hence $(E \cdot \beta - \beta) \in L_1(G)$. Since $L_1(G)$ is the centre of $L_1(G)^{**}$, for each $n \in L_1(G)^{**}$ we have

$$(E \cdot \beta - \beta) \cdot n = n \cdot (E \cdot \beta - \beta) = n \cdot \beta - n \cdot \beta = 0.$$

Then $E \cdot \beta - \beta = 0$, because $L_1(G)^{**}$ is a Banach algebra without order, thus $E \cdot \beta = \beta$.

Let $\nu \in M(G)$ denote the restriction of $\beta \in L_\infty(G)^*$ to $C(G)$. By [6, Proposition 2.5] we have that $\beta - \Gamma_E(\nu)$ is zero on $C(G)$. Moreover, by formula (3), we have

$$E \cdot (\beta - \Gamma_E(\nu)) = \beta - \Gamma_E(\nu).$$

The right identity $E \in L_1(G)^{**}$ is the w*-limit of $(i(e_\gamma))$ for some bounded approximate identity $(e_\gamma) \subset L_1(G)$, where $i : L_1(G) \to L_1(G)^{**}$ is the canonical inclusion. Therefore, for $f \in L_\infty(G)$ we have

$$\begin{aligned}
\langle \beta - \Gamma_E(\nu), f\rangle &= \langle E \cdot (\beta - \Gamma_E(\nu)), f\rangle \\
&= \langle E, (\beta - \Gamma_E(\nu)) \cdot f\rangle \\
&= \lim_\gamma \langle i(e_\gamma), (\beta - \Gamma_E(\nu)) \cdot f\rangle \\
&= \lim_\gamma \langle (\beta - \Gamma_E(\nu)) \cdot f, e_\gamma\rangle \\
&= \lim_\gamma \langle (\beta - \Gamma_E(\nu)), f \cdot e_\gamma\rangle = 0
\end{aligned}$$

because, by (1) and [18, Theorem 1.1.6], $f \cdot e_\gamma \in C(G)$, thus $\beta = \Gamma_E(\nu)$.

Note that $I = T_{\delta_0}$, where I is the identity operator on $L_1(G)$ and δ_0 is the unit measure concentrated at $\{0\}$. Then $T_{\delta_0}^{**}$ is the identity operator on $L_1(G)^{**}$, and thus $E = \Gamma_E(\delta_0)$. Hence, since $\Gamma_E : M(G) \to L_1(G)^{**}$ is an algebra homomorphism,

$$\Gamma_E(\nu \star \mu - \delta_0) = \Gamma_E(\nu) \cdot \Gamma_E(\mu) - \Gamma_E(\delta_0) = \beta \cdot \Gamma_E(\mu) - E \in L_1(G). \tag{4}$$

By formula (4) and [6, proposition 2.3], $\mu \star \nu - \delta_0 \in L_1(G)$, and thus

$$T_{\mu\star\nu-\delta_0} = T_\mu T_\nu - I$$

is a compact operator [4, Lemma 2.1.4]. Thus $T_\mu T_\nu = T_\nu T_\mu$ is a Fredholm operator, hence T_μ is Fredholm, and it has index zero by [1, Theorem 5.97].

For the converse implication, note that Fredholm operators are trivially cotauberian. □

Next we study the convolution operator $M_\mu : M(G) \to M(G)$ and its relation with $T_\mu : L_1(G) \to L_1(G)$.

Theorem 3.7. *Let G be a LCA group and let $\mu \in M(G)$. Then*

(1) *M_μ cotauberian implies M_μ tauberian.*
(2) *T_μ is tauberian if and only if M_μ is tauberian.*

Proof. (1) It was shown in [17, Main theorem] that, as in the case of $L_1(G)$, the measure algebra $M(G)$ is the centre of $M(G)^{**}$. Since $M(G)$ has an identity, it is without order. Thus the result follows from Proposition 3.4.

(2) Suppose that T_μ is tauberian, and let E be a right identity in $L_1(G)^{**}$. Then the following diagram is commutative:

$$\begin{array}{ccc}
L_1(G)^{**} & \xrightarrow{T_\mu^{**}} & L_1(G)^{**} \\
\Gamma_E \uparrow & & \uparrow \Gamma_E \\
M(G) & \xrightarrow{M_\mu} & M(G)
\end{array}$$

Indeed, given $\nu \in M(G)$ and $h \in L_\infty(G)$, we have

$$\begin{aligned}
\langle \Gamma_E M_\mu(\nu), h\rangle &= \langle \Gamma_E(\mu \star \nu), h\rangle \\
&= \langle T^{**}_{\mu\star\nu} E, h\rangle \\
&= \langle E, T^{*}_{\mu\star\nu} h\rangle \\
&= \langle E, T^{*}_{\nu}(T^{*}_{\mu} h)\rangle \\
&= \langle T^{**}_{\nu} E, T^{*}_{\mu} h\rangle \\
&= \langle \Gamma_E(\nu), T^{*}_{\mu} h\rangle \\
&= \langle T^{**}_{\mu}\Gamma_E(\nu), h\rangle.
\end{aligned}$$

Thus

$$\Gamma_E M_\mu = T^{**}_\mu \Gamma_E. \tag{5}$$

Now T_μ tauberian implies T^{**}_μ tauberian [9, Theorem 4.4.2]. Therefore $T^{**}_\mu \Gamma_E = \Gamma_E M_\mu$ is tauberian, and hence M_μ is tauberian, in both cases by [9, Proposition 2.1.3].

Similarly, denoting by $J : L_1(G) \to M(G)$ the natural isomorphic embedding, we have $JT_\mu = M_\mu J$. Hence, by [9, Proposition 2.1.3], if M_μ is tauberian, so is T_μ. □

Corollary 3.8. *Let G be a LCA group. If the convolution operators T_μ is cotauberian, then M_μ is Fredholm.*

Proof. Let T_μ be cotauberian, then T_μ is Fredholm (Theorem 3.6). Therefore T^{**}_μ is Fredholm and $T^{**}_\mu \Gamma_E$ is upper semi-Fredholm. Then, by formula (5), M_μ is upper semi-Fredholm. Since $\mu = \lambda * \nu$ where $\lambda * \lambda = \lambda$ and ν is invertible [12, Theorem 1], we have that $M(G) = R(M_\mu) \oplus N(M_\mu)$. Hence M_μ is Fredholm. □

Recall that an operator $T : L_1(G) \to L_1(G)$ is tauberian if and only if $m \in L_1(G)^{**}$ and $T^{**}m \in L_1(G)$ imply $m \in L_1(G)$.

Observation 3.9. *It was asked in* [7] *whether a convolution operator T_μ acting on $L_1(G)$ is tauberian when the measure μ satisfies the following condition:*

$$\nu \in M(G), \mu \star \nu \in L_1(G) \Rightarrow \nu \in L_1(G). \tag{6}$$

Next we will show that the answer to this question is negative.

Indeed, it was proved in [3] that there exists an atomic measure $\mu_0 \in M(\mathbb{T})$ such that T_{μ_0} is an injective non-tauberian operator, where $\mathbb{T}$ denotes the unit circle. It is enough to choose μ_0 such that its Fourier-Stieltjes transform $\widehat{\mu}_0$ satisfies $0 \in \overline{\widehat{\mu}_0(\mathbb{Z})} \setminus \widehat{\mu}_0(\mathbb{Z})$. The following argument, due to Doss [5], shows that μ_0 satisfies condition (6):

Every $\nu \in M(\mathbb{T})$ can be written as $\nu = \nu_1 + \nu_2$ with $\nu_1 \ll m$ and $\nu_2 \perp m$, where m is the Haar measure on $\mathbb{T}$. Since $\mu_0 \star \nu_1 \in L_1(\mathbb{T})$ and $\mu_0 \star \nu_2$ is supported in a m-null set, $T_{\mu_0}\nu \in L_1(\mathbb{T})$ if and only if $\nu_2 = 0$. Thus μ_0 satisfies (6).

It is not difficult to show that $\mu \star \widetilde{f} = \widetilde{\mu} \star f$ for $\mu \in M(G)$ and $f \in L_1(G)$. Also, for every normalized disjoint sequence $(f_n) \subset L_1(G)$, the sequence $(\widetilde{f_n})$ is also normalized and disjoint. Therefore, it follows from [9, Theorem 4.1.3] that T_μ is tauberian if and only if so is $T_{\widetilde{\mu}}$. Hence, by Theorem 3.7, the same happens for M_μ and $M_{\widetilde{\mu}}$, and we get the following result:

Proposition 3.10. *Let G be a non-compact LCA group. Then*

(i) $L_\mu : L_\infty(G) \to L_\infty(G)$ *is tauberian if and only if it is cotauberian, and this is equivalent to L_μ invertible;*

(ii) $M_\mu : M(G) \to M(G)$ *is tauberian if and only if it is invertible;*

(iii) $S_\mu : C_0(G) \to C_0(G)$ *is cotauberian if and only if it is invertible.*

Proof. (i) Note that $L_\mu = T^*_{\widetilde{\mu}}$ and $T_{\widetilde{\mu}}$ is tauberian if and only if so is its second conjugate $T^{**}_{\widetilde{\mu}}$ [9, Theorem 4.4.2]. So the result follows from Theorem 2.1 and Corollary 3.5.

(ii) It is a consequence of Theorems 3.7 and 2.1, and Theorem 1.1.3 in [15].

(iii) Since $S^*_\mu = M_{\widetilde{\mu}}$, the result follows from (ii). □

Acknowledgements.

We are indebted to the referee for several comments that improved the paper.

References

[1] P. Aiena. *Fredholm and local spectral theory, with applications to multipliers.* Kluwer Academic Press, 2004.

[2] R. Arens. *The adjoint of a bilinear operation.* Proc. Amer. Math. Soc. 2 (1951), 839–848.

[3] L. Cely, E. M. Galego and M. González. *Tauberian convolution operators acting on $L_1(G)$.* J. Math. Anal. Appl. 446 (2017), 299–306.

[4] C-H. Chu. *Matrix convolution operators on groups.* Lecture Notes in Math. 1956. Springer, 2008.

[5] R. Doss. *Convolution of singular measures.* Studia Math. 45 (1973), 111–117.

[6] F. Ghahramani, A. T. Lau and V. Losert. *Isometric isomorphisms between Banach algebras related to locally compact groups.* Trans. Amer. Math. Soc. 321 (1990), 273–283.

[7] M. González. *Tauberian operators. Properties, applications and open problems.* In "Concrete operators, spectral theory, operators in harmonic analysis and approximation", pp. 231–242. Operator Theory: Advances & Appl. 236. Birkhäuser, 2014.

[8] M. González and A. Martínez-Abejón. *Tauberian operators on $L_1(\mu)$ spaces.* Studia Math. 125 (1997), 289–303.

[9] M. González and A. Martínez-Abejón. *Tauberian operators.* Operator Theory: Advances and applications 194. Birkhäuser, 2010.

[10] M. González and V. M. Onieva. *Characterizations of tauberian operators and other semigroups of operators.* Proc. Amer. Math. Soc. 108 (1990), 399–405.

[11] M. González, E. Saksman and H.-O. Tylli. *Representing non-weakly compact operators.* Studia Math. 113 (1995), 265–282.

[12] B. Host and F. Parreau. *Sur un problème de I. Glicksberg: Les idéaux fermés de type fini de $M(G)$.* Ann. Inst. Fourier (Grenoble) 28 (1978), 143–164.

[13] W. B. Johnson, A. B. Nasseri, G. Schechtman and T. Tkocz. *Injective tauberian operators on L_1 and operators with dense range on ℓ_∞*. Canad. Math. Bull. 58 (2015), 276–280.
[14] N. Kalton, A. Wilansky. *Tauberian operators on Banach spaces.* Proc. Amer. Math. Soc. 57 (1976), 251–255.
[15] R. Larsen. *An introduction to the theory of multipliers.* Springer, 1971.
[16] A. T. Lau and V. Losert. *On the second conjugate algebra of $L_1(G)$ of a locally compact group.* J. London Math. Soc. 37 (1988), 464–470.
[17] V. Losert, M. Neufang, J. Pachl and J. Steprāns. *Proof of the Ghahramani-Lau conjecture.* Advances in Math. 290 (2016), 709–738.
[18] W. Rudin. *Fourier analysis on groups.* Wiley & Sons, 1962.
[19] D. G. Tacon. *Generalized semi-Fredholm transformations.* J. Austral. Math. Soc. 347 (1983), 60–70.
[20] S. Watanabe. *A Banach algebra which is an ideal in the second dual space.* Sci. Rep. Niigata Univ. Ser. A 11 (1974), 95–101.

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, 05508-090 São Paulo SP, Brazil
E-mail address: mlcelyp@unal.edu.co

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, 05508-090 São Paulo SP, Brazil
E-mail address: eloi@ime.usp.br

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain
E-mail address: manuel.gonzalez@unican.es