Entropy inequalities and Bell inequalities for two-qubit systems

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Sufficient conditions for the nonviolation of the Bell-Clauser-Horne-Shimony-Holt inequalities in a mixed state of a two-qubit system are: (1) the linear entropy of the state is not smaller than 0.457; (2) the sum of the conditional linear entropies is not smaller than -0.086; (3) the von Neumann entropy is not smaller than 0.833; and (4) the sum of the conditional von Neumann entropies is not smaller than 0.280.

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I. INTRODUCTION

As is well known, entangled quantum states give rise to most counterintuitive features. For instance, in classical physics, as well as in all other branches of science except quantum mechanics, complete knowledge of a composite system requires knowledge of every one of its parts. Indeed this is a common definition of "complete knowledge." In sharp contrast, in quantum mechanics if we know that two particles are in a state of zero total spin, our knowledge about the spin of the system is complete, the quantum state being pure, but we have no information at all about the individual spin of each particle. If the (lack of) information about a system consisting of two subsystems is formalized by means of the Shannon entropy, S_{12}, and the information about the first (second) subsystem by S_1 (S_2), the above-mentioned characteristic of classical information implies the fulfilment of the entropy inequalities

$$S_{12} \geqslant S_1, S_{12} \geqslant S_2, \tag{1}$$

which mean that the ignorance about the whole cannot be smaller than the ignorance about a part. In the rest of this paper we shall name Eq. (1) "entropy inequalities."

In quantum mechanics several definitions of entropy have been proposed with the property that the inequalities analogous to Eq. (1) are violated in some cases, e.g., in the singlet spin state mentioned above. (For a review of quantum entropies see Vedral [1] and references therein.) The most popular quantum entropy is due to von Neumann, but the most simple one is the so-called linear entropy which, for a system consisting of two subsystems, is defined as

$$S_{12} := \mathrm{Tr}[\rho(1-\rho)] \equiv 1-\mathrm{Tr}(\rho^2),$$

$$S_j := 1-\mathrm{Tr}(\rho_j^2), \tag{2}$$

where ρ is the density matrix of the whole system, and ρ_j is the reduced density matrix of subsystem j ($j=1,2$). An interesting property of the linear entropy is that the violation of the inequality (1) is a necessary condition for entanglement. It holds true in general, not only for two-qubit systems. For the sake of clarity we give the proof, which is very simple. In fact, a quantum state of the system is separable if, and only if, its density matrix may be written in the form

$$\rho = \sum_k w_k \rho_{1k}\rho_{2k}, \quad \omega_k > 0, \quad \sum_k w_k = 1, \tag{3}$$

where ρ_{1k} (ρ_{2k}) are density matrices of the first (second) subsystem. If we put Eq. (3) into Eq. (2) we get, using well-known properties of the density matrices,

$$\begin{aligned} S_{12} &= 1-\sum_k \sum_l w_k w_l \,\mathrm{Tr}_1(\rho_{1k}\rho_{1l})\mathrm{Tr}_2(\rho_{2k}\rho_{2l}) \\ &\geqslant 1-\sum_k \sum_l w_k w_l \,\mathrm{Tr}_1(\rho_{1k}\rho_{1l}) \\ &= 1-\mathrm{Tr}_1\left[\left(\sum w_k \rho_{1k}\right)^2\right] \\ &= S_1, \end{aligned}$$

where $\mathrm{Tr}_1(\mathrm{Tr}_2)$ is the trace in the Hilbert space of the first (second) subsystem, and the inequality derives from $\mathrm{Tr}_2(\rho_{2k}\rho_{2l}) \leqslant 1$. This completes the proof that separability is a sufficient condition for the fulfilment of Eq. (1) for quantum linear entropy. Thus the entropy inequalities give a partial characterization of entanglement, partial because separability, although sufficient, is not necessary for the fulfilment of the inequalities.

Another method for the characterization of nonclassical states of physical systems or, more specifically, to discover whether two distant physical systems are entangled is the use of Bell's inequalities. They have the advantage of connecting quantities which may be measured, at least in principle. As is well known, the violation of a Bell inequality is a sufficient condition for entanglement (nonseparability). The more general theoretical question of fully characterizing quantum states compatible with every Bell inequality is still unsolved (it is solved for pure states, which may violate a Bell inequality if and only if there is entanglement [2]). In this paper we shall consider only the most popular Bell inequalities, namely the CHSH (Clauser-Horne-Shimony-Holt [3]) or the equivalent Clauser-Horne [4] inequalities [for the proof of equivalence see below, after Eq. (6)]. Actually there are other Bell type inequalities, for instance entropic Bell inequalities, which involve classical entropy, hold true in any classical theory, but may be violated by quantum mechanics [5,6].

In summary, it is known that separability implies the fulfilment of both Bell inequalities and quantum entropy inequalities. Therefore a natural question is to ask whether the entropy inequalities are stronger or weaker than the Bell inequalities. That question may also have practical relevance for the applications of quantum information theory [7]. The attempt to get an answer is the main motivation for the present paper. The problem has been already investigated using quantum linear entropy. In fact it has been shown [8] that the inequality (1) for linear entropy is a sufficient condition for all CHSH inequalities. A slightly more powerful result is also true, namely that

$$S_{2/1}+S_{1/2}\geqslant 0, \quad S_{i/j}:=S_{12}-S_j, \tag{4}$$

where $S_{i/j}$ are called conditional entropies, is sufficient [9]. This means that, for quantum linear entropy

$$\text{separability}\Rightarrow\text{entropy inequalities}\Rightarrow\text{Bell inequality}. \tag{5}$$

The specific aim of the present paper is to generalize these results deriving inequalities weaker than Eq. (1), involving quantum (linear and von Neumann) entropy, which are sufficient for the nonviolation of the CHSH or CH inequalities for a two-qubit system in any mixed state.

The CHSH inequality is

$$-2\leqslant\beta\leqslant 2, \quad \beta\equiv\langle a_1a_2\rangle+\langle a_1b_2\rangle+\langle b_1a_2\rangle-\langle b_1b_2\rangle, \tag{6}$$

a_1, b_1 (a_2, b_2) being dichotomic observables, which may take only the values $+1$ or -1, for the first (second) qubit and $\langle x\rangle$ means the average of the observable x over many runs of the same experiment. As is well known the four averages should be measured in different experiments, all of them using the same preparation for the two-qubit system. I point out that any sufficient condition for the CHSH inequality is also valid for the Clauser-Horne inequality [4]

$$p(A_1)+p(A_2)\geqslant p(A_1A_2)+p(A_1B_2)+p(B_1A_2)-p(B_1B_2), \tag{7}$$

where A_j, B_j are observables which may take only the values 1 or 0, and $p(X)$ [or $p(XY)$] is the probability that X (or both X and Y) takes the value 1. In fact, it is enough to put

$$a_j=2A_j-1, \quad b_j=2B_j-1,$$

in Eq. (6) in order to check that $\beta\leqslant 2$ implies Eq. (7).

II. BELL INEQUALITIES AND LINEAR ENTROPY

Theorem 1. In a two-qubit system, a sufficient condition for the fulfilment of all CHSH inequalities is that the linear entropy of the state fulfils $S_{12}\geqslant\sqrt{2}/2-1/4\simeq 0.457$. For any smaller value of S_{12} there are states able to violate the inequalities.

Proof We consider quantum observables (Hermitian traceless 2×2 matrices) $\{a_1,b_1\}$ for the first qubit and $\{a_2,b_2\}$ for the second, all observables having eigenvalues 1 or -1. We define a Bell operator [10], B, by

$$B=a_1\otimes a_2+a_1\otimes b_2+b_1\otimes a_2-b_1\otimes b_2. \tag{8}$$

Hence it is easy to check that (see the Appendix)

$$\text{Tr}\,B=0, \quad \text{Tr}(B^2)=16, \tag{9}$$

and that the inequality (6) is violated if, for some choice of the Bell operator B, $|\beta|>2$, where

$$\beta=\text{Tr}(\rho B), \tag{10}$$

while quantum mechanics just predicts $|\beta|\leqslant 2\sqrt{2}$. [Equation (10) follows from Eq. (6) and the linearity of the trace.]

It is the case that not all values of β and S_{12} are compatible. In fact, the inequality

$$\text{Tr}\left(\rho-\frac{1}{4}I+\eta B\right)^2\geqslant 0, \quad \eta\in R,$$

where I is the 4×4 unit matrix, holds true for all η, which implies

$$\beta^2+16S_{12}\leqslant 12, \tag{11}$$

where Eq. (9) has been used. This means that there are no ρ and B such that S_{12}, Eq. (2), and β, Eq. (10), violate the inequality (11). However, this inequality provides just a necessary condition. In order to fully define a region of compatibility in the $\{\beta,S_{12}\}$ plane we need a condition which, together with the obvious one $S_{12}\geqslant 0$, is also sufficient. In order to get that condition we must search for the Bell operator, B, and the density matrix, ρ (Hermitean, positive, and having unit trace) that give a maximum of the linear entropy constrained by Eq. (10) with fixed β. To achieve the goal we start fixing B [see Eq. (8)] whose eigenvalues we shall label ξ_1,ξ_2,ξ_3,ξ_4, written in decreasing order. These eigenvalues fulfil [10] (see the Appendix)

$$\xi_3=-\xi_2, \quad \xi_4=-\xi_1, \xi_1^2+\xi_2^2=8, \tag{12}$$

so that the first one, $\xi_1\in[2,2\sqrt{2}]$, determines all of them. Now we will solve the said variational problem with ρ written in a basis of the eigenvectors of B, that is,

$$\sum_{j=1}^{4} r_{jj}=1, \quad \sum_{j=1}^{4}\xi_j r_{jj}=0,$$
$$S_{12}=1-\sum_{j=1}^{4}\sum_{k=1}^{4} r_{jk}r_{kj}=1-\sum_{j=1}^{4}\sum_{k=1}^{4}|r_{jk}|^2=\max, \tag{13}$$

where r_{jk} are the components of the matrix ρ in that basis. (The last equality follows from the Hermitean character of ρ.) It is easy to see that the maximum of S_{12}, for fixed β, happens when all nondiagonal elements are zero and, consequently, our problem is reduced to finding the diagonal elements, which I shall label $\{r_j\}$ from now on.

In the following we shall assume $\beta\geqslant 0$, the case $\beta<0$ being similar. Thus it is possible to solve the variational problem either searching for the maximum S_{12} compatible with a given β, or the maximum β compatible with a given S_{12}, and the second method will be used now. It may be realized that, given the Bell operator B and a set of four non-negative numbers whose sum is unity, we may define as many as $4!=24$ different density matrices having these numbers as diagonal elements. The linear entropy, S_{12}, is the same for all these density matrices, but the value of β is different, the choice giving the maximum β being $r_1 \geqslant r_2 \geqslant r_3 \geqslant r_4$, that is the diagonal elements of ρ decreasing with the eigenvalues of B. For this choice we get

$$\beta=(r_1-r_4)\xi_1+(r_2-r_3)\xi_2=(r_1-r_4)\xi_1+(r_2-r_3)\sqrt{8-\xi_1^2}, \tag{14}$$

where we have taken into account Eq. (12). Now we choose B, i.e., ξ_1, in order to maximize β and we find

$$\xi_1=\frac{2\sqrt{2}(r_1-r_4)}{\sqrt{(r_1-r_4)^2+(r_2-r_3)^2}}$$
$$\Rightarrow \beta^2=8(1-S_{12})-16(r_1r_4+r_2r_3). \tag{15}$$

After that we shall search for the set $\{r_j\}$ of non-negative numbers, adding to one, which make β^2 Eq. (15) a maximum with $S_{12}=1-\Sigma r_j^2$ fixed. The solution, written in terms of β, is

$$\begin{aligned} &r_1=\frac{1}{4}+\frac{\sqrt{2}}{8}\beta, \quad r_2=r_3=\frac{1}{4}, \\ &r_4=\frac{1}{4}-\frac{\sqrt{2}}{8}\beta \quad \text{if } \beta \leqslant \sqrt{2}, \\ &r_1=\frac{\sqrt{2}}{4}\beta, \quad r_2=r_3=\frac{1}{2}-\frac{\sqrt{2}}{8}\beta, \\ &r_4=0 \quad \text{if } \sqrt{2}\leqslant \beta \leqslant 2\sqrt{2}. \end{aligned} \tag{16}$$

I point out that, in both cases, $\xi_1=2\sqrt{2}$, which I shall express saying that the Bell operator is "maximal." This leads to [compare with Eq. (11)]

$$\begin{aligned} &S_{12}^{\max}=\frac{3}{4}-\frac{1}{16}\beta^2 \quad \text{if } |\beta|\leqslant \sqrt{2}, \\ &S_{12}^{\max}=\frac{1}{2}+\frac{\sqrt{2}}{4}|\beta|-\frac{3}{16}\beta^2 \quad \text{if } \sqrt{2}\leqslant |\beta| \leqslant 2\sqrt{2}, \end{aligned} \tag{17}$$

where we have included the results for negative β. The state (16) saturates the bound so that Eq. (17), plus $S_{12}\geqslant 0$, fully define the region of compatibility in the $\{\beta, S_{12}\}$ plane. They also imply that $\beta \leqslant 2$ whenever $S_{12}\geqslant \sqrt{2}/2-1/4$, which proves the theorem.

Theorem 2. In a two-qubit system, a sufficient condition for the fulfilment of all CHSH inequalities is that the sum of the conditional linear entropies of the state fulfils

$$S_{2/1}+S_{1/2}\geqslant \sqrt{2}-\frac{3}{2}. \tag{18}$$

For any smaller value, there are states violating the inequalities.

Proof. Using the Bell state basis it is not difficult to show [10] that the reduced density matrices corresponding to states (16) are

$$\rho_j=\frac{1}{2}I_j, \tag{19}$$

where I_j is the unit 2×2 matrix associated to the qubit j. The sum of conditional entropies for this state fulfils

$$\begin{aligned} S_{2/1}+S_{1/2}&=\frac{1}{2}-\frac{1}{8}\beta^2 \quad \text{if } |\beta|\leqslant \sqrt{2}, \\ &=\frac{\sqrt{2}}{2}|\beta|-\frac{3}{8}\beta^2 \quad \text{if } \sqrt{2}\leqslant |\beta|\leqslant 2\sqrt{2}. \end{aligned}$$

Now we must show that, for a given β, this value is a maximum in order to ensure that Eq. (18) implies $|\beta|\leqslant 2$. The condition for a maximum is that $\delta(S_{2/1}+S_{1/2})\leqslant 0$ for an arbitrary variation, $\delta\rho$, of the state ρ, Eq. (16), such that

$$\mathrm{Tr}(\delta\rho)=0, \quad \mathrm{Tr}(\delta\rho B)=0. \tag{20}$$

The first (second) equality guarantees the unit trace of the density matrix (that the value of β does not change). We have

$$\delta(S_{2/1}+S_{1/2})=2\,\delta S_{12}-\delta S_1-\delta S_2, \tag{21}$$

with

$$\begin{aligned} \delta S_{12}&=\mathrm{Tr}(\rho^2)-\mathrm{Tr}[(\rho+\delta\rho)^2] \\ &=-2\,\mathrm{Tr}(\rho\,\delta\rho)-\mathrm{Tr}(\delta\rho^2) \\ &=-\mathrm{Tr}(\delta\rho^2), \end{aligned}$$

$$\delta S_j=-2\,\mathrm{Tr}(\rho_j\delta\rho_j)-\mathrm{Tr}(\delta\rho_j^2)=-\mathrm{Tr}(\delta\rho_j^2).$$

In the former equation we have removed the first order term because S_{12} is stationary when β is fixed for the state ρ, Eq. (16), in the latter equation due to Eq. (19) and the first Eq. (20). Thus we get

$$\delta(S_{2/1}+S_{1/2})=\mathrm{Tr}(\delta\rho_1^2)+\mathrm{Tr}(\delta\rho_2^2)-2\,\mathrm{Tr}(\delta\rho^2). \tag{22}$$

A useful bound for the sum of terms involving $\delta\rho_1$ and $\delta\rho_2$ may be found from the obvious inequality

$$\mathrm{Tr}(\delta\rho-\delta\rho_1\otimes I_2-I_1\otimes\delta\rho_2)^2\geqslant 0,$$

where I_j is the unit 2×2 matrix for qubit j. After some algebra, taking the first Eq. (20) into account, this becomes

$$\mathrm{Tr}[\delta\rho_1^2]+\mathrm{Tr}[\delta\rho_2^2]\leqslant \mathrm{Tr}[\delta\rho^2], \tag{23}$$

which, put in Eq. (22), shows that the sum of conditional linear entropies of the state (16) is indeed a maximum for every β, thus proving the theorem.

III. BELL INEQUALITIES AND VON NEUMANN ENTROPY

In the following we shall derive similar theorems using, instead of the linear entropy, the von Neumann entropy

$$S_{12} := -\mathrm{Tr}(\rho \ln \rho), \quad S_j := -\mathrm{Tr}(\rho_j \ln \rho_j). \tag{24}$$

We begin proving that an inequality like Eq. (4), in terms of the von Neumann entropy, is not a sufficient condition for the CHSH inequalities. We consider the following family of states:

$$\rho = Z(\lambda)^{-1} \exp(\lambda B), \quad Z(\lambda) := \mathrm{Tr}\exp(\lambda B), \tag{25}$$

with B an arbitrary Bell operator. It is straightforward to compute β and S_{12} from the function $Z(\lambda)$ and we get

$$\beta = \frac{d \ln Z}{d\lambda},$$

$$S_{12} = -\frac{\mathrm{Tr}\{\exp(\lambda B)[\lambda B - \ln \mathrm{Tr}\exp(\lambda B)]\}}{\mathrm{Tr}\exp(\lambda B)} = \ln Z - \lambda \beta. \tag{26}$$

Now we consider more specifically the state

$$\rho_0 = Z_0(\lambda)^{-1} \exp(\lambda B_0), \quad Z_0(\lambda) := \mathrm{Tr}\exp(\lambda B_0), \tag{27}$$

B_0 being a maximal Bell operator (that is having $2\sqrt{2}$ as an eigenvalue). We obtain, writing $\exp(\lambda B_0)$ in the basis of the Bell states,

$$Z_0(\lambda) = \exp(2\sqrt{2}\lambda) + \exp(-2\sqrt{2}\lambda) + 2 = 4\cosh^2(\sqrt{2}\lambda), \tag{28}$$

whence

$$\beta = 2\sqrt{2}\tanh x, \quad S_{12} = 2\ln 2 + 2\ln\cosh x - 2x\tanh x,$$

$$x \equiv \sqrt{2}\lambda.$$

From these equations we may get a relation between β and S_{12} for the family of states (27), namely

$$S_{12} = 5\ln 2 - \frac{\sqrt{2}}{4}[(2\sqrt{2}+\beta)\ln(2\sqrt{2}+\beta) + (2\sqrt{2}-\beta)\ln(2\sqrt{2}-\beta)]. \tag{29}$$

Using the Bell state basis it is easy to prove that the reduced density matrices of Eq. (27) are a multiple of the identity, that is

$$\rho_j = \frac{1}{2} I_j \Rightarrow S_1 = S_2 = \ln 2. \tag{30}$$

From Eqs. (29) and (30) we derive that $S_{2/1} + S_{1/2} = 0$ corresponds to $\beta \simeq \pm 2.206$, so that an inequality like Eq. (4), but using von Neumann entropy, is not a sufficient condition for the CHSH inequalities. On the other hand $\beta = \pm 2$ corresponds to $S_{12} \simeq 0.833$ and $S_{2/1} + S_{1/2} \simeq 0.280$, which implies that, if there are sufficient conditions for the CHSH inequalities of the form $S_{12} \geqslant K_1$ and $S_{2/1} + S_{1/2} \geqslant K_2$, then $K_1 \geqslant 0.833$ and $K_2 \geqslant 0.280$.

Theorem 3. If a two-qubit system is in a state with density matrix ρ, the inequality

$$S_{12} \geqslant 3\ln 2 - \sqrt{2}\ln(\sqrt{2}+1) \simeq 0.833,$$

where S_{12} is the von Neumann entropy, is a sufficient condition for the fulfilment of all CHSH inequalities. For any smaller value of S_{12} there are states violating the inequalities.

Proof. We shall prove the theorem in three steps: (1) Fixing a number $\beta \in [-2\sqrt{2}, 2\sqrt{2}]$ and (the eigenvalues of) a Bell operator B, we shall search for a density matrix, ρ, making S_{12} a maximum compatible with $\mathrm{Tr}(\rho B) = \beta$. (2) Now we fix only β and search for (the eigenvalues of) the Bell operator providing the greatest S_{12}. Let us label $K_1(\beta)$ that value of S_{12}. (3) We shall show that $K_1(\beta) = K_1(-\beta)$ and that $K_1(\beta)$ decreases when $|\beta|$ increases. After that, it becomes clear that $K_1(2)$ gives the desired sufficient condition for the CHSH inequalities. In fact, any ρ and any B leading to $S_{12} \geqslant K_1(2)$ would give $|\beta| \leqslant 2$ so that the CHSH inequality will be satisfied.

In the first step we begin fixing the Bell operator B and the number β and we search for the density matrix ρ making the von Neumann entropy, S_{12}, stationary with the constraints $\mathrm{Tr}\,\rho = 1$, $\mathrm{Tr}(\rho B) = \beta$. This is a standard variational problem which, introducing Lagrange multipliers λ and χ, may be stated

$$\delta\{\mathrm{Tr}(\rho \ln \rho) - \lambda\, \mathrm{Tr}(\rho B) - \chi\, \mathrm{Tr}\,\rho\} = \mathrm{Tr}\{\delta\rho[\ln \rho - \lambda B - \chi + 1]\} = 0, \tag{31}$$

whose solution is of the form of Eq. (25), B being the given Bell operator and λ fixed by the first Eq. (26). Still it is necessary to prove that the solution found for the variational problem actually gives a maximum of S_{12} (rather than, e.g., a minimum). To do that we use the density operator $\rho' = \rho + \delta\rho$ where ρ is given by Eq. (25) with $\sqrt{2}\lambda = 0.883$ and $\delta\rho$ fulfils Eq. (20). Hence we obtain

$$\delta S_{12} = -\mathrm{Tr}[(\rho + \delta\rho)\ln(\rho + \delta\rho)] + \mathrm{Tr}(\rho \ln \rho).$$

We may expand $\ln(\rho + \delta\rho)$ in powers of $\delta\rho$ up to second order. The expansion is well defined because all integer powers or ρ, Eq. (25), either with positive or negative exponent, are well defined. Also, to second order there is no problem with the possible noncommutativity of the operators ρ and $\delta\rho$. Taking Eq. (20) into account we obtain no term of first order in $\delta\rho$, as it should, S_{12} being stationary. The second order term is

$$\delta^{(2)} S_{12} = -\frac{1}{2}\mathrm{Tr}[\rho^{-1}\delta\rho^2] \leqslant 0, \tag{32}$$

which proves that a density operator of the form of Eq. (25) makes S_{12} a maximum.

In the second step we shall prove that Eq. (25) gives the maximum value of S_{12} compatible with the fixed β, if we choose the Bell operator to be maximal. In fact, from the eigenvalues of any Bell operator we may get the function $Z(\lambda)$ [see Eqs. (25) and (12)] in terms of the eigenvalues

$$Z(\lambda)=\exp(\lambda\zeta_1)+\exp(-\lambda\zeta_1)+\exp(\lambda\zeta_2)+\exp(-\lambda\zeta_2)$$
$$=4\cosh\mu\cosh\nu,$$

with $\mu=1/2(\zeta_1+\zeta_2)$, $\nu=1/2(\zeta_1-\zeta_2)$. Hence it is straightforward to obtain β and S_{12} using Eq. (26), but we omit the results. This leads to the variational problem of finding ζ_1, ζ_2, and λ which make S_{12} a maximum for fixed β [with Eq. (12) fulfilled]. The solution is $\zeta_1=2\sqrt{2}$, $\zeta_2=0$, $\sqrt{2}\lambda=0.881$, which corresponds to the density operator of Eq. (27).

The third step, that is proving that $K_1(\beta)=K_1(-\beta)$ and that S_{12} increases when $|\beta|$ decreases, is trivial taking into account Eq. (29).

Finally, the state given by Eq. (27) saturates the bound of the theorem, which completes the proof.

The function $S_{12}=S_{12}(\beta)$, given by Eq. (29), provides the upper limit, and $S_{12}=0$ the lower limit, of the region of compatibility in the $\{\beta,S_{12}\}$ plane, this time in terms of von Neumann's entropy [compare with Eq. (17), defining a similar region in the case of linear entropy].

Theorem 4. If a two-qubit system is in a state with density matrix ρ, the inequality

$$S_{2/1}+S_{1/2}\geqslant 4\ln 2-2\sqrt{2}\ln(\sqrt{2}+1)\simeq 0.280$$

in terms of von Neumann entropy, is a sufficient condition for the fulfilment of all CHSH inequalities. For any smaller value, there are states violating the inequalities.

Proof. The previous results suggest that Eq. (27) provides the density matrix giving the maximum value of $S_{2/1}+S_{1/2}$ for a given β. Here we show that this is the case by just proving that $\delta(S_{2/1}+S_{1/2})$ is negative up to second order in $\delta\rho$ for that state. We get an equality like Eq. (21), but in terms of von Neumann's entropy, with δS_{12} given to second order by Eq. (32) and

$$\delta S_j=-\mathrm{Tr}[(\rho_j+\delta\rho_j)\ln(\rho_j+\delta\rho_j)]+\mathrm{Tr}(\rho_j\ln\rho_j)$$
$$=-\mathrm{Tr}[\delta\rho_j^2]+O(\delta\rho_j^3), \qquad (33)$$

where

$$\rho_1=\mathrm{Tr}_2\,\rho, \quad \delta\rho_1=\mathrm{Tr}_2(\delta\rho),$$

and is similar for ρ_2 and $\delta\rho_2$. In the second Eq. (33) we have taken into account Eqs. (20) and (30), the latter implying $\rho_j^{-1}=2I_j$ and $\ln\rho_j=-\ln 2I_j$. Hence using Eqs. (32) and (33) we get

$$\delta(S_{2/1}+S_{1/2})=\mathrm{Tr}[\delta\rho_1^2]+\mathrm{Tr}[\delta\rho_2^2]-\mathrm{Tr}[\rho^{-1}\delta\rho^2]+O(\delta\rho^3). \qquad (34)$$

Now we use the inequality (23) giving, to second order in $\delta\rho$,

$$\delta^{(2)}(S_{2/1}+S_{1/2})\leqslant\mathrm{Tr}[\delta\rho^2]-\mathrm{Tr}[\rho^{-1}\delta\rho^2].$$

The right-hand side may be calculated in a basis of Bell states and we obtain

$$\delta^{(2}(S_{2/1}+S_{1/2})\leqslant\sum_{k=1}^{4}\langle\chi_k|\delta\rho^2[1-\rho^{-1}]|\chi_k\rangle$$
$$=\sum_{k=1}^{4}\langle\chi_k|\delta\rho^2|\chi_k\rangle[1-Z_0(\lambda)\exp(-\lambda\xi_k)],$$

where we have labeled $|\chi_k\rangle$ the Bell states and ξ_k the corresponding eigenvalues, $Z_0(\lambda)$ being given by Eq. (28). We see that the right-hand side is negative if the following inequality holds for every k:

$$Z_0(\lambda)\exp(-\lambda\xi_k)>1,$$

and this is true if the inequality is fulfilled, for any β, for the largest eigenvalue $\xi_1=2\sqrt{2}$. A simple calculation proves that this is indeed the case, which shows that $S_{2/1}+S_{1/2}$ presents a maximum, thus proving the theorem.

It is interesting that, according to this theorem, the second implication (5) does not hold true in the case of the von Neumann entropy.

IV. ENTROPY AND LOCAL HIDDEN VARIABLES

I shall finish with a comment about how specific for the CHSH inequalities are the results here presented, that is whether they may be extended to other Bell inequalities [i.e., inequalities characteristic of local hidden variables (LHV) models]. The question, stated more generally, is whether the entropy inequalities considered in the previous theorems are sufficient for the existence of LHV models. The answer seems to be negative, although a more detailed study is necessary. In fact, it is known that the CHSH inequalities are necessary conditions for the existence of LHV theories, but they are not sufficient. It has been proven that, having chosen four observables a_1, a_2, b_1, b_2 as in Eq. (6), the fulfilment of the four CHSH inequalities obtained by changing the place of the minus sign is a sufficient condition for the existence of a LHV model involving these four observables [11], but there are counterexamples proving that the condition is not sufficient for more than four [12].

APPENDIX

For the sake of clarity *I* present here a short rederivation of some properties of the Bell operator (see the paper by Braunstein *et al.* [10]).

The square of the Bell operator (8) may be written, taking into account that the square of any of the operators a_1, a_2, b_1, or b_2 is the unit operator in the corresponding Hilbert space,

$$B^2=4I_1\otimes I_2-[a_1,b_1]\otimes[a_2,b_2].$$

Now we remember that any operator, a, in a two-dimensional space having eigenvalues ±1 may be written in the form

$$a=\mathbf{a}\cdot\boldsymbol{\sigma},$$

where **a** is a unit vector in ordinary, three-dimensional space and $\boldsymbol{\sigma}$ the vector of the Pauli matrices. Thus we may write

$$B^2 = 4I_1 \otimes I_2 + 4(\mathbf{a}_1 \times \mathbf{b}_1) \cdot \boldsymbol{\sigma}_1 \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \cdot \boldsymbol{\sigma}_2$$
$$\equiv 4I_1 \otimes I_2 + 4|\mathbf{a}_1 \times \mathbf{b}_1||\mathbf{a}_2 \times \mathbf{b}_2| \sigma_{1z} \otimes \sigma_{2z},$$

where the last expression corresponds to taking reference frames with the z axis in the direction $\mathbf{a}_1 \times \mathbf{b}_1$ ($\mathbf{a}_2 \times \mathbf{b}_2$) for the first (second) particle. From the latter representation it is easy to see that B^2 possesses eigenvectors which may be represented, with an obvious notation,

$$|\uparrow\uparrow\rangle \text{ and } |\downarrow\downarrow\rangle \text{ with the same eigenvalue } (\xi_1)^2 = (\xi_4)^2 = 4 + 4|\mathbf{a}_1 \times \mathbf{b}_1||\mathbf{a}_2 \times \mathbf{b}_2|,$$

$$|\uparrow\downarrow\rangle \text{ and } |\downarrow\uparrow\rangle \text{ with the same eigenvalue } (\xi_2)^2 = (\xi_3)^2 = 4 - 4|\mathbf{a}_1 \times \mathbf{b}_1||\mathbf{a}_2 \times \mathbf{b}_2|,$$

Hence Eqs. (12) and (9) follow without difficulty.

[1] V. Vedral, Rev. Mod. Phys. **74**, 197 (2002).

[2] F. Capasso, D. Fortunato, and F. Selleri, Int. J. Theor. Phys. **7**, 319 (1973); D. Fortunato and F. Selleri, *ibid.* **15**, 333 (1976); N. Gisin, Phys. Lett. A **154**, 201 (1991).

[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[4] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).

[5] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **61**, 662 (1988); Ann. Phys. (N.Y.) **202**, 22 (1990).

[6] N. J. Cerf and C. Adami, Phys. Rev. A **55**, 3371 (1997); Phys. Rev. Lett. **79**, 5194 (1997).

[7] M. A. Nielsen and I. L. Chang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[8] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996); R. Horodecki and M. Horodecki, Phys. Rev. A **54**, 1838 (1996).

[9] E. Santos and M. Ferrero, Phys. Rev. A **62**, 024101 (2000).

[10] S. L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).

[11] A. Fine, Phys. Rev. Lett. **48**, 291 (1982).

[12] A. Garg and N. D. Mermin, Phys. Rev. Lett. **49**, 1220 (1982).