Constructing Positive Reliable Numerical Solution for American Call Options: A New Front-Fixing Approach

R. Company, V.N. Egorova[1], L. Jódar

Instituto de Matemática Multidisciplinar, Universitat Politécnica de Valéncia, Camino de Vera s/n, 46022 Valencia, Spain

Abstract

A new front-fixing transformation is applied to the Black-Scholes equation for the American call option pricing problem. The transformed non-linear problem involves homogeneous boundary conditions independent of the free boundary. The numerical solution by an explicit finite-difference method is positive and monotone. Stability and consistency of the scheme is studied. The explicit proposed method is compared with other competitive implicit ones from the points of view accuracy and computational cost.

Keywords: American call option pricing, finite difference scheme, front-fixing transformation, numerical analysis, positivity.

1. Introduction

Free boundary problems appear in plasma physics, semiconductors, financial markets and other fields [2], [6], [13]. The free boundary has to be determined as a part of the solution. Crank in [6] systematized the knowledge about moving and free boundary problems and presented a front-fixing method for such problems. The method is based on Landau's transform [15] that let the unknown boundary be included into equation in exchange for a fixed boundary.

American option pricing leads to the free boundary problem [18]. Wu and Kwok in [19] introduced a logarithmic front-fixing transformation for solving

Email addresses: rcompany@imm.upv.es (R. Company), egorova.vn@gmail.com (V.N. Egorova), ljodar@imm.upv.es (L. Jódar)

[1]Corresponding author.

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such problems to the field of option pricing. Recently this technique has been treated in [4] and [20], [11]. Another transformation related to a synthetic portfolio is presented in [17], [1] involving the first spatial derivative of the option price. The transformed equation can be numerically solved by a finite element method (see [21]).

In this paper we introduce a new front-fixing transformation for American call option on dividend-paying assets. Under this transformation a nonlinear PDE with homogeneous boundary conditions independent of the free boundary is obtained. This fact simplifies the numerical analysis of the finite difference scheme. The proposed explicit finite difference scheme preserves theoretical properties of the solution mentioned in [12]. Dealing with prices it is important to guarantee that the proposed numerical solutions be non-negative. Our scheme guarantees this property as well as monotonicity of the free boundary and the option price. Numerical experiments show that the method is efficient and accurate in comparison with other implicit methods.

The paper is organized as follows. In Section 2 we introduce a new front-fixing transformation for the American call option problem and an explicit finite difference scheme is constructed. In Section 3 we study properties of the numerical solution, such as the non-negativity and monotonicity of the option price, increasing monotonicity and concave behaviour of the optimal exercise boundary. In Section 4 stability and consistency are treated. In last section we present implicit scheme and compare proposed method with other approaches as well as illustrate efficiency and convergence of the method.

Throughout the paper we will denote for a given $x = [x_1, x_2, ..., x_N]^T \in \mathbb{R}^N$ its supremum norm as $||x||_\infty = \max\{|x_i| :\ 1 \leq i \leq N\}$.

2. Front-Fixing Method

In this section we introduce a new front-fixing transformation similar to the ones used by [19], [17], [18]. This transformation translate the moving domain to the fixed one and changes the boundary conditions on the left boundary to the homogeneous ones. It allows to apply finite-difference method for the numerical solution. The discretization of the transformed problem and constructing the explicit finite-difference method are presented in this section.

American call option price model is given by [18] as the moving free boundary PDE

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC, \qquad 0 < S < B(\tau), \qquad 0 < \tau \le T, \tag{2.1}$$

together with the boundary and initial conditions

$$C(S,0) = \max(S-E,0), \tag{2.2}$$

$$\frac{\partial C}{\partial S}(B(\tau),\tau) = 1, \tag{2.3}$$

$$C(B(\tau),\tau) = B(\tau) - E, \tag{2.4}$$

$$C(0,\tau) = 0, \tag{2.5}$$

$$B(0) = \begin{cases} E, & r \le q, \\ \frac{r}{q}E, & r > q. \end{cases} \tag{2.6}$$

where $\tau = T - t$ denotes the time to maturity T, S is the asset's price, $C(S,\tau)$ is the option price, $B(\tau)$ is the unknown early exercise boundary, σ is the volatility of the asset, r is the risk free interest rate, q is the continuous dividend yield and E is the strike price.

It is well known that if there is no dividend payment ($q = 0$), then the optimal strategy is to exercise option at the maturity (see [18], chapter 7.7, [8]). In that case the American call becomes European one. Because of that we consider problem (2.1)-(2.6) with $q > 0$ [8].

Let us consider the dimensionless transformation with two targets: to fix the computational domain as in [19] and to simplify the boundary conditions like [18], p. 122,

$$x = \ln\frac{B(\tau)}{S}, \qquad c(x,\tau) = \frac{C(S,\tau) - S + E}{E}, \quad S_f(\tau) = \frac{B(\tau)}{E}. \tag{2.7}$$

Under transformation (2.7) the problem (2.1) - (2.6) can be rewritten in normalized form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2} - \left(r - q - \frac{\sigma^2}{2} + \frac{S_f'}{S_f}\right)\frac{\partial c}{\partial x} - rc - qS_f e^{-x} + r, \quad x > 0,\ 0 < \tau \le T, \tag{2.8}$$

with new initial and boundary conditions

$$c(x,0)=\begin{cases}1-e^{-x}, & r\le q,\\ g(x), \quad r>q, & \end{cases} \qquad x\ge 0, \tag{2.9}$$

$$g(x)=\max\left(1-\frac{r}{q}e^{-x},0\right), \tag{2.10}$$

$$\frac{\partial c}{\partial x}(0,\tau)=0, \tag{2.11}$$

$$c(0,\tau)=0, \tag{2.12}$$

$$\lim_{x\to\infty} c(x,\tau)=1, \tag{2.13}$$

$$S_f(0)=\begin{cases}1, & r\le q,\\ \frac{r}{q}, & r>q.\end{cases} \tag{2.14}$$

Following the ideas of [19], [13] and in order to solve the numerical difficulties derived from the discretization at the numerical boundary, we assume that (2.8) holds true at $x=0$,

$$\frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2}(0^+,\tau)-qS_f(\tau)+r=0. \tag{2.15}$$

The equation (2.8) is a non-linear differential equation on the domain $(0,\infty)\times(0,T]$. In order to solve numerically problem (2.8)-(2.14) at the point (x,τ) in the domain $(0,\infty)\times(0,T]$, one has to consider a bounded numerical domain. Let us introduce $x_{\max}$ large enough to translate the boundary condition (2.13). Then the problem (2.8)-(2.14) can be numerically studied on the fixed domain $[0,x_{\max}]\times[0,\tau]$. The value $x_{\max}$ is chosen following the criterion pointed out in [9].

Let us introduce the computational grid of $M+2$ space points and $N+1$ time levels with respective stepsizes h and k

$$h=\frac{x_{\max}}{M+1},\quad k=\frac{\tau}{N}, \tag{2.16}$$

$$x_j=hj,\quad j=0,..,M+1,\quad \tau^n=kn,\quad n=0,..,N. \tag{2.17}$$

The approximate value of option price at the point x_j and time τ^n is denoted by $c_j^n\approx c(x_j,\tau^n)$ and the approximate value of the free boundary is denoted by $S_f^n\approx S_f(\tau^n)$. Then a forward two-time level and centred in a space explicit scheme is constructed for internal spacial nodes as follows

$$\frac{c_j^{n+1} - c_j^n}{k} = \frac{1}{2}\sigma^2 \frac{c_{j-1}^n - 2c_j^n + c_{j+1}^n}{h^2} -$$

$$\left(r - q - \frac{\sigma^2}{2} + \frac{S_f^{n+1} - S_f^n}{kS_f^n} \right) \frac{c_{j+1}^n - c_{j-1}^n}{2h} - rc_j^n + r - qS_f^n e^{-x_j}. \qquad (2.18)$$

The equation (2.18) can be rewritten in the form

$$c_j^{n+1} = a_1^n c_{j-1}^n + bc_j^n + a_2^n c_{j+1}^n + k\left(r - qS_f^n e^{-x_j}\right), \quad 1 \le j \le M, \qquad (2.19)$$

where

$$a_1^n = \frac{k}{2h^2}\left(\sigma^2 + \left(r - q - \frac{\sigma^2}{2}\right)h\right) + \frac{S_f^{n+1} - S_f^n}{2hS_f^n} = a + \frac{S_f^{n+1} - S_f^n}{2hS_f^n},$$

$$b = 1 - \sigma^2\frac{k}{h^2} - rk, \qquad (2.20)$$

$$a_2^n = \frac{k}{2h^2}\left(\sigma^2 - \left(r - q - \frac{\sigma^2}{2}\right)h\right) - \frac{S_f^{n+1} - S_f^n}{2hS_f^n} = f - \frac{S_f^{n+1} - S_f^n}{2hS_f^n}.$$

From (2.12) and using the second order centred approximation of the boundary conditions (2.11) and (2.15) one gets

$$c_0^n = 0, \quad \frac{c_1^n - c_{-1}^n}{2h} = 0, \qquad (2.21)$$

$$\frac{\sigma^2}{2}\frac{c_{-1}^n - 2c_0^n + c_1^n}{h^2} - qS_f^n + r = 0, \qquad (2.22)$$

where c_{-1}^n means the value of the solution at the fictitious point $x = -h$, that should be eliminated later. From (2.21) and (2.22) the connection between the free boundary S_f^n and option value c_1^n on the same time level n is presented as

$$c_1^n = \frac{h^2}{\sigma^2}\left(qS_f^n - r\right), \quad n \ge 1. \qquad (2.23)$$

For the right boundary ($x = x_{\max}$) from (2.13) the Dirichlet's boundary condition is $c_{M+1}^n = 1$ for any $n \ge 0$.

We use together (2.19) for $j = 1$ and (2.23) at the time level $n+1$ to obtain the nonlinear law of the free boundary motion

$$S_f^{n+1} = d^n S_f^n, \tag{2.24}$$

$$d^n = \frac{bc_1^n + fc_2^n + \frac{c_2^n}{2h} + \frac{rh^2}{\sigma^2} + k\left(r - qS_f^n e^{-h}\right)}{\frac{c_2^n}{2h} + \frac{qh^2}{\sigma^2} S_f^n}. \tag{2.25}$$

3. Numerical Analysis

3.1. Positivity and monotonicity

In this section we will show qualitative scheme properties such as the free boundary non-decreasing monotonicity as well as the positivity and non-decreasing spacial monotonicity of the numerical option price under transformation by using the induction principle. The positivity of the coefficients a, b and f appearing in (2.20) will play an important role for obtaining this purpose. Note, that using expressions (2.20) it is easy to obtain that the constants of the scheme a, b and f are positive for both cases: $r \le q$ and $r > q$ under following conditions

$$h < \frac{\sigma^2}{\left|r - q - \frac{\sigma^2}{2}\right|}, \quad r \neq q + \frac{\sigma^2}{2}, \tag{3.1}$$

$$k < \frac{h^2}{\sigma^2 + rh^2}, \tag{3.2}$$

If $r = q + \frac{\sigma^2}{2}$, then under the condition (3.2), coefficients a, b and f are positive.

In order to show that the numerical free boundary S_f^n is increasing, from (2.24) we need to prove that $d^n > 1$. The case $n = 0$ deserves a special treatment because of the initial conditions (2.9) and (2.14). We have $c_j^0 \ge 0$ and $c_j^0 \le c_{j+1}^0$.

In order to provide numerical analysis of the scheme we have to estimate value c_2^n using the values on $n-th$ time level. Suppose, that the solution $c(x,\tau)$ is continuously differentiable up to fourth order. Then the Taylor's expansion in the node x_2 has the following form

$$c(x_2, \tau^n) = c(0,\tau^n) + 2h\frac{\partial c}{\partial x}(0,\tau^n) + 2h^2\frac{\partial^2 c}{\partial x^2}(0,\tau^n) + \frac{4h^3}{3}\frac{\partial^3 c}{\partial x^3}(0,\tau^n) + O(h^4) \tag{3.3}$$

From boundary conditions (2.12), (2.11) approximations (2.21) and (2.22), one gets

$$c_2^n = 4c_1^n + O(h^4). \tag{3.4}$$

For the sake of clarity in order to prove that $d^0 > 1$ and $\{c_j^1\}$ is an increasing sequence we distinguish two cases: $r > q$ and $r \leq q$.

Case $r > q$

From the initial conditions (2.9) it follows that

$$d^0 = 1 + \frac{\sigma^2 k}{h^2}\left(1 - e^{-h}\right) > 1. \tag{3.5}$$

Note that from the boundary conditions (2.12) and expressions (2.23), (3.4) one gets

$$c_1^1 = \frac{rh^2}{\sigma^2}\left(\frac{q}{r}S_f^1 - 1\right) > c_0^1 = 0, \quad c_2^1 = 4c_1^1 + O(h^4) > c_1^1. \tag{3.6}$$

From initial conditions (2.9), the values of the solution at interior mesh points are

$$c_j^1 - c_{j-1}^1 = a\left(c_{j-1}^0 - c_{j-2}^0\right) + b\left(c_j^0 - c_{j-1}^0\right) + f\left(c_{j+1}^0 - c_j^0\right) - \frac{d^0 - 1}{2h}\left(c_{j+1}^0 - c_{j-1}^0 - c_j^0 + c_{j-2}^0\right) + rke^{-jh}\left(e^h - 1\right), \quad j = 3, ..M. \tag{3.7}$$

Note that $c(x, 0)$, defined by (2.9), is a concave function for $x_{j-1} \geq \ln\frac{r}{q}$ and consequently verifies [3]

$$g(tx_{j-2} + (1-t)x_{j+1}) \geq tg(x_{j-2}) + (1-t)g(x_{j+1}). \tag{3.8}$$

Since $c_j^0 = g(x_j)$ by choosing $t = \frac{2}{3}$ and $t = \frac{1}{3}$ for the condition (3.8) one gets

$$c_{j-1}^0 \geq \frac{2}{3}c_{j-2}^0 + \frac{1}{3}c_{j+1}^0, \quad c_j^0 \geq \frac{1}{3}c_{j-2}^0 + \frac{2}{3}c_{j+1}^0. \tag{3.9}$$

If $x_{j+1} \leq \ln\frac{r}{q}$ function $c(x, 0)$ is a constant. If $x_j \leq \ln\frac{r}{q} < x_{j+1}$ or $x_{j-1} \leq \ln\frac{r}{q} < x_j$, then

$$c_{j-1}^0 + c_j^0 \geq c_{j-2}^0 + c_{j+1}^0. \tag{3.10}$$

Summarizing all possible cases, (3.10) holds true and from (3.7) it follows that $c_j^1 - c_{j-1}^1 \geq 0,\ j = 3, ...M$.

From the scheme (2.19) for $j = M$, since $\{c_j^0\}$ is increasing, one gets

$$c_M^1 \le (1 - rk)c_{M+1}^0 + k(r - qS_f^0 e^{-x_M}) = 1 - kqS_f^0 e^{-x_M} \le c_{M+1}^1 = 1. \quad (3.11)$$

Case $r \le q$

In that case initial conditions (2.9) are different, then d^0 has the form

$$d^0 - 1 = \frac{b\left(1 - e^{-h}\right) + f\left(1 - e^{-2h}\right) + k\left(r - qe^{-h}\right) + \frac{(r-q)h^2}{\sigma^2}}{\frac{1-e^{-2h}}{2h} + \frac{qh^2}{\sigma^2}} \ge$$

$$\frac{h\left(1 + (q - r)k\right) + k\frac{\sigma^2}{2}}{\frac{1-e^{-2h}}{2h} + \frac{qh^2}{\sigma^2}} = O(h), \quad (3.12)$$

since

$$e^{-jh} < 1 - jh, \quad \frac{1 - e^{-2h}}{2h} > 1. \quad (3.13)$$

It means that $S_f^1 > S_f^0 = 1$, then from boundary conditions (2.23), (3.4), $c_2^1 = 4c_1^1 > c_1^1 > c_0^1 = 0$. For $j > 2$ one gets $c_{j+1}^1 \ge c_j^1$ since initial function is concave for any $x \in [0; x_{\max}]$.

Assume, that for any $n > 1$

$$d^{n-1} > 1; \quad c_j^n \ge 0,\ j = 0, ..M + 1; \quad c_j^n \le c_{j+1}^n,\ j = 0, ..M. \quad (3.14)$$

Let us prove, that $d^n > 1$. From (2.25), denominator of d^n is positive. To guarantee $d^n > 1$, it is necessary that

$$(b - 1)c_1^n + fc_2^n + k\left(r - qS_f^n e^{-h}\right) > 0. \quad (3.15)$$

Using (2.23), (3.4) and Taylor's expansion for exponent function, the left-hand side of (3.15) can be presented for small enough k and h in form

$$(b - 1)c_1^n + fc_2^n + k\left(r - qS_f^n e^{-h}\right) \ge$$

$$\left((b - 1 + 4f)\frac{h^2}{\sigma^2} - k\right)\left(qS_f^n - r\right) + khqS_f^n + O(kh^2) \ge$$

$$khqS_f^n\left(1 - \frac{2\left(r - q - \frac{\sigma^2}{2}\right)}{\sigma^2}\right) + rkh\frac{2\left(r - q - \frac{\sigma^2}{2}\right)}{\sigma^2} + O(kh^2). \quad (3.16)$$

If $r-q-\frac{\sigma^2}{2} \le 0$, (3.16) is positive. If $r-q-\frac{\sigma^2}{2} > 0$, then by dividing last expression in (3.16) by $\frac{2kh\left(r-q-\frac{\sigma^2}{2}\right)}{\sigma^2} > 0$, one has to prove that

$$qS_f^n \frac{\sigma^2 - r + q}{r - q - \frac{\sigma^2}{2}} + r > 0. \tag{3.17}$$

If $r - q - \sigma^2 \le 0$ is fulfilled. Otherwise it holds true if

$$S_f^n < \frac{r\left(r-q-\frac{\sigma^2}{2}\right)}{q\left(r-q-\sigma^2\right)} = \frac{r}{q}\left(1 + \frac{\sigma^2}{2\left(r-q-\sigma^2\right)}\right). \tag{3.18}$$

Let us denote the critical asset price for perpetual American calls and puts respectively by $S_f^+(\infty)$ and $S_f^-(\infty)$, see [12]

$$S_f^+(\infty) = \frac{\alpha_+}{\alpha_+ - 1}, \quad S_f^-(\infty) = \frac{\alpha_-}{\alpha_- - 1}, \tag{3.19}$$

where

$$\alpha_\pm = \frac{1}{2\sigma^2}\left(\sigma^2 - 2(r-q) \pm \sqrt{4(r-q)^2 + 4(r+q)^2\sigma^2 + \sigma^4}\right). \tag{3.20}$$

If we consider polynomial $F(x) = (x-\alpha_-)(x-\alpha_+)$ and value

$$\alpha_* = \frac{S_f^*}{S_f^* - 1}, \tag{3.21}$$

where S_f^* is equal to right-hand side of inequality (3.18), then

$$F(\alpha_*) = -\frac{2\sigma^2 qr(2r - q - \sigma^2)}{\left(2(r-q)^2 + \sigma^2(2q-r)^2\right)^2} < 0. \tag{3.22}$$

Since $\alpha_- < \alpha_+$ and both are roots of convex polynomial $F(x)$, then from (3.22), it is clear that $\alpha_* < \alpha_+$. Using definitions (3.19) and (3.21), it can be shown that

$$\frac{1}{1 - \frac{1}{S_f^*}} < \frac{1}{1 - \frac{1}{S_f^+(\infty)}} \Rightarrow S_f^* > S_f^+(\infty). \tag{3.23}$$

Then the condition (3.18) can be presented in the following form

$$S_f^n < S_f^+(\infty) < S_f^*, \tag{3.24}$$

that is always fulfilled because critical asset price for perpetual American calls $S_f^+(\infty)$ represents an upper bound for the optimal exercise boundary [12].

We proved that $d^n > 1$. Moreover, from (3.16) and (3.4), $d^n = 1 + O(k)$. From (2.20) $a_1^n > a > 0$, and

$$a_2^n = \frac{k\sigma^2}{2h^2} - \left(r - q - \frac{\sigma^2}{2}\right)\frac{k}{2h} - \frac{d^n - 1}{2h} > 0. \tag{3.25}$$

Since all coefficients in the scheme (2.19) are positive, under (3.14)

$$c_{j+1}^{n+1} - c_j^{n+1} = a_1^n(c_j^n - c_{j-1}^n) + b(c_{j+1}^n - c_j^n) + a_2^n(c_{j+2}^n - c_{j+1}^n) + kqS_f^{n+1}e^{-jh}(1 - e^{-h}) > 0. \tag{3.26}$$

The positivity of the values c_j^{n+1} follows from the increasing behavior in index j and boundary condition (2.21).

Let us denote

$$y(z) = 1 + \frac{\left((b - 1 + 4f)\frac{qh^2}{\sigma^2} - kqe^{-h}\right)z + rk}{(2 + h)\frac{qh}{\sigma^2}z - \frac{2rh}{\sigma^2}}, \tag{3.27}$$

with negative derivative

$$\frac{dy}{dz} = -\frac{rqh}{\sigma^2}\frac{\left((b - 1 + 4f)\frac{h^2}{\sigma^2} - ke^{-h}\right) + (2 + h)k}{\left((2 + h)\frac{qh}{\sigma^2}z - \frac{2rh}{\sigma^2}\right)^2}. \tag{3.28}$$

for small enough values of h. Since $y(S_f^n) = d^n$, and $S_f^n > S_f^{n-1}$, then d^n is a decreasing discrete function of n. That means that $\{S_f^n\}$ has a concave behaviour.

Now the following results have been established:

Theorem 1. *Let $\{c_j^n, S_f^n\}$ be the numerical solution of scheme (2.19), (2.20), (2.21) for the transformed American call option problem (2.8) and let d^n be defined by (2.25). Then under conditions (3.1), (3.2), the numerical solution presents the following properties:*

i) *Increasing monotone concave behaviour and positivity of values S_f^n, $\quad n = 0, ..., N$;*
ii) *Non-negativity of the vectors $c^n = (c_0^n, ...c_{M+1}^n)$, $\quad n = 0, ..., N$;*
iii) *Increasing monotonicity of the vectors c^n with respect to space index for each fixed $n = 0, ..., N$.*

3.2. Stability and Consistency

In this section we are going to study the stability and consistency of the scheme. For the sake of clarity in the presentation knowing that several different concepts of the stability are used in the literature we begin the section with the following definition.

Definition 1. The numerical scheme (2.18) is said to be $||\cdot||_\infty$-stable in the fixed station sense in the domain $[0, x_\infty] \times [0, \tau]$, if for every partition with $k = \Delta\tau$, $h = \Delta x$, $Nk = \tau$ and $(M+1)h = x_\infty$,

$$||c^n||_\infty \leq A, \quad 0 \leq n \leq N, \tag{3.29}$$

where A is independent of h, k and $n = 0, .., N$ (see [16]).

Theorem 2. *Under assumptions of Theorem 1 the numerical scheme (2.19) for solving the transformed problem (2.8)-(2.14) is* $||\cdot||_\infty$*-stable.*

Proof of Theorem 2. Since c_j^n is non-decreasing vectors for each fixed n and from boundary condition (2.13),

$$||c^n||_\infty = c_{M+1}^n = 1. \tag{3.30}$$

The scheme is stable by the definition.

Consistency of the scheme

Consistency of a numerical scheme with respect to a partial differential equation means that the exact theoretical solution of the PDE approximates well the exact theoretical solution of the difference scheme as the step size discretization tends to zero [5]. Let us write the numerical scheme (2.18) in the form

$$F(c_j^n, S_f^n) = \frac{c_j^{n+1} - c_j^n}{k} - \frac{1}{2}\sigma^2 \frac{c_{j-1}^n - 2c_j^n + c_{j+1}^n}{h^2} + \left(r - q - \frac{\sigma^2}{2}\right) \frac{c_{j+1}^n - c_{j-1}^n}{2h} +$$

$$rc_j^n + \frac{S_f^{n+1} - S_f^n}{kS_f^n} \frac{c_{j+1}^n - c_{j-1}^n}{2h} + qS_f^n e^{-x_j} - r = 0. \tag{3.31}$$

In order to study the consistency let us take an arbitrary point (x,τ) in the domain $(0,\infty)\times(0,T]$ and consider the mesh points (x_j,τ^n) given by (2.17). Let us denote by $\tilde{c}_j^n = c(x_j,\tau^n)$ the exact theoretical solution value of the PDE (2.8) at the mesh point (x_j,τ^n), and let $\tilde{S}_f^n = S_f(\tau^n)$ be the exact solution of the free boundary at time τ^n. The scheme (3.31) is said to be consistent with

$$L(c,S_f) = \frac{\partial c}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 c}{\partial x^2} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial c}{\partial x} + rc + \frac{S_f'}{S_f}\frac{\partial c}{\partial x} + qS_f e^{-x} - r = 0, \tag{3.32}$$

if the local truncation error

$$T_j^n(\tilde{c},\tilde{S}_f) = F(\tilde{c}_j^n,\tilde{S}_f^n) - L(\tilde{c}_j^n,\tilde{S}_f^n), \tag{3.33}$$

satisfies

$$T_j^n(\tilde{c},\tilde{S}_f) \to 0, \quad as \quad h\to 0, \quad k\to 0. \tag{3.34}$$

Let us assume that the function $c(x,\tau)$ admits four times continuous partial derivatives with respect to x and twice continuous partial derivatives with respect to τ as well as the function $S_f(\tau)$ is twice differentiable. Using Taylor's expansion about (x_j,τ^n) one gets

$$T_j^n(\tilde{c},\tilde{S}_f) = kE_j^n(3) - \frac{\sigma^2}{2}h^2E_j^n(2) + \left(r - q - \frac{\sigma^2}{2}\right)h^2E_j^n(1) +$$

$$kE_j^n(4)\frac{\partial c}{\partial x}(x_j,\tau^n) + h^2E_j^n(1)\frac{1}{S_f(\tau^n)}\frac{dS_f}{d\tau}(\tau^n) + kh^2E_j^n(4)E_j^n(1), \tag{3.35}$$

where

$$E_j^n(1) = \frac{1}{6}\frac{\partial^3 c}{\partial x^3}(\xi_1,\tau^n), \quad x_i - h < \xi_1 < x_i + h, \tag{3.36}$$

$$\left|E_j^n(1)\right| \le \frac{1}{6}\max\left\{\left|\frac{\partial^3 c}{\partial x^3}(\xi,\tau^n)\right|; \quad 0\le\xi\le x_{\max}\right\} = \frac{1}{6}\left|W^n(1)\right|_{\max}; \tag{3.37}$$

$$E_j^n(2) = \frac{1}{12}\frac{\partial^4 c}{\partial x^4}(\xi_2,\tau^n), \; x_i - h < \xi_2 < x_i + h \tag{3.38}$$

$$\left|E_j^n(2)\right| \le \frac{1}{12}\max\left\{\left|\frac{\partial^4 c}{\partial x^4}(\xi,\tau^n)\right|; \quad 0\le\xi\le x_{\max}\right\} = \frac{1}{12}\left|W^n(2)\right|_{\max}; \tag{3.39}$$

$$E_j^n(3) = \frac{1}{2}\frac{\partial^2 c}{\partial \tau^2}(x_j, \eta^1), \ \tau^n < \eta^1 < \tau^{n+1}, \tag{3.40}$$

$$\left|E_j^n(3)\right| \le \frac{1}{2}\max\left\{\left|\frac{\partial^2 c}{\partial \tau^2}(x_j, \eta)\right|; \quad \tau^n < \eta < \tau^{n+1}\right\} = \frac{1}{2}\left|W_j^n(3)\right|_{\max}; \tag{3.41}$$

$$E_j^n(4) = \frac{1}{2S_f(\eta^2)}\frac{d^2 S_f}{d\tau^2}(\eta^2), \ \tau^n < \eta^2 < \tau^{n+1}, \tag{3.42}$$

$$\left|E_j^n(4)\right| \le \frac{1}{2}\max\left\{\left|\frac{1}{S_f(\eta)}\frac{d^2 S_f}{d\tau^2}(\eta)\right|; \quad \tau^n < \eta < \tau^{n+1}\right\} = \frac{1}{2}\left|W^n(4)\right|_{\max}; \tag{3.43}$$

Then the truncation error

$$\left|T_j^n(\tilde{c}, \tilde{S}_f)\right| \le \left(\frac{1}{2}\left|W_j^n(3)\right|_{\max} + \frac{1}{2}\left|W^n(4)\right|_{\max}\left|\frac{\partial c}{\partial x}(x_j, \tau^n)\right|\right)k +$$
$$\left(\frac{1}{6}\left|r - q - \frac{\sigma^2}{2}\right|\left|W^n(1)\right|_{\max} - \frac{\sigma^2}{24}\left|W^n(2)\right|_{\max} + \frac{1}{6}\left|W^n(1)\right|_{\max}\left|\frac{1}{S_f(\tau^n)}\frac{dS_f}{d\tau}(\tau^n)\right|\right)h^2 + \tag{3.44}$$
$$\frac{1}{12}\left|W^n(4)\right|_{\max}\left|W^n(1)\right|_{\max} kh^2.$$

Hence,

$$T_j^n(\tilde{c}, \tilde{S}_f) = O(h^2) + O(k). \tag{3.45}$$

With respect to the additional boundary condition (2.15), let us denote

$$L_{bc}(c, S_f) = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2}(0, \tau) - qS_f(\tau) + r = 0, \tag{3.46}$$

$$F_{bc}(c^n, S_f^n) = \frac{\sigma^2}{2}\frac{c_{-1}^n - 2c_0^n + c_1^n}{h^2} - qS_f^n + r = 0. \tag{3.47}$$

Truncation error satisfies $F_{bc}(\tilde{c}, \tilde{S}_f) - L_{bc}(\tilde{c}, \tilde{S}_f) = O(h^2)$. The truncation error for the boundary condition behaves as h^2. Analogously, truncation error for the boundary condition (2.11) satisfies second order in space because of the central difference approximation (2.21).

Theorem 3. *Assuming that the solution of the PDE problem (2.8)-(2.14) admits two times continuous partial derivative with respect to time and up to order four with respect to space, the numerical solution computed by the scheme (2.19), (2.20) is consistent with the equation (2.8) and boundary conditions (2.11), (2.15) of order two in space and order one in time.*

4. Numerical Results and Discussions

4.1. Efficiency and convergence rate

In order to compare computational efficiency of the method and to study the convergence rate, we consider the problem with the parameters [7]:

$$r = 0.03, \quad q = 0.03, \quad \sigma = 0.4, \quad T = 0.5, \quad E = 100. \tag{4.1}$$

Table 1 shows the comparison of the proposed method with other methods in [7]. Since exact values are not known, the results of the binomial method with large steps (15000) are used for "True Value". FDP stands for the Crank-Nicolson finite-difference method with projected SOR iteration to impose the free boundary condition. FDE stands for the Crank-Nicolson finite difference method with elimination-back substitution. HW stands for the Han and Wu method (see [7]). FF stands for the proposed explicit finite difference method combined with the front-fixing transformation with stepsizes: FF1 for $h = 2 \cdot 10^{-3}$ and $k = 2 \cdot 10^{-5}$, FF2 for $h = 2 \cdot 10^{-3}$ and $k = 6.25 \cdot 10^{-6}$, FF3 for $h = 10^{-3}$ and $k = 6.25 \cdot 10^{-6}$. The root-mean-square error (RMSE) is used to measure the accuracy of the scheme. The last row presents the CPU-time in seconds for each experiment.

Results presented in Table 1 show the competitiveness of the proposed method.

It was theoretically proved in previous section that the scheme has order of approximation $O(h^2) + O(k)$. To check numerically the order of approximation in space we fix the time step and introduce the convergence rate

$$\gamma(h_1, h_2) = \frac{\ln RMSE(FF1) - \ln RMSE(FF2)}{\ln h_1 - \ln h_2}. \tag{4.2}$$

From the Table 1 and (4.2), one obtains $\gamma(2 \cdot 10^{-3}, 10^{-3}) = 1.9036$, that is close to 2. To check the order of approximation in time the space step h is fixed ($h = 2 \cdot 10^{-3}$) and time step k is variable. From the Table 2 by analogous to (4.2) formula one gets $\gamma(2 \cdot 10^{-5}, 6.25 \cdot 10^{-6}) = 0.8343$, that is close to 1. It proves the second order of approximation in space and the first order in time.

Table 1: Comparison of the computational efficiency for the problem with the parameters (4.1).

Asset Price	True Value	FDP	FDE	HW	FF1	FF2	FF3
40	0.002792	0.0025	0.0025	0.0028	0.0025	0.0027	0.0028
50	0.045594	0.0457	0.0457	0.0457	0.0451	0.0453	0.0456
60	0.301387	0.3014	0.3015	0.3015	0.3005	0.3011	0.3015
70	1.145799	1.1459	1.1461	1.1461	1.1442	1.1451	1.1456
80	3.041536	3.0414	3.0415	3.0415	3.0401	3.0411	3.0413
90	6.328677	6.3285	6.3287	6.3287	6.3266	6.3274	6.3284
100	11.108407	11.1066	11.1068	11.1068	11.1051	11.1072	11.1080
110	17.266726	17.2664	17.2665	17.2665	17.2632	17.2653	17.2663
120	24.564972	24.5654	24.5655	24.5655	24.5603	24.5641	24.5648
	RMSE	6.4217-4	5.8822-4	5.8012-4	2.4771-3	9.3865-4	2.5088-4
	CPU-time, sec	37.130	15.760	11.500	7.169	27.805	32.794

4.2. Newton's method

In this subsection we present an implicit finite-difference scheme for the problem (2.8) - (2.14).

$$\frac{c_j^{n+1} - c_j^n}{k} = \frac{1}{2}\sigma^2 \frac{c_{j-1}^{n+1} - 2c_j^{n+1} + c_{j+1}^{n+1}}{h^2} -$$

$$\left(r - q - \frac{\sigma^2}{2} + \frac{S_f^{n+1} - S_f^n}{kS_f^{n+1}} \right) \frac{c_{j+1}^{n+1} - c_{j-1}^{n+1}}{2h} - rc_j^{n+1} + r - qS_f^{n+1}e^{-x_j}, \quad j = 1, ...M, \tag{4.3}$$

$$c_0^{n+1} = 0, \quad c_{M+1}^{n+1} = 1, \tag{4.4}$$

$$\frac{\sigma^2}{h^2}c_1^{n+1} - qS_f^{n+1} + r = 0. \tag{4.5}$$

Writing the finite-difference equations (4.3) and introducing the boundary conditions (4.4) and the discretization of the free boundary (4.5), a nonlinear system of equation is obtained. We denote by $Y^l = [c_1^{n+1} \; ... \; c_M^{n+1} \; S_f^{n+1}]^T$

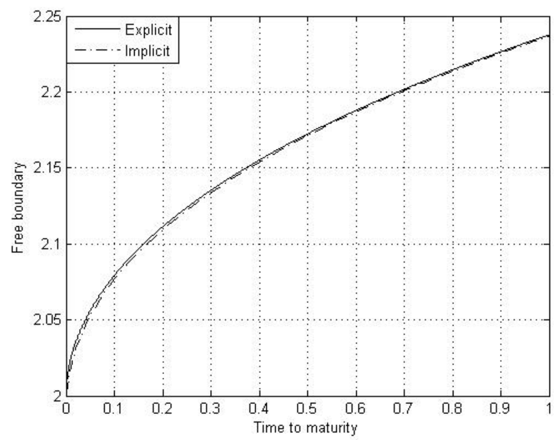

Figure 1: Free boundary motion for the problem with the parameters $r = 0.1, \quad q = 0.05, \quad \sigma = 0.2, \quad T = 1$ by explicit and implicit method.

vector of $M + 1$ unknowns on the l-th iteration. This nonlinear system is solved by widely used Newton method and extensions [10], [14].

$$Y^{l+1} = Y^l - J^{-1}F^l, \tag{4.6}$$

where F^l is matrix, obtained from (4.3) and (4.5) by substituting Y^l. J is Jacobian of the system. The iteration process is done until $\|Y^{l+1} - Y^l\| < \varepsilon$ for a given error tolerance ε.

As a numerical experiment we compare explicit and implicit method with $h = 0.01$ and different time steps: $k = 10^{-4}$ for the explicit method to guarantee condition (3.2), $k = 0.01$ for the implicit method. The results are presented on the Figure 1.

It is well known that implicit method is unconditionally stable and there is no any restrictions on the time step k. This fact allows to reduce computational time. But, there are additional calculations of the inverse Jacobian matrix on each iteration. The computational time for both methods is presented in Table 2. It is shown that for the same step sizes the explicit method is ten times faster than implicit one. In the case of the smaller space steps for the explicit method we have to choose time steps much smaller, but the total computational time is almost ten times less.

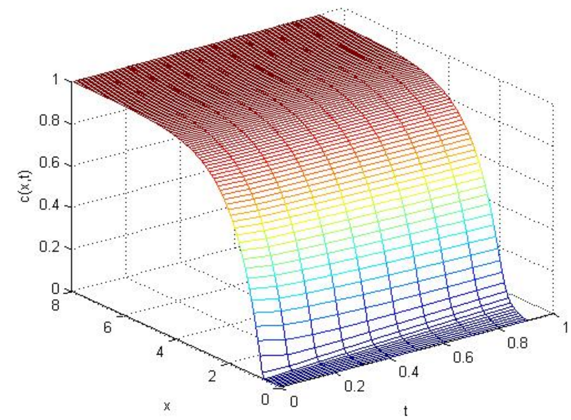

Figure 2: The function $c(x,\tau)$ calculated by the implicit method.

Method	Space step	Time step	Computational time (sec.)	Free Boundary
Explicit	10^{-1}	10^{-2}	0.016	2.2283
	10^{-2}	10^{-4}	3.693	2.2375
	10^{-3}	10^{-6}	28.918	2.2376
Implicit	10^{-1}	10^{-2}	0.179	2.2269
	10^{-2}	10^{-2}	16.029	2.2368
	10^{-3}	10^{-2}	107.435	2.2375

Table 2: Comparison of the computational time and accuracy for explicit and implicit methods.

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