Using Sparse Control Methods to Identify Sources in Linear Diffusion-Convection Equations

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Abstract. Techniques from sparse control theory are proposed to approximate initial conditions for diffusion-convection equations. Existence and uniqueness of optimal controls are proven, and necessary and sufficient optimality conditions are derived. From these conditions the sparsity structure of the solutions is derived, which relates to identification of the sources to be reconstructed.

Keywords: measure controls, sparsity, parabolic equations, source identification

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1. Introduction

The goal of this paper is to describe the use of optimal control tools to approximate the solution to a class of inverse source problems related to diffusion-convection equations. In particular we study the identification of the initial conditions for the parabolic equation

$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t} + Ay & = & f & \text{in } Q = \Omega \times (0,T) \\ y(x,0) & = & u & \text{in } \bar{\Omega} \\ \partial_n y(x,t) & = & 0 & \text{on } \Sigma = \Gamma \times (0,T), \end{array}\right. \tag{1.1}$$

where Γ is the boundary of Ω, u is a Borel measure in $\bar{\Omega}$, and A is the elliptic operator defined by

$$Ay = -a\Delta y + b(x,t) \cdot \nabla y + c(x,t)y \tag{1.2}$$

for a constant $a > 0$ and functions b and c. The goal is to identify u from the observation y_d corresponding to the state at the final time $y_u(T)$. For this purpose we consider the following optimal control problem

$$(\mathrm{P}_\alpha) \qquad \min_{u \in U_{ad}} J(u) = \frac{1}{2}\|y_u(T) - y_d\|^2_{L^2(\Omega)},$$

where

$$U_{ad} = \{u \in \mathcal{M}(\bar{\Omega}) : \|u\|_{\mathcal{M}(\bar{\Omega})} \leq \alpha\},$$

with $\alpha > 0$, and where y_u is the solution to (1.1) corresponding to the control u.

There is a vast amount of mathematical contributions to inverse source problems. The reader is referred to the monographs [1] and [2] and the references therein. Here we are inspired by applications whose aim is the identification of pointwise pollution sources, see e.g. [3], [4]. The total amount or an upper bound of pollution could be known in some cases, which justifies the consideration of the control constraint. In this context, the use of sparse control techniques suggests itself as a powerful tool. The choice of controls as measures whose supports indicate the source locations appears in a natural way. In some investigations the problem is formulated as searching for combinations of Dirac measures given by

$$u = \sum_{k=1}^{m} \alpha_k \delta_{x_k},$$

where the number m of locations is fixed and α_k and x_k are the optimization variables. This leads to a non convex optimization problem with practical difficulties related to the computation of the derivatives with respect to x_k. Our formulation is a convex optimization problem and in certain situations can be proved that the solutions are of the above type; see Corollary 2.8 and Remark 2.10. The use of measures as controls has been exploited in previous papers and its numerical realization has been achieved in an efficient way; see [5], [6], [7]. [8], [9], [10].

The plan of this paper is as follows. In the next section, we analyze the control problem (P_α), which uses deep results from the theory of diffusion-convection equations such as uniqueness of backward parabolic equations with time dependent coefficients, analyticity of the solutions, Hölder regularity, and approximate controllability. In some applications, it is known a priori that the source is non-negative. Section 3 is dedicated to this issue. Finally, in §4 we consider a related problem where, instead of imposing a control constraint, we penalize the norm of the control, in a Tikhonov type style. We subsequently compare both formulations, which provides a direct relation between the constraint parameter α and the penalty parameter.

2. Analysis of (P_α)

In this section we analyze the problem (P_α). First, we study the state equation. Then, existence and uniqueness of an optimal solution $\bar{u}$ is proved. Finally, we derive the optimality conditions and deduce structural properties of $\bar{u}$, in particular, sparsity.

The following regularity assumptions will be assumed throughout this section.

- Ω is an open, connected and bounded subset of $\mathbb{R}^n$, $1 \le n \le 3$, with a Lipschitz boundary Γ.
- $f \in L^1(0,T;L^2(\Omega))$, a is positive real number, $b \in L^\infty(Q)^n$, $c \in L^\infty(Q)$, and $y_d \in L^2(\Omega)$.

With $\mathcal{M}(\bar{\Omega})$ we denote the space of real and regular Borel measures in $\bar{\Omega}$ endowed with the norm

$$\|u\|_{\mathcal{M}(\bar{\Omega})} = \sup_{\|\phi\|_{C(\bar{\Omega})}\le 1} \int_{\bar{\Omega}} \phi(x)\, du(x) = |u|(\bar{\Omega}),$$

where $C(\bar{\Omega})$ is the space of continuous functions in $\bar{\Omega}$ and $|u|$ represents the total variation measure of u. $C(\bar{\Omega})$ is a separable Banach space for the supremum norm and $\mathcal{M}(\bar{\Omega})$ is its dual space; see [11, page 130].

2.1. Analysis of the State Equation

Definition 2.1 *We say that a function $y \in L^1(Q)$ is a solution of* (1.1) *if the following identity holds*

$$\int_Q (-\frac{\partial \phi}{\partial t} + A^*\phi) y \, dxdt = \int_Q f\phi \, dx\, dt + \int_{\bar{\Omega}} \phi(0)\, du \quad \forall \phi \in \Phi, \tag{2.1}$$

where

$$\Phi = \{\phi \in L^2(0,T;H^1(\Omega)) : -\frac{\partial \phi}{\partial t} + A^*\phi \in L^\infty(Q),\ \partial_n \phi = 0 \ on\ \Sigma,\ \phi(T) = 0 \ in\ \Omega\}$$

and

$$A^*\phi = -a\Delta\phi - \mathrm{div}[b(x,t)\phi] + c\phi.$$

Let us observe that the problem

$$\left\{\begin{array}{rcll} -\dfrac{\partial \phi}{\partial t} + A^*\phi &=& g & \text{in } Q \\ \phi(x,T) &=& 0 & \text{in } \Omega \\ \partial_n \phi(x,t) &=& 0 & \text{on } \Sigma \end{array}\right. \tag{2.2}$$

has a unique solution $\phi \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ for every $g \in L^\infty(Q)$. Moreover, the regularity $\phi \in C(\bar{Q})$ holds. This continuity property follows, for instance, from the results in [12]; see also [13, Chapter III].

Theorem 2.2 *There exists a unique solution y of (1.1). Moreover, the regularity $y \in L^r(0,T;W_0^{1,p}(\Omega)) \cap C((0,T];L^2(\Omega))$ holds for all $p,r \in [1,2)$ with $(2/r)+(n/p) > n+1$, and we have the following estimate for some constant $C_{r,p}$*

$$\|y\|_{L^r(0,T;W_0^{1,p}(\Omega))} + t\|y(t)\|_{L^2(\Omega)} \leq C_{r,p}(\|u\|_{\mathcal{M}(\bar{\Omega})} + \|f\|_{L^1(0,T;L^2(\Omega))}) \quad \forall t \in [0,T]. \tag{2.3}$$

Proof. Let us prove first the uniqueness. If y_1, y_2 are two solutions associated with the control u, then from (2.1) we deduce that $y = y_2 - y_1$ satisfies

$$\int_Q (-\frac{\partial \phi}{\partial t} + A^*\phi) y \, dxdt = 0 \quad \forall \phi \in \Phi.$$

Choosing $g = \text{sign}(y)$ in (2.2) and inserting the corresponding solution ϕ in the above identity we infer that $y = 0$. To prove the existence and the regularity we choose a sequence $\{u_k\}_k \subset C(\bar{\Omega})$ such that $u_k \overset{*}{\rightharpoonup} u$ in $\mathcal{M}(\bar{\Omega})$ and $\|u_k\|_{L^1(\Omega)} \leq \|u\|_{\mathcal{M}(\bar{\Omega})}$. This can be achieved by taking the convolution with sequences of mollifiers. Associated with $\{u_k\}_k$ we define the sequence of solutions $\{y_k\}_k \subset L^2(0,T;H^1(\Omega))$ of (1.1). Then, using the regularity of y_k we integrate by parts to obtain for every $\phi \in \Phi$

$$\begin{aligned} \int_Q (-\frac{\partial \phi}{\partial t} + A^*\phi) y_k \, dxdt &= \int_Q (\frac{\partial y_k}{\partial t} + Ay_k)\phi \, dxdt + \int_\Omega \phi(0) u_k \, dx \\ &= \int_Q f\phi \, dx \, dt + \int_\Omega \phi(0) u_k \, dx. \end{aligned} \tag{2.4}$$

Let us obtain the estimate on the first summand in (2.3) for y_k. To this end, we take $\{\psi_j\}_{j=0}^d \subset \mathcal{D}(Q)$ and $\phi \in \Phi$ satisfying

$$\left\{\begin{array}{rcll} -\dfrac{\partial \phi}{\partial t} + A^*\phi &=& \psi_0 - \dfrac{\partial \psi_j}{\partial x_j} & \text{in } Q \\ \phi(x,T) &=& 0 & \text{in } \Omega \\ \partial_n \phi(x,t) &=& 0 & \text{on } \Sigma. \end{array}\right. \tag{2.5}$$

Following [12] and [13, Chapter III], we know that there exists a constant C such that

$$\|\phi\|_{C(\bar{Q})} \leq C \sum_{j=0}^d \|\psi_j\|_{L^{r'}(0,T;L^{p'}(\Omega))}. \tag{2.6}$$

Indeed, let us observe that the condition $\frac{2}{r} + \frac{n}{p} > n+1$ is equivalent to $\frac{2}{r'} + \frac{n}{p'} < 1$, where r' and p' denote the conjugates of r and p. The later inequality is sufficient for the regularity and a priori estimate (2.6).

Now, using distributional derivatives we obtain from (2.4)-(2.6)

$$\langle y_k, \psi_0\rangle + \sum_{j=1}^{d}\langle \partial_{x_j} y_k, \psi_j\rangle = \int_Q y_k(\psi_0 - \sum_{j=0}^{d}\partial_{x_j}\psi_j)\,dx\,dt$$

$$= \int_Q (-\frac{\partial\phi}{\partial t} + A^*\phi)y_k\,dxdt = \int_Q f\phi\,dx\,dt + \int_\Omega \phi(0)u_k\,dx$$

$$\leq C(\|f\|_{L^1(Q)} + \|u_k\|_{L^1(\Omega)})\sum_{j=0}^{d}\|\psi_j\|_{L^{r'}(0,T;L^{p'}(\Omega))}$$

$$\leq C'(\|f\|_{L^1(0,T;L^2(\Omega))} + \|u\|_{\mathcal{M}(\bar\Omega)})\sum_{j=0}^{d}\|\psi_j\|_{L^{r'}(0,T;L^{p'}(\Omega))}.$$

This proves that $\{y_k\}_k \subset L^r(0,T;W^{1,p}(\Omega))$ and every y_k satisfies (2.3). By taking a subsequence, we deduce the existence of $y \in L^r(0,T;W^{1,p}(\Omega))$ such that $y_k \rightharpoonup y$ in this space. Then, passing to the limit in (2.4) we infer that y is solution of (1.1) and the estimate on the first term in (2.3) holds.

To deduce the second estimate we introduce the function $z(x,t) = (t-t_0)y(x,t)$ with $0 < t_0 < T$. Then, z satisfies the equation

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} + Az &=& (t-t_0)f + y & \text{in } \Omega\times(t_0,T)\\ z(x,t_0) &=& 0 & \text{in } \Omega\\ \partial_n z(x,t) &=& 0 & \text{on } \Gamma\times(t_0,T).\end{array}\right. \tag{2.7}$$

We have already proved that $y \in L^r(0,T;W^{1,p}(\Omega))$ for all $p,r \in [1,2)$ with $(2/r) + (n/p) > n+1$. Choosing $r=1$ and $p < \frac{n}{n-1}$ big enough so that $W^{1,p}(\Omega) \subset L^2(\Omega)$ we deduce that $y \in L^1(0,T;L^2(\Omega))$ and

$$\|y\|_{L^1(0,T;L^2(\Omega))} \leq C_1(\|u\|_{\mathcal{M}(\bar\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}) \tag{2.8}$$

for some constant $C_1 > 0$. It is standard that $z \in L^2(t_0,T;H^1(\Omega)) \cap C([t_0,T];L^2(\Omega))$ and

$$\|z\|_{C([t_0,T];L^2(\Omega))} \leq C_2(T\|f\|_{L^1(0,T;L^2(\Omega))} + \|y\|_{L^1(0,T;L^2(\Omega))})$$

for a constant C_2 independent of t_0. For a method of proof the reader is referred to [14, pp. 264-265]. Now, inserting (2.8) in the above inequality we obtain

$$\|z\|_{C([t_0,T];L^2(\Omega))} \leq C_3(\|u\|_{\mathcal{M}(\bar\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}) \tag{2.9}$$

for a constant C_3. Since $y(x,t) = \frac{1}{t-t_0}z(x,t)$ for $t > t_0$ and $t_0 > 0$ is arbitrary, we infer that $y \in C((0,T],L^2(\Omega))$ and, in particular, we deduce from (2.9)

$$(t-t_0)\|y(t)\|_{L^2(\Omega)} = \|z(t)\|_{L^2(\Omega)} \leq C_3(\|u\|_{\mathcal{M}(\bar\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}).$$

Since $t_0 > 0$ was arbitrary, this inequality implies (2.3). □

Though the previous theorem implies the continuity of the relation control-to-state with respect to the strong topology in $\mathcal{M}(\bar\Omega)$, this is not enough to prove the solvability of (P_α). For this purpose we establish the following result.

Theorem 2.3 *Let $\{u_k\}_k \subset \mathcal{M}(\bar\Omega)$ be such that $u_k \overset{*}{\rightharpoonup} u$ in $\mathcal{M}(\bar\Omega)$ and let $\{y_k\}_k$ and y be the associated states. Then, $y_k \rightharpoonup y$ in $L^r(0,T;W^{1,p}(\Omega))$ and*

$$\lim_{k\to\infty} \|y_k - y\|_{C([t_0,T];L^2(\Omega))} = 0 \quad \forall t_0 \in (0,T). \tag{2.10}$$

In particular, the convergence $y_k(T) \to y(T)$ in $L^2(\Omega)$ holds.

In the above statement, r and p must satisfy the relations specified in Theorem 2.2.

Proof. From (2.3) we get the boundedness of $\{y_k\}_k$ in $L^r(0,T;W^{1,p}(\Omega))$. This implies the existence of subsequences weakly converging in $L^r(0,T;W^{1,p}(\Omega))$. For these subsequences it is immediate to pass to the limit in (2.1) and to deduce that the state y corresponding to the control u is the unique limit. Hence, the whole sequence converges weakly to y.

To prove (2.10) we introduce the functions $z_k = (t-t_0)y_k$ and $z = (t-t_0)y$. We observe that $e_k = z_k - z$ satisfies the equation

$$\begin{cases} \dfrac{\partial e_k}{\partial t} + Ae_k &=& y_k - y & \text{in } \Omega\times(t_0,T) \\ e_k(x,t_0) &=& 0 & \text{in } \Omega \\ \partial_n e_k(x,t) &=& 0 & \text{on } \Gamma\times(t_0,T). \end{cases}$$

Using again (2.3) we know that $\{y_k - y\}_k$ is bounded in $L^\infty(t_0,T;L^2(\Omega))$. From [12] we have that $\{e_k\}_k$ is bounded in a space of Hölder functions $C^{0,\mu}(\bar\Omega\times[t_0,T])$ for some $\mu\in(0,1)$. Since $y_k - y \rightharpoonup 0$ weakly in $L^r(0,T;W^{1,p}(\Omega))$ and the embedding $C^{0,\mu}(\bar\Omega\times[t_0,T]) \subset C(\bar\Omega\times[t_0,T])$ is compact, we infer that $e_k \to 0$ strongly in $C(\bar\Omega\times[t_0,T])$, which proves (2.10). □

2.2. Analysis of (P_α)

In this section, we are going to prove the existence and uniqueness of a solution of (P_α). We will also derive the optimality conditions satisfied by this solution and discuss its sparsity structure.

Theorem 2.4 *Problem* (P_α) *has a unique solution $\bar u$.*

Proof. For the existence of a solution we observe that U_{ad} is bounded and weakly* closed in $\mathcal{M}(\bar\Omega)$. Actually, from Banach-Alaoglu-Bourbaki Theorem we know that it is weakly* compact; see, for instance, [15, Theorem 3.16]. Therefore, any minimizing sequence is bounded in $\mathcal{M}(\bar\Omega)$ and any weak* limit belongs to U_{ad}. Finally, using Theorem 2.3 we conclude that any of these limits is a solution of (P_α).

Let us prove the uniqueness. Let $\bar u_1$ and $\bar u_2$ be two solutions of (P_α) with associated states $\bar y_1$ and $\bar y_2$. From the convexity of U_{ad} and the strict convexity of the $L^2(\Omega)$ norm we deduce that $\bar y_1(T) = \bar y_2(T)$. Let us set $\bar y = \bar y_2 - \bar y_1$ and take $t_0\in(0,T)$ arbitrary. We have that $\bar y(x,t) = \frac{1}{t-t_0} z(x,t)$ for every $t\in(t_0,T]$, where z satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az &=& \bar y & \text{in } \Omega\times(t_0,T) \\ z(x,t_0) &=& 0 & \text{in } \Omega \\ \partial_n z(x,t) &=& 0 & \text{on } \Gamma\times(t_0,T). \end{cases}$$

From Theorem 2.2 we know that $\bar{y} \in L^2(t_0, T; L^2(\Omega))$ holds. Hence, $z \in L^2(t_0, T; H^1(\Omega))$ and

$$\left\{ \begin{array}{rcll} \dfrac{\partial z}{\partial t} - a\Delta z & = & \bar{y} + g & \text{in } \Omega \times (t_0, T) \\ z(x, t_0) & = & 0 & \text{in } \Omega \\ \partial_n z(x, t) & = & 0 & \text{on } \Gamma \times (t_0, T), \end{array} \right.$$

where

$$g = -b \cdot \nabla z - cz \in L^2(t_0, T; L^2(\Omega)).$$

As usual we denote

$$D(\Delta) = \{\phi \in H^1(\Omega) : \Delta\phi \in L^2(\Omega) \text{ and } \partial_n \phi = 0 \text{ in } \Gamma\}.$$

Now, from standard results on evolution equations, see, for instance, [16, pp. 113–114], we infer that $z \in C([t_0, T]; H^1(\Omega)) \cap L^2(t_0, T; D(\Delta))$. Taking into account that $z(t) = (t - t_0)\bar{y}(t)$, we deduce that $\bar{y} \in C([\delta, T]; H^1(\Omega)) \cap L^2(\delta, T; D(\Delta))$ for all $t_0 < \delta < T$. Since t_0 is arbitrary in $(0, T)$ we conclude that this regularity of $\bar{y}$ holds for arbitrary $0 < \delta < T$. Moreover, from the state equation satisfied by $\bar{y}$ we get for almost every $t \in (\delta, T)$

$$\|\frac{\partial \bar{y}}{\partial t}(t) - a\Delta\bar{y}(t)\|_{L^2(\Omega)} = \|b(t) \cdot \nabla\bar{y}(t) + c(t)\bar{y}(t)\|_{L^2(\Omega)} \leq C\|\bar{y}(t)\|_{H^1(\Omega)}.$$

Now, since $\bar{y}(T) = 0$ we deduce from the backward uniqueness of the parabolic equation [17, Theorem 1.1] that $\bar{y}(t) = 0$ for all $t \in [\delta, T]$. Due to the fact that $\delta > 0$ can be taken arbitrarily small, we conclude that $\bar{y}(t) = 0 \ \forall t \in (0, T]$. Finally, since $\bar{y} \in L^1(0, T; W^{1,p}(\Omega))$ for $p < \frac{n}{n-1}$ and $\frac{\partial \bar{y}}{\partial t} = f - A\bar{y} \in L^1(0, T; W^{1,p'}(\Omega)^*)$, it follows that $\bar{y} : [0, T] \longrightarrow W^{1,p'}(\Omega)^*$ is continuous. Hence, we conclude that $\bar{u}_2 - \bar{u}_1 = \bar{y}(0) = \lim_{t\to 0} \bar{y}(t) = 0$. □

Next, we establish the optimality conditions satisfied by the solution $\bar{u}$ of (P_α). First, we introduce the adjoint state associated with $\bar{u}$ as the solution to

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \bar{\varphi}}{\partial t} + A^*\bar{\varphi} & = & 0 & \text{in } Q \\ \bar{\varphi}(T) & = & \bar{y}(T) - y_d & \text{in } \Omega \\ \partial_n \bar{\varphi}(x, t) & = & 0 & \text{on } \Sigma, \end{array} \right. \tag{2.11}$$

where $\bar{y}$ is the state corresponding to $\bar{u}$. According to Theorem 2.2, we know that $\bar{y}(T) \in L^2(\Omega)$, hence $\bar{\varphi} \in L^2(0, T; H^1(\Omega)) \cap C(\bar{\Omega} \times [0, T))$. The norm of $\bar{\varphi}$ in these spaces can be estimated by the norm of $\bar{y}(T) - y_d$ in $L^2(\Omega)$; see, for instance, [13, §III-7 to §III-10]. In particular, there exists a constant C_0 such that

$$\|\bar{\varphi}(0)\|_{C(\bar{\Omega})} \leq C_0\|\bar{y}(T) - y_d\|_{L^2(\Omega)}. \tag{2.12}$$

Theorem 2.5 *Let $\bar{u}$ be the solution of (P_α) with $\bar{y}$ and $\bar{\varphi}$ the associated state and adjoint state. Then, the following properties hold*

(i) If $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} < \alpha$, then $\bar{y}(T) = y_d$ and $\bar{\varphi} = 0$ in Q.

(ii) If $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} = \alpha$, *then*

$$\begin{aligned} \operatorname{supp}(\bar{u}^+) &\subset \{x \in \bar{\Omega} : \bar{\varphi}(x,0) = -\|\bar{\varphi}(0)\|_{C(\bar{\Omega})}\}, \\ \operatorname{supp}(\bar{u}^-) &\subset \{x \in \bar{\Omega} : \bar{\varphi}(x,0) = +\|\bar{\varphi}(0)\|_{C(\bar{\Omega})}\}, \end{aligned} \tag{2.13}$$

where $\bar{u} = \bar{u}^+ - \bar{u}^-$ *is the Jordan decomposition of* $\bar{u}$.

Conversely, if $\bar{u}$ *is an element of* U_{ad} *satisfying* (i) *or* (ii), *then* $\bar{u}$ *is the solution to* (P_α).

Before proving the above theorem we establish two lemmas.

Lemma 2.6 *Given* $u \in \mathcal{M}(\bar{\Omega})$, *the solution* $z_u \in L^r(0,T;W^{1,p}(\Omega))$ *to*

$$\left\{ \begin{array}{rcll} \dfrac{\partial z}{\partial t} + Az & = & 0 & \text{in } Q \\ z(0) & = & u & \text{in } \Omega \\ \partial_n z(x,t) & = & 0 & \text{on } \Sigma \end{array} \right. \tag{2.14}$$

satisfies

$$\int_\Omega (y_u(T) - y_d) z_u(T)\, dx = \int_{\bar{\Omega}} \bar{\varphi}(0)\, du. \tag{2.15}$$

Proof. Let us consider two sequences $\{g_k\}_k, \{u_k\}_k \subset C(\bar{\Omega})$ such that $g_k \to \bar{y}(T) - y_d$ strongly in $L^2(\Omega)$ and $u_k \overset{*}{\rightharpoonup} u$ in $M(\bar{\Omega})$. Now we introduce the solutions $\{\varphi_k\}_k, \{z_k\}_k \subset L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*)) \cap C(\bar{Q})$ to the equations

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \varphi_k}{\partial t} + A^*\varphi_k & = & 0 & \text{in } Q \\ \varphi_k(x,T) & = & g_k(x) & \text{in } \Omega \\ \partial_n \varphi_k(x,t) & = & 0 & \text{on } \Sigma \end{array} \right. \qquad \left\{ \begin{array}{rcll} \dfrac{\partial z_k}{\partial t} + Az_k & = & 0 & \text{in } Q \\ z_k(0) & = & u_k & \text{in } \Omega \\ \partial_n z_k(x,t) & = & 0 & \text{on } \Sigma \end{array} \right.$$

Because of the regularity of φ_k and z_k we are justified to integrate by parts getting

$$\int_\Omega g_k z_k(T)\, dx = \int_\Omega \varphi_k(0) u_k\, dx.$$

Finally, using that $\|z_k(T) - z(T)\|_{L^2(\Omega)} \to 0$ due to Theorem 2.3, and $\|\varphi_k(0) - \bar{\varphi}(0)\|_{C(\bar{\Omega})} \to 0$ by (2.12), we can pass to the limit in the above identity and deduce (2.15). ∎

The following lemma establishes the approximate controllability of system (1.1) by the initial condition. We prove it here because the coefficients b and c are time dependent.

Lemma 2.7 *For every* $\varepsilon > 0$ *there exists a control* $u \in L^2(\Omega)$ *such that* $\|y_u(T) - y_d\|_{L^2(\Omega)} < \varepsilon$.

Proof. We argue by contradiction and assume that the statement of the lemma is false. Then, in particular, the reachable set

$$\mathcal{R} = \{y_u(T) : u \in L^2(\Omega)\}$$

is not dense in $L^2(\Omega)$. Therefore, there exists an element $g \in L^2(\Omega)$, $g \neq 0$, such that

$$\int_\Omega g(x) y_u(x,T)\,dx = 0 \quad \forall u \in L^2(\Omega). \tag{2.16}$$

Denote by y_0 the solution to (1.1) corresponding to the control 0 and by z_u the solution of (2.14). Then, the identity $y_u = y_0 + z_u$ obviously holds for every $u \in L^2(\Omega)$. If we take $u = 0$ in (2.16) we deduce that

$$\int_\Omega g(x) y_0(x,T)\,dx = 0.$$

Hence, with (2.16) we get the identity

$$\int_\Omega g(x) z_u(x,T)\,dx = 0 \quad \forall u \in L^2(\Omega). \tag{2.17}$$

Let us take φ as solution of

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi &= 0 \quad \text{in } Q, \\ \varphi(T) &= g \quad \text{in } \Omega, \\ \partial_n \varphi &= 0 \quad \text{on } \Sigma. \end{cases}$$

From (2.17) and Lemma 2.6 with $\bar{y}(T) - y_d$ replaced by g, we infer

$$0 = \int_\Omega \varphi(x,T) z_u(x,T)\,dx = \int_\Omega \varphi(x,0) u(x)\,dx \quad \forall u \in L^2(\Omega).$$

Consequently, $\varphi(0) = 0$ is satisfied. Arguing as in the proof of Theorem 2.11 we can apply the backward uniqueness to φ to deduce that $\varphi(t) = 0\ \forall t \in [0,T]$. Hence $g = 0$ and we obtain a contradiction. □

Proof of Theorem 2.5. Let u be an arbitrary element of U_{ad}. Denote by $z_{u-\bar{u}}$ the solution to (2.14) with u replaced by $u - \bar{u}$. From Lemma 2.6 we get

$$\lim_{\rho \searrow 0} \frac{J(\bar{u} + \rho(u - \bar{u})) - J(\bar{u})}{\rho} = \int_\Omega (\bar{y}(T) - y_d) z_{u-\bar{u}}(T) dx = \int_{\bar{\Omega}} \bar{\varphi}(0)\, d(u - \bar{u}).$$

Since (P_α) is a convex problem, the next condition is necessary and sufficient for optimality of an element $\bar{u} \in U_{ad}$

$$J'(\bar{u})(u - \bar{u}) = \int_{\bar{\Omega}} \bar{\varphi}(0)\, d(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad},$$

or equivalently

$$-\int_{\bar{\Omega}} \bar{\varphi}(0)\, du \leq -\int_{\bar{\Omega}} \bar{\varphi}(0)\, d\bar{u} \quad \forall u \in U_{ad}.$$

Taking the supremum in $u \in U_{ad}$ we infer the equivalent expression.

$$\alpha \|\bar{\varphi}(0)\|_{C(\bar{\Omega})} = -\int_{\bar{\Omega}} \bar{\varphi}(0)\, d\bar{u}.$$

In the case $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} = \alpha$ this identity is the same as

$$\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} \|\bar{\varphi}(0)\|_{C(\bar{\Omega})} = -\int_{\bar{\Omega}} \bar{\varphi}(0)\, d\bar{u}. \tag{2.18}$$

Therefore, (2.18) is a necessary and sufficient condition for optimality if $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} = \alpha$. Now, from [6, Lemma 3.4] we infer (2.13). Conversely, if $\bar{u} \in \mathcal{M}(\bar{\Omega})$ satisfies *(ii)*, then it is immediate to check that (2.18) holds as well and, hence, $\bar{u}$ is the solution of (P_α).

Let us study the case $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} < \alpha$. Obviously if $\bar{y}(T) = y_d$, then $\bar{u}$ is the solution of (P_α). Conversely, assume now that $\bar{u}$ is the solution and $\bar{y}(T) \neq y_d$. From Lemma 2.7 we deduce the existence of an element $u \in \mathcal{M}(\bar{\Omega})$ such that $J(u) < J(\bar{u})$. Since $\bar{u}$ is a solution of (P_α), then $u \notin U_{ad}$. Let us take a number λ satisfying

$$0 < \lambda < \min\left\{\frac{\alpha - \|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}}{\|u - \bar{u}\|_{\mathcal{M}(\bar{\Omega})}}, 1\right\}.$$

Then, $v = \bar{u} + \lambda(u - \bar{u}) \in U_{ad}$ holds and

$$J(v) = J(\lambda u + (1-\lambda)\bar{u}) \leq \lambda J(u) + (1-\lambda)J(\bar{u}) < J(\bar{u}),$$

which contradicts the optimality of $\bar{u}$. Hence $\bar{y}(T) = y_d$, and using (2.11) we conclude that $\bar{\varphi} = 0$. □

The relationships (2.13) suggest the idea that the optimal control is supported in a small region. Actually, this can be proved under certain assumptions.

Corollary 2.8 *Let us assume that $c \equiv 0$ and b is a function only depending on time and analytic in $(-\varepsilon, T+\varepsilon)$ for some $\varepsilon > 0$. If $\bar{u}$ is the solution of (P_α) with associated state $\bar{y}$ and $\bar{y}(T) - y_d$ is not constant, then the following properties hold:*

(i) The support of $\bar{u}$ has zero Lebesgue measure.

(ii) If $n = 1$, then there exists a countable set of points $\{x_i\}_{i\in I} \subset \bar{\Omega}$ and real numbers $\{\bar{\lambda}_i\}_{i\in I}$ such that

$$\bar{u} = \sum_{i\in I} \bar{\lambda}_i \delta_{x_i} \quad \text{with } \bar{\lambda}_i = \begin{cases} > 0 & \text{if } \bar{\varphi}(x_i, 0) = -\|\bar{\varphi}(0)\|_{C(\bar{\Omega})} \\ < 0 & \text{if } \bar{\varphi}(x_i, 0) = +\|\bar{\varphi}(0)\|_{C(\bar{\Omega})} \end{cases} \tag{2.19}$$

If the cardinality of I is not finite, then the accumulation points of $\{x_i\}_{i\in I}$ are points of Γ.

Proof. (i) Since b is analytic, the solution $\bar{\varphi}$ to (2.11) is analytic with respect to the variable x in Ω for every $t \in [0, T)$. In particular, $\bar{\varphi}(0)$ is analytic in Ω; see, for instance, [18, page 324] or [19] and the references therein. As a consequence, either $\bar{\varphi}(0)$ is a constant function in $\bar{\Omega}$ or the set of points $S = \{x \in \bar{\Omega} : |\bar{\varphi}(x, 0)| = \|\bar{\varphi}(0)\|_{C(\bar{\Omega})}\}$ has zero Lebesgue measure. Let us prove that the first case is not possible. Indeed, if $\bar{\varphi}(x, 0) = \kappa$ for some $\kappa \in \mathbb{R}$ and for every $x \in \bar{\Omega}$, then $\psi = \bar{\varphi} - \kappa$ satisfies

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} + A^*\psi &= 0 \quad \text{in } Q \\ \psi(0) &= 0 \quad \text{in } \Omega \\ \partial_n \psi &= 0 \quad \text{on } \Sigma, \end{cases}$$

due to the assumptions on b and c. Then, arguing as in the proof of Theorem 2.11, we deduce the backward uniqueness of the equation and, hence, that $\psi(x, t) = 0$ in Q.

Therefore, the identity $\bar\varphi(x,t)=\kappa$ holds in Q. This implies that $\bar y(T)-y_d=\bar\varphi(T)=\kappa$, which contradicts our assumption.

(ii) In the case $n=1$, the analyticity of $\bar\varphi$ implies that the set S defined above is countable and the possible accumulation points must be on the boundary Γ. Moreover, (2.19) is an immediate consequence of (2.13). □

Remark 2.9 *The assumption that $\bar y(T)-y_d$ is not identically equal to a constant is obviously satisfied provided that $y_d\notin H^1(\Omega)$. In the case of a regular function y_d, if its normal derivative on Γ is not identically equal to zero, then for a regular boundary Γ and a function $f\in L^p(Q)$ with $p>4$, the equality $\bar y(T)-y_d=\kappa$ with $\kappa\in\mathbb{R}$ is again not possible. Indeed, it is enough to observe that in this case $\partial_n\bar y(t)=0$ is satisfied for every $t\in(0,T]$. Finally, the identity $\bar y(T)-y_d=\kappa$ is not possible either if y_d is not attainable by any control $u\in\mathcal{M}(\bar\Omega)$. Indeed, if $\bar y(T)-y_d=\kappa$, then $y_u(T)=\bar y(T)+\kappa=y_d$ for $u=\bar u+\kappa$, which contradicts the assumption.*

Remark 2.10 *In the state equation* (1.1) *we have considered a Neumann boundary condition. All the results proved in this paper remain valid if we replace this condition by an homogeneous Dirichlet or Robin condition. Concerning Corollary 2.8, the statement holds under the weaker assumption $\bar y(T)\neq y_d$. In particular, the condition $c=0$ is not necessary. Indeed, following the above proof, if $\bar\varphi(0)=\kappa$ in Ω, then κ must be zero due to the homogeneous boundary condition. Once again, the backward uniqueness property of the adjoint state equation implies that $\bar\varphi=0$ in Q, therefore $\bar y(T)-y_d=\bar\varphi(T)=0$, contradicting the assumption.*

Moreover, in the case $n=1$, for a Dirichlet boundary condition, the set of points $\{x_i\}_{i\in I}$ is finite because if there is an accumulation point in Γ, then $\bar\varphi$ vanishes there. This means that $\bar\varphi(0)=0$, which is not possible under our assumptions as we have proved in the previous paragraph.

Remark 2.11 *Let us briefly discuss the case, when the terminal tracking function which was used in* (P_α) *is replaces by the temporally distributed cost $\frac12\int_{t_1}^T\|y_u(t)-y_d\|^2_{L^2(\Omega)}\,dt$, where $t_1\in[0,T)$ and $y_d\in\Omega\times(t_1,T)$. Again the associated optimal control problem has a unique solution $\bar u$, i.e. the analogue of Theorem holds. The adjoint equation for this case is given by*

$$\left\{\begin{array}{rcll} -\dfrac{\partial\bar\varphi}{\partial t}+A^*\bar\varphi &=& (\bar y-y_d)\chi_{Q_1} & \text{in } Q\\ \bar\varphi(T) &=& 0 & \text{in } \Omega\\ \partial_n\bar\varphi(x,t) &=& 0 & \text{on } \Sigma, \end{array}\right. \tag{2.20}$$

where χ_{Q_1} is the characteristic function of the set $Q_1=\Omega\times(t_1,T)$.

To obtain the analogue of Theorem 2.5 we assume that $\int_Q c\,dx\,dt=0$, $b\in C(Q)^n$, and additionally that $\int_\Sigma b\cdot n\,dx\,dt=0$, and $\int_{Q_1}(\bar y-y_d)\,dxdt\neq0$. Under these conditions, integrating the adjoint equation given in (2.20) *over Q and using the boundary conditions for $\bar\varphi$ and b, and the fact that $c=0$, we obtain that*

$$\int_\Omega\bar\varphi(0)\,dx=\int_{Q_1}(\bar y-y_d)\,dxdt\neq0.$$

Hence $\bar\varphi(0) \neq 0$ holds. Supposing that $\|\bar u\|_{\mathcal{M}(\bar\Omega)} < \alpha$, one can choose $u_\rho = \bar u + \rho\,\bar\varphi(0)$ with $\rho < 0$ and $|\rho|$ sufficiently small such that $u_\rho \in U_{ad}$, and then $J'(\bar u)(u_\rho - \bar u) = \rho\|\bar\varphi\|^2_{L^2(\Omega)} < 0$, hence the necessary and sufficient optimality condition is violated. Therefore, the case $\|\bar u\|_{\mathcal{M}(\bar\Omega)} < \alpha$ cannot occur and the constraint is necessarily active. The case $\|\bar u\|_{\mathcal{M}(\bar\Omega)} = \alpha$ can be treated exactly as in the proof of Theorem 2.5.

3. Identification of positive sources

In this section, we concentrate on the situation where a priori knowledge on the system suggests to consider non-negative measures. Then, the identification problem is formulated as follows

$$(\mathrm{P}^+_\alpha) \qquad \min_{u\in U^+_{ad}} J(u) = \frac{1}{2}\|y_u(T) - y_d\|^2_{L^2(\Omega)},$$

where

$$U^+_{ad} = \{u \in \mathcal{M}^+(\bar\Omega) : u(\bar\Omega) \leq \alpha\}$$

with $\mathcal{M}^+(\bar\Omega)$ the subspace of $\mathcal{M}(\bar\Omega)$ formed by all non-negative measures. Observe that for non-negative measures the equality $\|u\|_{\mathcal{M}(\bar\Omega)} = u(\bar\Omega)$ holds. For problem (P^+_α) we have the following result.

Theorem 3.1 *(P^+_α) has a unique solution. Moreover, $\bar u \in U^+_{ad}$ is a solution of (P^+_α) if and only if*

$$\int_{\bar\Omega} \bar\varphi(x,0)\, d\bar u \leq \int_{\bar\Omega} \bar\varphi(x,0)\, du \quad \forall u \in U^+_{ad}. \tag{3.1}$$

If $\bar u(\bar\Omega) = \alpha$, then the following properties are fulfilled:

(i) Inequality (3.1) is equivalent to the identity

$$\int_{\bar\Omega} \bar\varphi(x,0)\, d\bar u = \alpha\bar\mu := \alpha \min_{\omega\in\bar\Omega} \bar\varphi(\omega,0), \tag{3.2}$$

where $\bar\mu \leq 0$.

(ii) $\bar u$ is the solution of (P^+_α) if and only if

$$\mathrm{supp}(\bar u) \subset \{x \in \bar\Omega : \bar\varphi(x,0) = \bar\mu\}. \tag{3.3}$$

Proof. Existence and uniqueness for problem (P^+_α) is proved as in Theorem 2.11. As in the proof of Theorem 2.5, we get that an element $\bar u$ of U^+_{ad} is the solution of (P^+_α) if and only if

$$J'(\bar u)(u - \bar u) = \int_{\bar\Omega} \bar\varphi(0)\, d(u - \bar u) \geq 0 \quad \forall u \in U^+_{ad},$$

which is equivalent to (3.1). Let us consider the case of $\bar u(\bar\Omega) = \alpha$ and prove *(i)*. First, if $\bar\mu > 0$, then taking $u = 0$ in (3.1) we deduce that $\bar u = 0$. Thus, $\bar\mu \leq 0$ holds.

Now, observe that (3.1) is equivalent to

$$\int_{\bar\Omega} \bar\varphi(x,0)\, d\bar u = \min_{u\in U^+_{ad}} \int_{\bar\Omega} \bar\varphi(x,0)\, du.$$

Let $\omega_0 \in \bar{\Omega}$ be a point where $\bar{\varphi}(\omega_0, 0) = \bar{\mu}$. Then, the above minimum is achieved for $u = \alpha\delta_{\omega_0}$, which directly leads to (3.2).The converse implication is immediate.

To prove *(ii)* we distinguish two cases. First we assume that $\bar{\mu} = 0$. In this case, $\bar{\varphi}(x, 0) \geq 0 \ \forall x \in \bar{\Omega}$ and hence (3.2) obviously implies (3.3). In the case of $\bar{\mu} < 0$, we set $\psi(x) = -\min\{\bar{\varphi}(x, 0), 0\}$. It is easy to check that $0 \leq \psi(x) \leq -\bar{\mu}$ and $\|\psi\|_{C(\bar{\Omega})} = -\bar{\mu}$. Now, using (3.1) and the fact that $\psi(x) \geq -\bar{\varphi}(x, 0)$ we get

$$\int_{\bar{\Omega}} \psi \, d\bar{u} \geq -\int_{\bar{\Omega}} \bar{\varphi}(x, 0) \, d\bar{u} \geq -\int_{\bar{\Omega}} \bar{\varphi}(x, 0) \, du \quad \forall u \in U_{ad}^{+}.$$

Taking $u = \alpha\delta_{\omega_0}$ we infer

$$\int_{\bar{\Omega}} \psi \, d\bar{u} \geq -\bar{\mu}\alpha = \|\psi\|_{C(\bar{\Omega})}\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}.$$

The converse inequality is obvious and therefore, we have that

$$\int_{\bar{\Omega}} \psi \, d\bar{u} = \|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}\|\psi\|_{C(\bar{\Omega})}.$$

Now, from [6, Lemma 3.4] we get (3.3).

Finally, the converse implication is a consequence of the fact that (3.3) implies (3.1) whenever $\bar{u}$ is a non-negative measure with $\bar{u}(\bar{\Omega}) = \alpha$. □

Remark 3.2 *Denote again by y_0 the solution of* (1.1) *corresponding to the control 0. If $y_d \leq y_0(T)$, then the unique solution to* (P_α^+) *is given by $\bar{u} = 0$. Indeed, observe that $y_u = z_u + y_0$, where z_u the solution of* (2.14)*. By the weak maximum principle we know that $z_u \geq 0$ in Q. Consequently, for every $u \neq 0$*

$$J(0) = \frac{1}{2}\|y_0(T) - y_d\|_{L^2(\Omega)}^2 < \frac{1}{2}\|z_u(T) - (y_d - y_0(T))\|_{L^2(\Omega)}^2 = J(u).$$

Remark 3.3 *Contrarily to Theorem 2.5, the case of an optimal control with $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} < \alpha$ has not been addressed in Theorem 3.1. In Theorem 2.5 we established that $\bar{y}(T) = y_d$ whenever $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} < \alpha$. This property fails if we restrict the controls to be non-negative. Indeed, we have established in the above remark that $\bar{u} = 0$ if $y_d \leq y_0(T)$.*

4. A related penalty formulation

An alternative formulation to identify the initial condition u is the following

$$(\mathrm{P}_\beta) \qquad \min_{u \in \mathcal{M}(\bar{\Omega})} J(u) = \frac{1}{2}\|y_u(T) - y_d\|_{L^2(\Omega)}^2 + \beta\|u\|_{\mathcal{M}(\bar{\Omega})},$$

where $\beta > 0$ is fixed. In this section, we analyze problem (P_β) and we compare it with (P_α). The reader is referred to [20] for a related problem. Let us introduce some notation. We set $J(u) = F(u) + \beta j(u)$, where

$$F(u) = \frac{1}{2}\|y_u(T) - y_d\|_{L^2(\Omega)}^2 \ \text{ and } \ j(u) = \|u\|_{\mathcal{M}(\bar{\Omega})}.$$

First, we state the well posedness of the problem and the optimality conditions.

Theorem 4.1 *Problem* (P_β) *has a unique solution. Moreover, an element* $\bar{u} \in \mathcal{M}(\bar{\Omega})$ *is the solution of* (P_β) *if and only if*

$$\begin{cases} \displaystyle\int_{\bar{\Omega}} \bar{\varphi}(0)\, d\bar{u} + \beta\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})} = 0, \\ \|\bar{\varphi}(0)\|_{C(\bar{\Omega})} \begin{cases} = \beta & \text{if } \bar{u} \neq 0, \\ \leq \beta & \text{if } \bar{u} = 0. \end{cases} \end{cases} \tag{4.1}$$

Proof. The two addends of the cost functional J define convex functions, j is lower semicontinuous with respect to the weak* topology in $\mathcal{M}(\bar{\Omega})$, and J is coercive in $\mathcal{M}(\bar{\Omega})$. Hence, we can argue similarly to the proof of Theorem 2.11 to deduce the existence and uniqueness of a solution of (P_β). Moreover, given $\bar{u} \in \mathcal{M}(\bar{\Omega})$, using the convexity of F and j, we get with Lemma 2.6 that $\bar{u}$ is a solution of (P_β) if and only if for all $u \in \mathcal{M}(\bar{\Omega})$

$$0 \leq \lim_{\rho \searrow 0} \frac{J(u + \rho(u - \bar{u})) - J(\bar{u})}{\rho} \leq \int_{\bar{\Omega}} \bar{\varphi}(0)\, d(u - \bar{u}) + \beta\|u\|_{\mathcal{M}(\bar{\Omega})} - \beta\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}.$$

This is obviously equivalent to $-\frac{1}{\beta}\bar{\varphi} \in \partial j(\bar{u})$, where $\partial j(\bar{u})$ denotes the subdifferential of j at $\bar{u}$ in the sense of the convex analysis. Finally, it is straightforward to check that $\partial j(\bar{u})$ is equivalent to (4.1). ∎

Corollary 4.2 *Let* $\bar{u} \in \mathcal{M}(\bar{\Omega})$ *be a nonzero measure. Then* $\bar{u}$ *is a solution of* (P_β) *if and only if the following property holds*

$$\begin{cases} \mathrm{Supp}(\bar{u}^+) \subset \{x \in \bar{\Omega} : \bar{\varphi}(x, 0) = -\beta\}, \\ \mathrm{Supp}(\bar{u}^-) \subset \{x \in \bar{\Omega} : \bar{\varphi}(x, 0) = +\beta\}, \end{cases} \tag{4.2}$$

where $\bar{u} = \bar{u}^+ - \bar{u}^-$ *is the Jordan decomposition of* $\bar{u}$.

This is an immediate consequence of (4.1) and [6, Lemma 3.4].

Corollary 4.3 *Let* φ_0 *be the adjoint state corresponding to the zero control, and set* $\beta_0 = \|\varphi_0(0)\|_{C(\bar{\Omega})}$. *Then,* $\bar{u} \neq 0$ *for all* $\beta \in (0, \beta_0)$ *and* $\bar{u} = 0$ *for every* $\beta \geq \beta_0$.

Proof. Using (4.1), we get that $\bar{u} \neq 0$ for all $\beta < \|\varphi_0(0)\|_{C(\bar{\Omega})}$. On the other hand, the control zero satisfies the optimality conditions (4.1) for every $\beta \geq \beta_0$. ∎

Now, we can compare the control problems (P_α) and (P_β).

Theorem 4.4 *Let* $\bar{u} \in \mathcal{M}(\bar{\Omega})$ *be different from 0. Then, we have*

(i) If $\bar{u}$ *is a solution to* (P_β), *then it is also a solution to* (P_α) *for* $\alpha = \|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}$.

(ii) If $\bar{u}$ *is a solution to* (P_α), *then it is also a solution to* (P_β) *for* $\beta = \|\bar{\varphi}(0)\|_{C(\bar{\Omega})}$.

Proof. It suffices to compare the necessary and sufficient optimality conditions (2.13) and (4.1) with α and β selected as indicated in the theorem to conclude the statement. ∎

Remark 4.5 *Though the problems* (P_α) *and* (P_β) *are equivalent in the above sense, the choice of* α *and* β *is a delicate issue. In some applications, there is an a priori knowledge of the total input* $\|\bar{u}\|_{\mathcal{M}(\bar{\Omega})}$*, which can be used for the choice of* α *in the formulation of* (P_α)*. However, the a priori knowledge of the data does not provide a comparable information for the choice of* β*.*

Now, we are going to study the dependence of the solution to (P_β) with respect to β. To this end, we denote by u_β the solution of (P_β) for $\beta > 0$, and by y_β and φ_β the associated state and adjoint state. In the remaining of the section we also make the following non-attainability assumption

$$y_u(T) \neq y_d \quad \forall u \in \mathcal{M}(\bar{\Omega}). \tag{4.3}$$

Lemma 4.6 *Let* $\{\beta_k\}_k \subset (0,\infty)$ *be a sequence converging to* $\beta > 0$*. Then, the following properties hold*

$$u_{\beta_k} \overset{*}{\rightharpoonup} u_\beta \text{ in } \mathcal{M}(\bar{\Omega}) \text{ and } \|u_{\beta_k}\|_{\mathcal{M}(\bar{\Omega})} \to \|u_\beta\|_{\mathcal{M}(\bar{\Omega})}, \tag{4.4}$$

$$\lim_{k\to\infty} \Big(\|y_{\beta_k}(T) - y_\beta(T)\|_{L^2(\Omega)} + \|\varphi_{\beta_k}(0) - \varphi_\beta(0)\|_{C(\bar{\Omega})} \Big) = 0. \tag{4.5}$$

Proof. From the inequalities $J_{\beta_k}(u_{\beta_k}) \leq J_{\beta_k}(0) = \frac{1}{2}\|y_0(T) - y_d\|^2_{L^2(\Omega)}$ we infer the boundedness of $\{u_{\beta_k}\}_k$ in $\mathcal{M}(\bar{\Omega})$. Hence, we can take a subsequence, denoted in the same way, and an element $u \in \mathcal{M}(\bar{\Omega})$ such that $u_{\beta_k} \overset{*}{\rightharpoonup} u$ in $\mathcal{M}(\bar{\Omega})$. From Theorem 2.3 and inequality (2.12) we deduce that $y_{\beta_k}(T) \to y_u(T)$ in $L^2(\Omega)$ and $\varphi_{\beta_k}(0) \to \varphi_u(0)$ in $C(\bar{\Omega})$ when $k \to \infty$.

Using the characterization of a solution given by (4.1) we have for every k

$$\begin{cases} \displaystyle\int_{\bar{\Omega}} \varphi_{\beta_k}(0)\, du_{\beta_k} + \beta_k \|u_{\beta_k}\|_{\mathcal{M}(\bar{\Omega})} = 0, \\ \|\varphi_{\beta_k}(0)\|_{C(\bar{\Omega})} = \beta_k. \end{cases}$$

Using the established convergence properties, we can pass to the limit above and deduce

$$\begin{cases} \displaystyle\int_{\bar{\Omega}} \varphi_u(0)\, du + \beta \|u\|_{\mathcal{M}(\bar{\Omega})} \leq 0, \\ \|\varphi_u(0)\|_{C(\bar{\Omega})} = \beta. \end{cases} \tag{4.6}$$

From the last identity we also get

$$-\int_{\bar{\Omega}} \varphi_u(0)\, du \leq \|\varphi_u(0)\|_{C(\bar{\Omega})} \|u\|_{\mathcal{M}(\bar{\Omega})} = \beta \|u\|_{\mathcal{M}(\bar{\Omega})}.$$

Combining this with the first inequality of (4.6) we obtain

$$\int_{\bar{\Omega}} \varphi_u(0)\, du + \beta \|u\|_{\mathcal{M}(\bar{\Omega})} = 0.$$

This identity and the second one of (4.6) implies with Theorem 4.1 that $u = u_\beta$ is the solution of (P_β). Since every converging subsequence converges to this solution, we have that the whole sequence converges as well. Finally, we also have

$$\lim_{k\to\infty} \|u_{\beta_k}\|_{\mathcal{M}(\bar{\Omega})} = -\lim_{k\to\infty} \frac{1}{\beta_k} \int_{\bar{\Omega}} \varphi_{\beta_k}(0)\, du_{\beta_k} = -\frac{1}{\beta} \int_{\bar{\Omega}} \varphi_\beta(0)\, du_\beta = \|u_\beta\|_{\mathcal{M}(\bar{\Omega})},$$

which completes the proof. □

Let us introduce the functions $h : (0, \beta_0] \longrightarrow [0, \infty)$ and $g : (0, \beta_0] \longrightarrow (0, \infty)$ given by $h(\beta) = \|u_\beta\|_{\mathcal{M}(\bar{\Omega})}$ and $g(\beta) = F(u_\beta)$, where F is the first summand of the cost functional J. We have the following properties for h and g.

Theorem 4.7 *The function h is continuous, strictly monotone decreasing and bijective. Moreover, the function g is continuous and strictly monotone increasing.*

Proof. The continuity of h and g follows from Lemma 4.6. Let us prove that h is strictly monotone decreasing. Let us take $0 < \beta_1 < \beta_2 \leq \beta_0$. First, we observe that $u_{\beta_1} \neq u_{\beta_2}$. Indeed, it follows from (4.1) that

$$\|\varphi_{\beta_1}(0)\|_{C(\bar{\Omega})} = \beta_1 < \beta_2 = \|\varphi_{\beta_2}(0)\|_{C(\bar{\Omega})}.$$

Hence, the adjoint states are different and therefore the controls as well. Let us set

$$J_\beta(u) = \frac{1}{2}\|y_u(T) - y_d\|^2_{L^2(\Omega)} + \beta\|u\|_{\mathcal{M}(\bar{\Omega})}.$$

Then, from the optimality and the uniqueness of u_{β_1} and u_{β_2} we obtain

$$J_{\beta_1}(u_{\beta_1}) < J_{\beta_1}(u_{\beta_2}) \quad \text{and} \quad J_{\beta_2}(u_{\beta_2}) < J_{\beta_2}(u_{\beta_1}).$$

Adding both inequalities and canceling the tracking terms we get

$$(\beta_2 - \beta_1)\|u_{\beta_2}\|_{\mathcal{M}(\bar{\Omega})} < (\beta_2 - \beta_1)\|u_{\beta_1}\|_{\mathcal{M}(\bar{\Omega})},$$

which says that $h(\beta_2) < h(\beta_1)$.

Let us prove that g is strictly increasing. For this purpose we use the optimality of u_{β_1} and the strict monotone decreasing property of h

$$g(\beta_1) + \beta_1 h(\beta_1) = J_{\beta_1}(u_{\beta_1}) < J_{\beta_1}(u_{\beta_2}) = g(\beta_2) + \beta_1 h(\beta_2) < g(\beta_2) + \beta_1 h(\beta_1),$$

which proves that $g(\beta_1) < g(\beta_2)$.

Finally, let us prove that h is bijective. Since $h(\beta_0) = 0$ as proved in Corollary 4.3, due to the strict monotonicity and the continuity of h it is enough to check that $h(\beta) \to \infty$ as $\beta \searrow 0$. Here, we argue by contradiction and assume that there exists a sequence $\{\beta_k\}_k \subset (0, \beta_0]$ with $\beta_k \searrow 0$ and $h(\beta_k) \leq \kappa$ for constant $\kappa < \infty$. Then, by taking a subsequence, that we denote in the same way, we can assume that $u_{\beta_k} \overset{*}{\rightharpoonup} u$ in $\mathcal{M}(\bar{\Omega})$. Then, using again Lemma 4.6 and (4.1) we have

$$\|\varphi_u(0)\|_{C(\bar{\Omega})} = \lim_{k\to\infty} \|\varphi_{\beta_k}(0)\|_{C(\bar{\Omega})} = \lim_{k\to\infty} \beta_k = 0.$$

Hence, using the backward uniqueness property of the adjoint state equation we conclude that $\varphi_u = 0$ in Q and then $y_u(T) - y_d = \varphi_u(T) = 0$, which contradicts our assumption (4.3). □

Remark 4.8 *Combining Theorems 4.4 and 4.7 we get that the inverse of h is given by $h^{-1}(\alpha) = \|\varphi_{u_\alpha}\|_{C(\bar{\Omega})}$, where u_α is the solution of problem (P_α) and φ_{u_α} is the associated adjoint state. As a consequence of this, we also have that u_α depends continuously with respect to α in the sense of* (4.4) *and* (4.5).

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