IMPROPER GAUSSIAN SIGNALING FOR THE TWO-USER BROADCAST CHANNEL TREATING INTERFERENCE AS NOISE

Christian Lameiro*, Ignacio Santamaría†, and Peter J. Schreier*

* Signal and System Theory Group, Universität Paderborn, Germany, http://sst.upb.de
† Advanced Signal Processing Group, Universidad de Cantabria, Spain, http://gtas.unican.es

ABSTRACT

Improper Gaussian signaling (IGS) has been shown to enlarge the rate region achievable by conventional proper Gaussian signaling (PGS) schemes in several interference-limited multiuser networks. In this work, we consider the 2-user broadcast channel (BC) when treating interference as noise "TIN" at every receiver. For this scenario, we derive a closed-form characterization of the rate region boundary with IGS. The Pareto-optimal points are achieved when at least one of the users employs maximally improper (rectilinear) signals. Differently from other interference-limited networks, our results show that IGS always outperforms PGS for the 2-user BC with TIN. Furthermore, IGS also enlarges the PGS rate region with time-sharing for this scenario.

Index Terms— Broadcast channel, improper Gaussian signaling, treating interference as noise.

1. INTRODUCTION

Improper Gaussian signaling (IGS) has recently been proposed as a low-complexity approach to handle interference in multiuser networks [1]. Contrary to the conventional proper Gaussian signaling (PGS), in IGS the real and imaginary parts of the transmit signals are correlated and/or have unequal power [2]. Such a statistical property has been shown to pay off in interference-limited scenarios when treating interference as noise (TIN). The benefits of IGS over PGS have been shown in, e.g., the interference channel [1, 3–9], relay channels [10], and cognitive radio networks [11–13], to name a few.

The broadcast channel (BC), where a transmitter sends distinct information to a set of receivers, is one of the fundamental multiuser communication scenarios. The capacity of the BC is achieved by PGS with superposition coding and successive interference cancellation [14, 15]. Such a nonlinear scheme, however, entails higher complexity than a linear transceiver that treats interference as noise and presents

also some issues regarding its practical implementation [16]. With TIN, IGS has been shown to outperform PGS in the multiantenna BC [17]. The parameters of the IGS scheme can then be optimized applying existing numerical methods. However, solutions based on numerical optimization do not permit drawing insights into the features of IGS. For example, an intriguing question is under which conditions IGS outperforms PGS. Such analytical results have been obtained, e.g., for the Z-interference channel in [9], where it was shown that IGS outperforms PGS when the ratio between the gain of the cross-link and that of the direct link is above a certain threshold. A similar result was also obtained for a cognitive radio scenario in [11].

In order to shed light onto the behavior of IGS in the BC with TIN, we consider a single-antenna 2-user BC. We analyze the boundary of the achievable rate region for this setting and derive closed-form expressions for the value of the parameters that attain the rate region boundary. Interestingly, our results show that all Pareto-optimal points are achieved when at least one of the users employs maximally improper signals and, unlike other interference-limited scenarios, IGS always outperforms PGS with TIN for the 2-user BC. Furthermore, there are boundary points for which one of the users transmits real-valued signals while the other uses purely imaginary signals. These boundary points strictly outperform PGS with time-sharing (TS).[1] Furthermore, we show that maximally improper (rectilinear) signaling is optimal for at least one user.

2. SYSTEM MODEL

We consider a single-antenna 2-user BC with TIN. The received signal at the ith user is given by

$$y_i = h_i(\sqrt{\alpha_1 p}s_1 + \sqrt{\alpha_2 p}s_2) + n_i, \ i = 1, 2, \qquad (1)$$

where h_i is the channel coefficient of the ith user, n_i is proper Gaussian noise with variance σ_i^2, and s_i is the signal intended to the ith user, which is Gaussian with unit variance. The parameter p is the transmit power, whereas α_i, with $\alpha_1 + \alpha_2 = 1$,

[1] We consider a TS scheme in which the power constraint is satisfied at each operating point. We refer the reader to [18] for an extended analysis of IGS with TS.

determines the percentage of transmit power devoted to the ith-user's message. Since the system is limited by interference, we allow the base station to transmit improper Gaussian signals.[2] Taking this into account, the achievable rate of the ith user, $i = 1, 2$, is [4]

$$R_i = \tfrac{1}{2} \log \left[\frac{\left(pg_i + \sigma_i^2\right)^2 - p^2 g_i^2 \left|\alpha_1 \kappa_1 e^{j\phi_1} + \alpha_2 \kappa_2 e^{j\phi_2}\right|^2}{\left(\alpha_{\bar{i}} pg_i + \sigma_i^2\right)^2 - \alpha_{\bar{i}}^2 p^2 g_i^2 \kappa_{\bar{i}}^2} \right],$$
(2)

where $g_i = |h_i|^2$, $\bar{i} = 2$ if $i = 1$ and $\bar{i} = 1$ if $i = 2$, and the logarithm is taken with base 2. The parameters κ_i and ϕ_i are the circularity coefficient and phase of the complementary variance, respectively, of s_i.

For this setting, we are interested in the Pareto boundary of the achievable rate region, which is the region obtained as the union of all achievable rate pairs. To this end, we first notice from (2) that both rates are simultaneously maximized by taking $\phi_1 = \phi_2 + \pi$. This choice can be interpreted by looking at the distribution of each user's signal. Their probability density contour are ellipses whose major axes are orthogonal to each other [2]. Therefore, since the desired signal for one user is interference for the other, this choice makes in turn the desired signal and the interference exhibit this property at both receivers simultaneously. Denoting $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$, we have three parameters to optimize, namely, α, κ_1, and κ_2. The achievable rates (2) can then be expressed as

$$R_1 = \frac{1}{2} \log \left[\frac{(pg_1 + \sigma_1^2)^2 - p^2 g_1^2 [\alpha \kappa_1 - (1 - \alpha)\kappa_2]^2}{[(1 - \alpha)pg_1 + \sigma_1^2]^2 - (1 - \alpha)^2 p^2 g_1^2 \kappa_2^2} \right], \quad (3)$$

$$R_2 = \frac{1}{2} \log \left[\frac{(pg_2 + \sigma_2^2)^2 - p^2 g_2^2 [\alpha \kappa_1 - (1 - \alpha)\kappa_2]^2}{(\alpha pg_2 + \sigma_2^2)^2 - \alpha^2 p^2 g_2^2 \kappa_1^2} \right], \quad (4)$$

and the rate region is

$$\mathcal{R} = \bigcup_{\substack{0 \le \alpha \le 1 \\ 0 \le \kappa_i \le 1}} (R_1, R_2) . \quad (5)$$

The Pareto boundary of the rate region defined above is comprised of rate pairs such that the rate of one user cannot increase without decreasing the other user's rate [19]. Therefore, each boundary point can be obtained as the solution of

$$\underset{0 \le \{\alpha, \kappa_1, \kappa_2\} \le 1}{\text{maximize}} \quad R_2, \quad (6a)$$

$$\text{subject to} \quad R_1 \ge \bar{R}_1, \quad (6b)$$

where $\bar{R}_1 = \tau \log\left(1 + \frac{pg_1}{\sigma_1^2}\right)$, with $0 \le \tau \le 1$. Varying τ between 0 and 1 and solving the above problem yields the boundary of the achievable rate region. Even though the above problem is a non-convex optimization problem, we will show in the next section that an analytical solution is possible for the global optimal point.

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[2]See [11] for an introduction on improper random variables.

3. BOUNDARY OF THE RATE REGION

Our first result about the optimal solution of (6) is stated in the following lemma.

Lemma 1. *Every point of the boundary of the rate region defined in (5) is achieved when at least one of the users employs maximally improper signaling, i.e., $\kappa_1 = 1$ and/or $\kappa_2 = 1$.*

Proof. First, we can rewrite the rate constraint $R_1 \ge \bar{R}_1$ as a constraint on κ_2, after some manipulations, as

$$\kappa_2 \ge \frac{-\alpha pg_1 \kappa_1}{(1 - \alpha)pg_1 \beta_1}$$
$$+ \frac{\sqrt{\beta_1^2 \gamma_1^2 - \beta_1(\beta_1 + 1)\alpha pg_1 [2\gamma_1 - \alpha pg_1] + (\beta_1 + 1)(\alpha pg_1 \kappa_1)^2}}{(1 - \alpha)pg_1 \beta_1},$$
(7)

where $\beta_1 = 2^{2\bar{R}_1} - 1$, and $\gamma_1 = pg_1 + \sigma_1^2$. We now take equality in the foregoing expression and plug it into (4), which yields an expression for R_2 that only depends on α and κ_1. Let us denote this expression as $R_2(\alpha, \kappa_1)$. Since the rate constraint on user 1 has already been accounted for with (7), the dependency of $R_2(\alpha, \kappa_1)$ on κ_1 can be analyzed by evaluating its derivative. By doing so, we obtain that $R_2(\alpha, \kappa_1)$ is increasing in κ_1 if and only if

$$-\left\{ 2p^2 g_2^2 \tilde{\kappa} \left[\alpha - (1 - \alpha)\frac{\partial \kappa_2}{\partial \kappa_1} \right] \right\} [(\alpha pg_2 + \sigma_2^2)^2 - (\alpha pg_2 \kappa_1)^2]$$
$$+ 2(\alpha pg_2)^2 \kappa_1 \left\{ \gamma_2^2 - p^2 g_2^2 \tilde{\kappa}^2 \right\} \ge 0,$$
(8)

where $\gamma_2 = pg_2 + \sigma_2^2$, $\tilde{\kappa} = \alpha \kappa_1 - (1 - \alpha)\kappa_2$, κ_2 is given by (7), and

$$\frac{\partial \kappa_2}{\partial \kappa_1} = \frac{\alpha \tilde{\kappa}}{(1 - \alpha)[\alpha \kappa_1 + (1 - \alpha)\kappa_2 \beta_1]} . \quad (9)$$

Now we will show that condition (8) is fulfilled regardless of the values of its parameters. To this end, we plug (9) into (8), yielding

$$\alpha \kappa_1 [\alpha \kappa_1 + (1 - \alpha)\kappa_2 \beta_1] \left\{ \gamma_2^2 - p^2 g_2^2 \tilde{\kappa}^2 \right\}$$
$$\ge (1 - \alpha)\kappa_2(\beta_1 + 1)\tilde{\kappa}[(\alpha pg_2 + \sigma_2^2)^2 - (\alpha pg_2 \kappa_1)^2] . \quad (10)$$

Let us recall that α, κ_1, and κ_2 belong to the interval $[0, 1]$, and then $-1 \le \tilde{\kappa} \le 1$. The left-hand side of (10) is always positive, while the right-hand side is equal to or smaller than zero when $\bar{\kappa} \le 0$ or, equivalently, $\alpha \kappa_1 \le (1 - \alpha)\kappa_2$. Then, (10) holds for $\alpha \kappa_1 \le (1 - \alpha)\kappa_2$. When this condition is not fulfilled, after some manipulations (10) can be rewritten as

$$\alpha \kappa_1 \tilde{\kappa} \left\{ \gamma_2^2 + p^2 g_2^2 (1 - \alpha)^2 \kappa_2^2 (\beta_1 + 1) - p^2 g_2^2 \tilde{\kappa}^2 \right\}$$
$$+ (1 - \alpha)^2 \kappa_2^2 (\beta_1 + 1)(\alpha pg_2 + \sigma_2^2)^2$$
$$+ \alpha \kappa_1 (1 - \alpha)\kappa_2(\beta_1 + 1)[\gamma_2^2 - (\alpha pg_2 + \sigma_2^2)^2] \ge 0 . \quad (11)$$

When $\alpha \kappa_1 > (1 - \alpha)\kappa_2$, the first line in (11) is always non-negative. The second line is also non-negative since all of its

terms are non-negative. In addition, the third line is always positive because $\gamma_2 \geq \alpha p g_2 + \sigma_2^2$. Thus, (10) holds as well for $\alpha \kappa_1 > (1 - \alpha)\kappa_2$. This means that $R_2(\alpha, \kappa_1)$ is non-decreasing in κ_1 for all values of α. Consequently, $R_2(\alpha, \kappa_1)$ is maximized for $\kappa_1 = 1$ except if (7) returns a value greater than one. In such a case, κ_1 must then be chosen as the maximum value for which $\kappa_2 = 1$. Since this is the case for any value of α, every point of the Pareto boundary is therefore achieved by $\kappa_1 = 1$ and/or $\kappa_2 = 1$, which concludes the proof.

□

Lemma 1 provides an interesting result: With TIN, improper signaling outperforms proper signaling for every point of the Pareto boundary.[3] Furthermore, maximally improper signaling is always optimal at least for one user. This behavior differs from other wireless scenarios, such as underlay cognitive radio networks [11], where improper signaling was shown to be optimal only when the ratio between the gains of the interference link and the direct link is above a certain threshold. In the BC channel, however, the signal and the interference for each user travel through the same channel. Furthermore, since we are considering a degraded BC, the condition for the optimality of IGS will always be met for at least one of the users in every Pareto optimal point. Therefore, IGS with at least one user transmitting maximally improper Gaussian codewords is optimal for the 2-user BC with TIN.

In the following, we obtain the optimal transmission parameters to attain a Pareto-optimal point, namely, the values of κ_1, κ_2, and α. From Lemma 1, at least one user must employ maximally improper signaling, therefore the rate region defined in (5) can be expressed as the union of two different rate regions, $\mathcal{R} = \mathcal{R}_1 \bigcup \mathcal{R}_2$, where $\mathcal{R}_i$ is the achievable rate region when $\kappa_i = 1$. The rate region $\mathcal{R}_1$ can be obtained by solving

$$\underset{0 \leq \{\alpha, \kappa_2\} \leq 1}{\text{maximize}} \quad R_2, \tag{12a}$$

$$\text{subject to} \quad R_1 \geq \bar{R}_1, \tag{12b}$$

with $\kappa_1 = 1$, and $\bar{R}_1 = \frac{\tau}{2} \log(1 + \frac{2pg_1}{\sigma_1^2})$. Again, varying τ between 0 and 1 we can obtain the boundary of $\mathcal{R}_1$. The optimal solution of (12) can be found in closed form as presented in the following lemma.

Lemma 2. *The optimal solution of* (12)*, in which $\kappa_1 = 1$, is given by*

$$\alpha^\star = \max(\alpha_0, \alpha_{th}), \tag{13}$$

$$\alpha_{th} = \frac{\beta_1 \sigma_1^2}{2pg_1}, \alpha_0 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}, \tag{14}$$

$$\kappa_2^\star = \frac{\beta_1 \gamma_1 - (\beta_1 + 2)\alpha^\star pg_1}{(1 - \alpha^\star)pg_1 \beta_1}, \tag{15}$$

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[3]An exception is the extreme points ($\tau = 0$ or $\tau = 1$), where only one user is active and thus proper signaling is optimal.
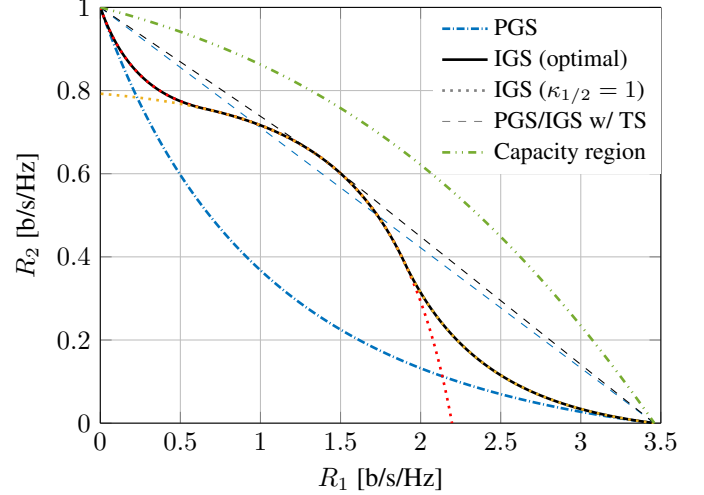

Fig. 1. Rate region boundaries for $g_1 = 10$ and $g_2 = 1$.

with

$$a_2 = - \left[\frac{2pg_2(\beta_1 + 1)}{\beta_1} \right]^2, a_1 = -4pg_2\sigma_2^2 \left(\frac{\beta_1 + 1}{\beta_1} \right)^2, \tag{16}$$

$$a_0 = \frac{g_2}{g_1}\gamma_1 \left[\frac{g_2}{g_1}\gamma_1 + 2\sigma_2^2 \frac{\beta_1 + 1}{\beta_1} \right] - \gamma_2^2. \tag{17}$$

Proof. For $\kappa_1 = 1$, (7) can be simplified to $\kappa_2 \geq \kappa_2^\star$. Since R_2 is decreasing in κ_2, we have that $\kappa_2 = \kappa_2^\star$ is optimal. This expression returns a valid value of κ_2 if it is equal to or smaller than 1, which implies $\alpha \geq \alpha_{th}$. Taking $\kappa_2 = \kappa_2^\star$ and plugging it into (4), with $\kappa_1 = 1$, we obtain an expression for R_2, denoted as $R_2(\alpha)$, that depends only on α, which yields the condition

$$\frac{\partial R_2(\alpha)}{\partial \alpha} \geq 0 \Leftrightarrow \alpha^2 a_2 + \alpha a_1 + a_0 \geq 0, \tag{18}$$

where a_2, a_1, and a_0 are given by (16), (17). First, we observe that, since $a_2 \leq 0$, the function on the right-hand side of (18) is concave. Second, we notice that a_2 and a_1 are both equal to or smaller than zero, which, by the Descartes' rule of signs, implies that the number of positive roots of the aforementioned function is at most 1. With these considerations, we conclude that $R_2(\alpha)$ is increasing in α for $0 \leq \alpha \leq \max(\alpha_0, 0)$, where α_0 is such that

$$\alpha_0^2 a_2 + \alpha_0 a_1 + a_0 = 0, \tag{19}$$

which yields the right-hand side of (14). Finally, since $R_2(\alpha)$ increases monotonically in the interval $0 \leq \alpha \leq \max(\alpha_0, 0)$, and decreases in $\max(\alpha_0, 0) \leq \alpha \leq 1$, we obtain (13). □

The second rate region, $\mathcal{R}_2$, can be obtained by solving

$$\underset{0 \leq \{\alpha, \kappa_1\} \leq 1}{\text{maximize}} \quad R_1, \tag{20a}$$

$$\text{subject to} \quad R_2 \geq \bar{R}_2, \tag{20b}$$

Fig. 2. Rate region boundaries for $g_1 = 1$ and $g_2 = 100$.

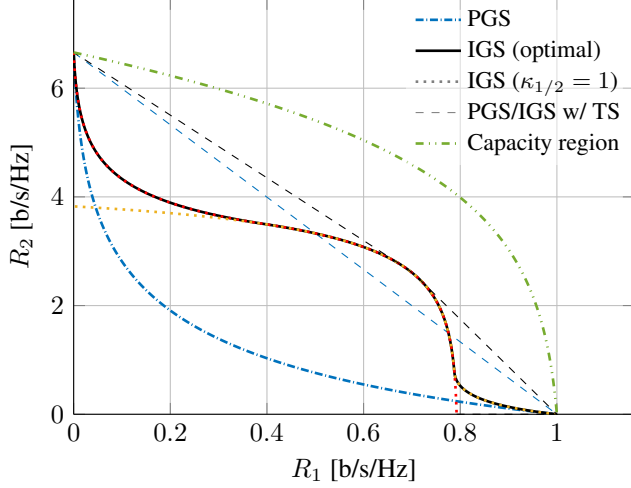**Fig. 3**. Optimal power splitting for $g_1 = 10$ and $g_2 = 1$.

with $\kappa_2 = 1$, and $\bar{R}_2 = \frac{\tau}{2} \log(1 + \frac{2pg_2}{\sigma_2^2})$. Since the above problem is analogous to (12), its optimal solution is given by Lemma 2, exchanging the roles of user 1 and user 2.

4. NUMERICAL RESULTS

We provide some simulation examples for $p = 1$, $\sigma_1^2 = \sigma_2^2 = 1$, and different channel gains g_1, g_2. Figure 1 shows the rate regions for $g_1 = 10$, $g_2 = 1$, and different strategies, namely: i) the capacity region achieved with superposition coding, ii) the PGS rate region, along with its convex hull (achieved with TS), and iii) the regions $\mathcal{R}_1$ and $\mathcal{R}_2$, which are obtained taking $\kappa_1 = 1$ and $\kappa_2 = 1$, respectively, along with their union and convex hull. When TS is not allowed, IGS significantly enlarges the PGS rate region. When PGS with TS is used, there is still some improvement by using IGS. Specifically, we observe that IGS enlarges the rate region for a subset of rate pairs for which both users employ maximally IGS (both users employ maximally IGS in the intersection between the rate regions corresponding to $\kappa_1 = 1$ and $\kappa_2 = 1$), which in turn implies an enlargement of the whole rate region if IGS with TS is employed. Indeed, the enlargement over PGS with TS is more prominent the higher the difference between both channel gains is. This observation is illustrated in Fig. 2, where $g_1 = 1$ and $g_2 = 100$. Since now user 2 is the strongest user, the enlargement of the rate region over PGS with TS is shifted to the right. Moreover, it can be observed that the enlargement is more significant than in the previous example. As a matter of fact, when both channel gains are equal, i.e., $g_1 = g_2$, the capacity region becomes that of PGS with TS.

Another interesting behavior is observed from the analysis of the optimal power splitting factor $\alpha^\star$. Figures 3 and 4 depict this quantity for the first and second example, respectively. While for PGS the rate region boundary can be parameterized by α, this is in general not the case for IGS.
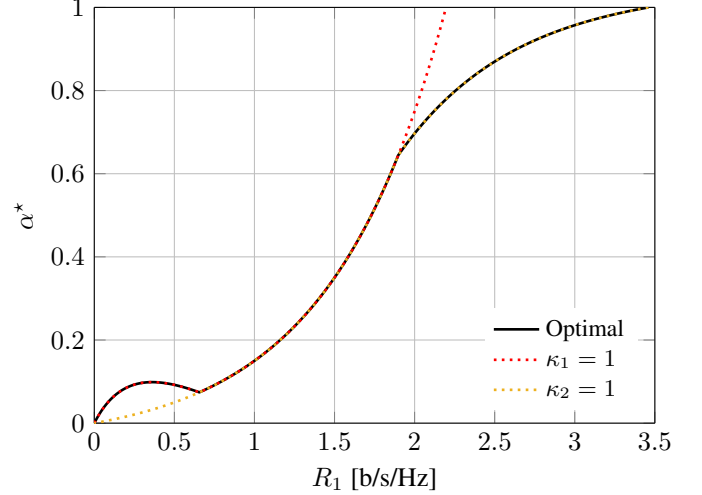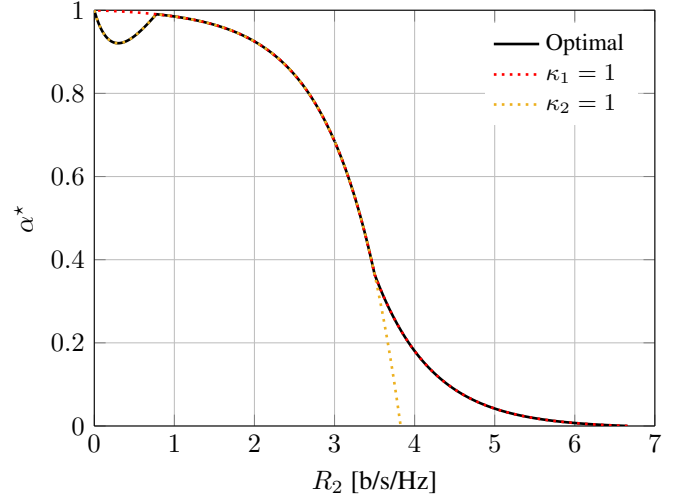

Fig. 4. Optimal power splitting for $g_1 = 1$ and $g_2 = 100$.

In particular, we observe that there are some values of α that correspond to up to three different points of the IGS rate region boundary. Furthermore, the protrusion observed for low values of R_1 (R_2) in Fig. 3 (Fig. 2) indicates that there is a subset of rate pairs for which a higher R_1 (R_2) is achieved by actually reducing the power allocated to this user.

5. CONCLUSION

We have derived a closed-form characterization of the rate region boundary for the 2-user BC with IGS+TIN. We have shown analytically that every boundary point is achieved when at least one user employs maximally IGS. Furthermore, our numerical results show that the IGS rate region is significantly larger than for PGS with and without TS.

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