Dispersion cancellation and quantum eraser experiments analyzed in the Wigner function formalism

Alberto Casado,[1,*] Agustin Fernandez-Rueda,[1,*] Trevor W. Marshall,[2] Ramon Risco-Delgado,[1,*] and Emilio Santos[1]
[1]*Departamento de Fisica Moderna, Universidad de Cantabria, 39005 Santander, Spain*
[2]*Department of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom*

(Received 20 March 1997)

We extend the Wigner function formalism for parametric down-conversion experiments presented in a previous paper [Phys. Rev. A **55** 3879 (1997)] to experiments involving propagation through a dispersive medium [Steinberg *et al.*, Phys. Rev. A **45**, 6659 (1992)], and polarization [Kwiat *et al.*, Phys. Rev. A **45**, 7729 (1992)]. [S1050-2947(97)02009-X]

PACS number(s): 42.50.Ar, 03.65.Sq, 42.50.Lc

I. INTRODUCTION

Parametric down-conversion experiments in the Wigner representation have been recently studied [1,2]. In this section we present a brief summary of the most importants results in order to apply them to new experiments. In the Wigner representation the electric field corresponding to a narrow light beam may be written (without taking the polarization into account)

$$E^{(+)}(\mathbf{r},t)=i\sum_{[\mathbf{k}]}\left(\frac{\hbar\omega_{\mathbf{k}}}{L^3}\right)^{1/2}\alpha_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{1}$$

where we assume that the light beam contains frequencies $\omega_{\mathbf{k}}$ in an interval $(\omega_{\min},\omega_{\max})$, and wave vectors with a limited transverse component $|\mathbf{k}^{\mathrm{tr}}|\ll\omega_{\min}/c$.

It is convenient to work with slowly varying amplitudes defined by

$$F^{(+)}(\mathbf{r},t)=e^{i\omega_a t}E^{(+)}(\mathbf{r},t), \tag{2}$$

ω_a being an average frequency more or less midway between $\omega_{\min}$ and $\omega_{\max}$. If the complex amplitude $\alpha_{\mathbf{k}}(t)$ has a free evolution of the form

$$\alpha_{\mathbf{k}}(t)=\alpha_{\mathbf{k}}(0)e^{-i\omega_{\mathbf{k}}t}, \tag{3}$$

then the amplitude $F^{(+)}(\mathbf{r}_B,t)$ in terms of the amplitude $F^{(+)}(\mathbf{r}_A,t)$ at another point of the light beam is

$$F^{(+)}(\mathbf{r}_B,t)=F^{(+)}\left(\mathbf{r}_A,t-\frac{r_{AB}}{c}\right)e^{i\omega_a r_{AB}/c}, \tag{4}$$

where $\mathbf{r}_{AB}=\mathbf{r}_B-\mathbf{r}_A$, $r_{AB}=|\mathbf{r}_{AB}|$, and it is assumed that

$$r_{AB}\ll\frac{c}{\omega_{\max}-\omega_{\min}}. \tag{5}$$

*Permanent address: Escuela Superior de Ingenieros, Departamento de Fisica Aplicada, Universidad de Sevilla, Sevilla, Spain. Electronic address: acasado@cica.es

On the other hand, to second order in perturbation theory, the signal beam leaving the crystal can be expressed as

$$F_s^{(+)}(\mathbf{r},t)=F_{0s}^{(+)}(\mathbf{r},t)+gVGF_{0i}^{(-)}(\mathbf{r},t)+g^2|V|^2JF_{0s}^{(+)}(\mathbf{r},t), \tag{6}$$

where we have represented the pumping laser beam by a plane wave of amplitude V and g is a dimensionless coupling constant. A similar expression holds for the idler beam by exchanging the indices "s" and "i." $F_{0s}^{(+)}$ is the vacuum field entering the crystal in the direction of the signal beam, and $F_{0i}^{(+)}$ is the vacuum field in the direction of the idler beam. ω_s and ω_i are the average frequencies of the beams with wave vectors $\mathbf{k}_s$, $\mathbf{k}_i$, respectively, fulfilling the matching conditions $\omega_s+\omega_i=\omega_0$, $\mathbf{k}_s+\mathbf{k}_i\approx\mathbf{k}_0$, with ω_0 and $\mathbf{k}_0$ being the frequency and wave vector of the pumping laser beam. G and J are linear operators expressing the interaction, within the crystal, of the laser with the zero-point field. The correlation properties of these fields are as follows:

(a) Autocorrelations. Taking the signal field at a point $\mathbf{r}$ and times t and t', we have

$$\begin{aligned}\langle F_s^{(+)}(\mathbf{r},t)F_s^{(-)}(\mathbf{r},t')\rangle-\langle F_{0s}^{(+)}(\mathbf{r},t)F_{0s}^{(-)}(\mathbf{r},t')\rangle\\ =2g^2|V|^2\langle GF_{0i}^{(-)}(\mathbf{r},t)G^*F_{0i}^{(+)}(\mathbf{r},t')\rangle\\ \equiv g^2|V|^2\mu_s(t'-t),\\ \langle F_s^{(+)}(\mathbf{r},t)F_s^{(+)}(\mathbf{r},t')\rangle=0.\end{aligned} \tag{7}$$

Here $\langle\rangle$ means an average using the Wigner function in the vacuum state as probability density. $\mu_s(t-t')$ is a correlation function that goes to zero when $|t'-t|$ is greater than the correlation time of the signal τ_s. Similar expressions hold for the idler field by exchanging the indices "s" and "i."

(b) Cross correlations. Taking the signal and idler fields at the center of the crystal $\mathbf{r}=\mathbf{r}'=\mathbf{0}$ and times t and t', we have

$$\begin{aligned}\langle F_s^{(+)}(\mathbf{0},t)F_i^{(+)}(\mathbf{0},t')\rangle=gV\nu(t'-t).\\ \langle F_s^{(+)}(\mathbf{0},t)F_i^{(-)}(\mathbf{0},t')\rangle=\langle F_s^{(-)}(\mathbf{0},t)F_i^{(+)}(\mathbf{0},t')\rangle=0.\end{aligned} \tag{8}$$

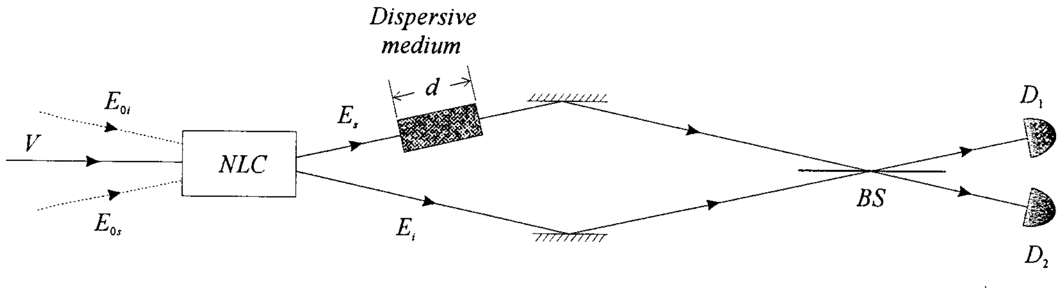

FIG. 1. Experiment of dispersion cancellation.

Here $\nu(t'-t)$ is a function that vanishes when $|t'-t|$ is greater than the coherence time between signal and idler. From Eq. (8) it is possible to derive all cross correlations at different points $\mathbf{r}\neq\mathbf{r}'$ by using Eq. (4).

Finally, the quantum theory of detection in the Wigner representation gives us the following results for single and joint detection probabilities:

(a) Single probability. The following result is a general expression for calculating single probabilities in the Wigner representation:

$$P_1(\mathbf{r}_1,t)\propto\langle I(\mathbf{r}_1,t)-I_0(\mathbf{r}_1)\rangle, \tag{9}$$

where $I(\mathbf{r}_1,t)=|E^{(+)}(\mathbf{r}_1,t)|^2$, and $I_0(\mathbf{r}_1)$ is the intensity of the vacuum field at the position of the detector.

(b) Joint probability. It can be proved that in parametric down-conversion experiments

$$P_{12}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)\propto\langle\{I(\mathbf{r}_1,t)-I_0(\mathbf{r}_1)\}\times\{I(\mathbf{r}_2,t+\tau)-I_0(\mathbf{r}_2)\}\rangle. \tag{10}$$

By taking into account that the Wigner field amplitudes are Gaussian, and neglecting fourth-order terms in g, we have

$$P_{12}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)\propto|\langle E^{(+)}(\mathbf{r}_1,t)E^{(+)}(\mathbf{r}_2,t+\tau)\rangle|^2. \tag{11}$$

Finally we point out that these expressions for the detection probabilities remain valid when we use the amplitudes $F^{(+)}$ and $F^{(-)}$ in place of $E^{(+)}$ and $E^{(-)}$.

II. DISPERSION CANCELLATION

In this section we present a study of dispersion cancellation in a fourth-order interferometer [4], using the Wigner formalism. This kind of process has been considered as an example of nonlocality in quantum mechanics, due to the fact that there is no dispersion cancellation in "classical" optics [5]. However, the Wigner formalism suggests a fully local interpretation of this and many other phenomena, in the sense that a description in terms of fields propagating in space time is possible without ever surpassing the velocity of light. This possibility rests upon the fact that, in parametric down-conversion, the Wigner function is positive definite [1,2] and it may be interpreted as a probability distribution. Consequently, the Wigner representation of the experiments offers a counterexample to the claim that no local realist model may account for the said experiments.

The experimental setup is shown in Fig. 1. This is similar to the Hong-Ou-Mandel interferometer [3], but with a dispersive medium inserted in one arm. In order to calculate the joint probability we are going to express the fields at the detectors D_1 and D_2, by propagating the slowly varying functions $F^{(+)}$ from the crystal to the beam splitter BS (the phase factor from BS to the detectors is dropped because it is not important). The main difference with the other experiments that we have explained in the Wigner formalism lies in the fact that we have to propagate the field $F_s^{(+)}$ through the dispersive medium. Let us start by writing the fields at the detectors D_1 and D_2 at different times t and $t+\tau$. Assuming, for simplicity, that $T=R=1/\sqrt{2}$ for the beam splitter, we have

$$F^{(+)}(\mathbf{r}_1,t)=\frac{1}{\sqrt{2}}[F_i^{(+)}(\mathbf{r}_1,t)+iF_s^{(+)}(\mathbf{r}_1,t)],$$

$$F^{(+)}(\mathbf{r}_2,t+\tau)=\frac{1}{\sqrt{2}}[F_s^{(+)}(\mathbf{r}_2,t+\tau)+iF_i^{(+)}(\mathbf{r}_2,t+\tau)], \tag{12}$$

where

$$F_i^{(+)}(\mathbf{r}_1,t)=F_i^{(+)}\left(\mathbf{0},t-\frac{\delta l}{c}\right)e^{i\omega_i\delta l/c}, \tag{13}$$

and similarly for $F_i^{(+)}(\mathbf{r}_2,t+\tau)$. δl is the optical path length in the lower arm of the interferometer.

In order to obtain $F_s^{(+)}(\mathbf{r}_1,t)$ we shall use the expressions (1) and (2). We have

$$F_s^{(+)}(\mathbf{r}_1,t)=E_s^{(+)}(\mathbf{r}_1,t)e^{i\omega_s t} = K\int_{-\infty}^{+\infty}d\omega_{\mathbf{k}_s}\alpha(\omega_{\mathbf{k}_s})e^{i\mathbf{k}(\omega_{\mathbf{k}_s})\cdot\mathbf{r}_1}e^{-i(\omega_{\mathbf{k}_s}-\omega_s)t}, \tag{14}$$

where we have replaced the sum by an integral and extended the range of the integral to $\pm\infty$ because the function $\alpha(\omega_{\mathbf{k}_s})$ is peaked at $\omega_{\mathbf{k}_s}\approx\omega_s$, and we have introduced a constant K, which includes some other constants that are irrelevant for our purposes. We may expand the wave number $k(\omega_{\mathbf{k}_s})$ to second order in a Taylor series about ω_s as follows:

$$k(\omega_{\mathbf{k}_s})=k_0+\alpha(\omega_{\mathbf{k}_s}-\omega_s)+\beta(\omega_{\mathbf{k}_s}-\omega_s)^2, \tag{15}$$

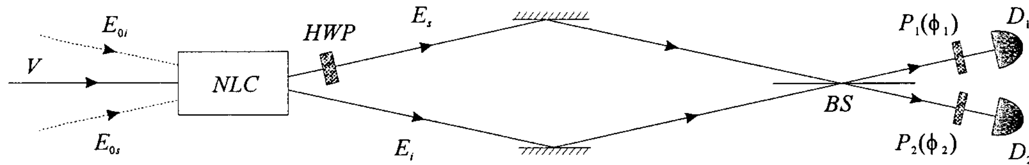

FIG. 2. Experiment of quantum eraser.

with α and β being constants appropriate for the dispersive medium.

In order to express $F_s^{(+)}$ at $\mathbf{r}=\mathbf{r}_1$ in terms of $F_s^{(+)}$ at $\mathbf{r}=\mathbf{0}$, we take into account that $\alpha(\omega_{\mathbf{k}_s})$ is the inverse Fourier transform of $E_s^{(+)}(\mathbf{0},t)$, that is,

$$\alpha(\omega_{\mathbf{k}_s})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt'E_s^{(+)}(\mathbf{0},t')e^{i\omega_{\mathbf{k}_s}t'}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt'F_s^{(+)}(\mathbf{0},t')e^{i(\omega_{\mathbf{k}_s}-\omega_s)t'}. \quad (16)$$

By taking into account Eqs. (16), (14), (12), and (13) we finally have

$$F^{(+)}(\mathbf{r}_1,t)=\frac{1}{\sqrt{2}}\left[F_i^{(+)}\left(\mathbf{0},t-\frac{\delta l}{c}\right)e^{i\omega_i\delta l/c}+\frac{iK}{2\pi}\int_{-\infty}^{+\infty}d\omega_{\mathbf{k}_s}\times\int_{-\infty}^{+\infty}dt'F_s^{(+)}(\mathbf{0},t')e^{i(\omega_{\mathbf{k}_s}-\omega_s)(t'-t)}e^{ik(\omega_{\mathbf{k}_s})d}\right], \quad (17)$$

with a similar expression for $F^{(+)}(\mathbf{r}_2,t+\tau)$.

The coincidence detection probability is given by the correlation

$$\langle F^{(+)}(\mathbf{r}_1,t)F^{(+)}(\mathbf{r}_2,t+\tau)\rangle=\frac{gVKe^{i\omega_i\delta l/c}}{2\pi i}\int_{-\infty}^{+\infty}d\omega_{\mathbf{k}_s}e^{ik(\omega_{\mathbf{k}_s})d}\times e^{-i(\omega_{\mathbf{k}_s}-\omega_s)\delta l/c}\,\overline{\nu}(\omega_{\mathbf{k}_s}-\omega_s)\times\sin[(\omega_{\mathbf{k}_s}-\omega_s)\tau], \quad (18)$$

where we have used Eqs. (7), (8), and defined the Fourier transform of ν

$$\overline{\nu}(\omega_{\mathbf{k}_s}-\omega_s)=\int_{-\infty}^{+\infty}du\,\nu(u)e^{-i(\omega_{\mathbf{k}_s}-\omega_s)u}. \quad (19)$$

Multiplying Eq. (18) by its complex conjugate and using Eq. (11) we can calculate the joint detection probability. After some easy algebra, making use of the relation

$$\int_0^{\infty}dx\ \sin ax\ \sin bx=\frac{\pi}{2}[\delta(a-b)-\delta(a+b)],$$

and assuming that $\overline{\nu}(\omega)$ is symmetric in ω, we have

$$P_{12}=C\int_{-\infty}^{+\infty}d\omega_{\mathbf{k}_s}|\overline{\nu}(\omega_{\mathbf{k}_s}-\omega_s)|^2\times[1-e^{-2i(\omega_{\mathbf{k}_s}-\omega_s)\delta l/c}e^{i[k(\omega_{\mathbf{k}_s})-k(2\omega_s-\omega_{\mathbf{k}_s})]d}], \quad (20)$$

C being a constant. Finally, by substituing Eq. (15) into Eq. (20) and defining $\omega=\omega_{\mathbf{k}_s}-\omega_s$ we obtain the final result for P_{12}:

$$P_{12}=C\int_{-\infty}^{+\infty}d\omega|\overline{\nu}(\omega)|^2[1-\cos[2\omega(\alpha-\delta l/c)]]. \quad (21)$$

This result is similar to the one obtained in Eq. (12) of [4].

III. THE QUANTUM ERASER

In 1992 Kwiat and co-workers [6] performed an experiment to show how the information may be erased from the state vector. This effect is known as the *quantum eraser* and shows the relation between quantum coherence and distinguishability. An outline of the experimental setup is shown in Fig. 2. A half wave plate at an angle ($\phi/2$) to the horizontal is placed in one arm of a Hong-Ou-Mandel interferometer giving rise to a change in the polarization state of the light in this arm. Two polarizers P_1 and P_2 at angles ϕ_1 and ϕ_2 to the horizontal are inserted in front of detectors D_1 and D_2, respectively.

We now present an analysis of this experiment in the Wigner formalism. This time we have to take into account the polarization of both the light beam and the vacuum field. The field is now represented by a vector

$$\mathbf{F}^{(+)}(\mathbf{r},t)=i\sum_{[\mathbf{k}],\lambda}\left(\frac{\hbar\omega_{\mathbf{k}}}{2L^3}\right)^{1/2}\alpha_{\mathbf{k},\lambda}(t)\boldsymbol{\epsilon}_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\omega_a t}, \quad (22)$$

$\boldsymbol{\epsilon}_{\mathbf{k},\lambda}$ being orthonormal polarization vectors ($\lambda=1$ denotes horizontal polarization and $\lambda=2$ vertical polarization).

It can easily be proved that the expressions for the detection probabilities (9), (10) remain valid. Moreover, the final expression for the joint detection probability when we deal with parametric down-conversion experiments involving polarization is

$$P_{12}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)\propto\sum_{\lambda}\sum_{\lambda'}|\langle F_{\lambda}^{(+)}(\mathbf{r}_1,t)F_{\lambda'}^{(+)}(\mathbf{r}_2,t+\tau)\rangle|^2. \quad (23)$$

In order to apply Eq. (23) we must calculate the fields at the detectors D_1 and D_2. For the sake of simplicity we shall consider that the half wave plate is placed at $\mathbf{r}=\mathbf{0}$. The signal beam coming out the half wave plate and the idler beam outgoing the crystal are

$$\mathbf{F}_s^{(+)}(\mathbf{0},t)=F_s^{(+)}(\mathbf{0},t)(\cos\phi,\sin\phi),$$

$$\mathbf{F}_i^{(+)}(\mathbf{0},t)=F_i^{(+)}(\mathbf{0},t)(1,0). \tag{24}$$

The field corresponding to the output port of the 50:50 beam splitter (placed at $\mathbf{r}=\mathbf{R}$) in the direction of detector D_1 is

$$\mathbf{F}^{(+)}(\mathbf{R},t)=\frac{1}{\sqrt{2}}[\mathbf{F}_i^{(+)}(\mathbf{R},t)+i\mathbf{F}_s(\mathbf{R},t)]=\frac{1}{\sqrt{2}}F_i^{(+)}(\mathbf{0},t-\tau_2)e^{i\omega_i\tau_2}+iF_s^{(+)}(\mathbf{0},t-\tau_1)e^{i\omega_s\tau_1}\cos\phi,iF_s^{(+)}(\mathbf{0},t-\tau_1)e^{i\omega_s\tau_1}\sin\phi. \tag{25}$$

Here τ_1 and τ_2 are the propagation times from the crystal to the beam splitter. When a polarizer oriented at angle ϕ_1 to the horizontal is placed in the output port of the interferometer, the field at the detector D_1 (placed at $\mathbf{r}_1$) at time t is

$$\begin{aligned}\mathbf{F}^{(+)}(\mathbf{r}_1,t)&=[\mathbf{F}^{(+)}(\mathbf{R},t)\cdot(\cos\phi_1,\sin\phi_1)](\cos\phi_1,\sin\phi_1)\\&=\frac{1}{\sqrt{2}}[F_i^{(+)}(\mathbf{0},t-\tau_2)e^{i\omega_i\tau_2}\cos\phi_1+iF_s^{(+)}(\mathbf{0},t-\tau_1)e^{i\omega_s\tau_1}\cos(\phi-\phi_1)](\cos\phi_1,\sin\phi_1),\end{aligned} \tag{26}$$

where we have dropped an irrelevant phase shift coming from the propagation between BS and D_1. In the same way, we write for the field at the detector D_2 (placed at $\mathbf{r}_2$) at time $t+\tau$

$$\begin{aligned}\mathbf{F}^{(+)}(\mathbf{r}_2,t+\tau)=&\frac{1}{\sqrt{2}}[iF_i^{(+)}(\mathbf{0},t+\tau-\tau_2)e^{i\omega_i\tau_2}\cos\phi_2+F_s^{(+)}(\mathbf{0},t+\tau-\tau_1)e^{i\omega_s\tau_1}\cos(\phi-\phi_2)]\\&\times(\cos\phi_2,\sin\phi_2).\end{aligned} \tag{27}$$

In order to calculate the joint probability we combine Eqs. (23), (26), and (27), and take into account the correlation properties of the field given by Eqs. (7) and (8). After some easy algebra and an integration of $P_{12}(\tau)$ over the time interval τ, we obtain the coincidence probability

$$\begin{aligned}P_{12}=C\Bigg[&[\cos^2\phi_1\cos^2(\phi-\phi_2)+\cos^2\phi_2\cos^2(\phi-\phi_1)]\\&\times\int_0^\infty|\nu(\tau)|^2d\tau-2\cos\phi_1\cos(\phi-\phi_2)\cos\phi_2\cos(\phi-\phi_1)\mathrm{Re}\int_0^\infty\nu(\delta\tau-\tau)\nu^*(\delta\tau+\tau)d\tau\Bigg],\end{aligned} \tag{28}$$

with $\delta\tau=\tau_1-\tau_2$, and C being a constant. When $\delta\tau=0$ we have

$$P_{12}=C'\sin^2\phi\sin^2(\phi_2-\phi_1), \tag{29}$$

C' being another constant. This expression is similar to the one obtained in the Appendix of [6].

ACKNOWLEDGEMENT

We acknowledge financial support of CAICYT Project No. PB-95-0594 (Spain).

[1] A. Casado, T. W. Marshall, and E. Santos, J. Opt. Soc. Am. B **14**, 494 (1997).

[2] A. Casado, A. Fernandez, T. W. Marshall, R. Risco-Delgado, and E. Santos, Phys. Rev. A **55**, 3879 (1997).

[3] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987). The analysis of this experiment with the Wigner function formalism is a trivial extension of the work in Ref. [2].

[4] A. M. Steinberg, P. G. Kwiat, and R. Y. Chiao, Phys. Rev. A **45**, 6659 (1992).

[5] J. D. Franson, Phys. Rev. A **45**, 3126 (1992).

[6] P. G. Kwiat, A. M. Steinberg, and R. Y. Chiao, Phys. Rev. A **45**, 7729 (1992).