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RECENT ADVANCES IN THE ANALYSIS OF POINTWISE STATE-CONSTRAINED ELLIPTIC OPTIMAL CONTROL PROBLEMS *

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Abstract. Optimal control problems for semilinear elliptic equations with control constraints and pointwise state constraints are studied. Several theoretical results are derived, which are necessary to carry out a numerical analysis for this class of control problems. In particular, sufficient second-order optimality conditions, some new regularity results on optimal controls and a sufficient condition for the uniqueness of the Lagrange multiplier associated with the state constraints are presented.

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1. Introduction

In this paper, we consider different issues of state-constrained optimal control problems for semilinear elliptic equations, which seem to be important for a related numerical analysis. In the last years, state-constrained optimal control problems have attracted increasing attention. There are several reasons for this remarkable activity in the research on state-constrained problems.

First of all, state constraints are very important in various applications of the optimal control of PDEs. Moreover, in contrast to control-constrained problems, many interesting questions are still open or not yet satisfactorily solved with state constraints. This concerns in particular second-order optimality conditions, the error analysis for finite element approximations of the problems, numerical algorithms, and the associated convergence analysis.

Let us motivate our results presented here in the context of recent developments in the numerical approximation of state-constrained optimal control problems by finite elements, where only a few results are known.

Keywords and phrases. Optimal control, pointwise state constraints, first and second order optimality conditions, Lagrange multipliers, Borel measures.

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In [3,5], error estimates for approximated locally optimal controls were shown for problems with semilinear elliptic equation and finitely many state constraints. Recently, in [17], higher order error estimates were established for a similar setting with control vectors instead of control functions.

A study of these three papers reveals that second-order sufficient conditions at (locally) optimal controls are indispensable to obtain results on convergence or approximation of optimal controls. This is due to the nonconvex character of the problems with nonlinear equations. It is meanwhile known that second-order sufficient optimality conditions are fairly delicate under the presence of state constraints. In [9], second-order sufficient conditions were established, which are, in some sense, closest to associated necessary ones and admit a form similar to the theory of nonlinear programming in finite-dimensional spaces. Here, we briefly discuss this result and show its equivalence to an earlier form stated in [8] that was quite difficult to explain.

Error estimates for the approximated optimal controls of problems with pointwise state-constraints were derived by Deckelnick and Hinze [10,11] and Meyer [18], who all consider linear-quadratic problems. It is known from the control-constrained case that optimal estimates need a precise information on the regularity of the optimal control. Therefore, the smoothness of optimal controls is a key question. We show for a problem with pointwise state constraints in the whole domain that the optimal control is Lipschitz, if the state constraints are only active at finitely many points.

We also present a counterexample that this result is not true for infinitely many active points. On the other hand, we prove the somehow surprising result that optimal controls belong to $H^1(\Omega)$ no matter how large the active set is. Moreover, we discuss the uniqueness of the Lagrange multiplier associated with the state-constraints.

2. The control problem

Let Ω be an open, connected and bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with a Lipschitz boundary Γ. In this domain we consider the following state equation

$$\left\{ \begin{array}{rcll} Ay + a_0(x, y) & = & u & \text{in } \Omega, \\ y & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{2.1}$$

where $a_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and A denotes a second-order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x))$$

and the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy

$$\lambda_A \|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n,\ \forall x \in \Omega$$

for some $\lambda_A > 0$. In (2.1), the function u denotes the control and we will denote by y_u the solution associated to u. We will state later the conditions leading to the existence and uniqueness of a solution of (2.1) in $C(\bar{\Omega}) \cap H^1(\Omega)$.

The optimal control problem is formulated as follows

$$
\text{(P)}\begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,\mathrm{d}x + \frac{N}{2}\int_\Omega u(x)^2\,\mathrm{d}x \\ \text{subject to } (y_u, u) \in (C(\bar\Omega) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ \alpha(x) \le u(x) \le \beta(x) \quad \text{for a.e. } \ x \in \Omega, \\ a(x) \le g(x, y_u(x)) \le b(x) \ \ \forall x \in K, \end{cases}
$$

where K is as compact subset of $\bar\Omega$.

We impose the following assumptions on the data of the control problem.

(A1) In the whole paper a real number $N > 0$ and functions α, β are given in $L^\infty(\Omega)$, with $\alpha \le \beta$ a.e. in Ω. We introduce the sets

$$
\begin{aligned} \mathcal{U}_{\alpha,\beta} &= \{u \in L^\infty(\Omega)\colon \alpha(x) \le u(x) \le \beta(x) \ \text{ a.e. in } \Omega\} \\ \mathcal{U}_{ad} &= \{u \in \mathcal{U}_{\alpha,\beta}\colon a(x) \le g(x, y_u(x)) \le b(x) \ \forall x \in K\}. \end{aligned}
$$

(A2) The mapping $a_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and there exists a real number $p > n/2$ such that

$$
a_0(\cdot, 0) \in L^p(\Omega), \qquad \frac{\partial a_0}{\partial y}(x, y) \ge 0 \qquad \text{for a.e. } x \in \Omega.
$$

For all $M > 0$, there exists a constant $C_{0,M} > 0$ such that

$$
\begin{gathered} \left|\frac{\partial a_0}{\partial y}(x, y)\right| + \left|\frac{\partial^2 a_0}{\partial y^2}(x, y)\right| \le C_{0,M} \text{ for a.e. } x \in \Omega \text{ and for all } |y| \le M, \\ \left|\frac{\partial^2 a_0}{\partial y^2}(x, y_2) - \frac{\partial^2 a_0}{\partial y^2}(x, y_1)\right| \le C_{0,M}|y_2 - y_1| \text{ for a.e. } x \in \Omega, \ \forall |y_1|, |y_2| \le M. \end{gathered}
$$

(A3) $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable, $L(\cdot, 0) \in L^1(\Omega)$, and for all $M > 0$ there exist a constant $C_{L,M} > 0$ and a function $\psi_M \in L^1(\Omega)$ such that

$$
\left|\frac{\partial L}{\partial y}(x, y)\right| \le \psi_M(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x, y)\right| \le C_{L,M},
$$

$$
\left|\frac{\partial^2 L}{\partial y^2}(x, y_2) - \frac{\partial^2 L}{\partial y^2} L(x, y_1)\right| \le C_{L,M}(|y_2 - y_1|),
$$

for a.e. $x \in \Omega$ and $|y|, |y_i| \le M$, $i = 1, 2$.

(A4) The function $g : K \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, together with its derivatives $(\partial^j g/\partial y^j) \in C(K \times \mathbb{R})$ for $j = 1, 2$. We also assume that $a, b : K \longrightarrow [-\infty, +\infty]$ are measurable functions, with $a(x) < b(x)$ for every $x \in K$, such that their domains

$$
\mathrm{Dom}(a) = \{x \in K\colon -\infty < a(x)\} \ \text{ and } \ \mathrm{Dom}(b) = \{x \in K\colon b(x) < \infty\}
$$

are closed sets and a and b are continuous on their respective domains. Finally, we assume that either $K \cap \Gamma = \emptyset$ or $a(x) < g(x,0) < b(x)$ holds for every $x \in K \cap \Gamma$. We will denote

$$\mathcal{Y}_{ab} = \{z \in C(K)\colon a(x) \le z(x) \le b(x)\ \forall x \in K\}.$$

Let us remark that a (b) can be identically equal to $-\infty$ ($+\infty$), which means that we only have upper (lower) bounds on the state. Thus the above framework define a quite general formulation for the pointwise state constraints.

Example. With fixed functions y_d, $e \in L^2(\Omega)$, a power $\lambda \ge 2$, and real constants $\alpha < \beta$, $a < b$, the following particular problem satisfies all of our assumptions:

$$\min J(u) := \frac{1}{2}\int_\Omega (y_u(x) - y_d(x))^2\,\mathrm{d}x + \frac{N}{2}\int_\Omega u(x)^2\,\mathrm{d}x$$

subject to

$$\begin{array}{rcl} Ay + |y|^\lambda y &=& u \quad \text{in } \Omega,\\ y &=& 0 \quad \text{on } \Gamma,\end{array}$$

$$\alpha \le u(x) \le \beta \quad \text{for a.e.} \quad x \in \Omega, \qquad a \le y_u(x) \le b \quad \forall x \in K.$$

Another interesting example for the semilinear term of the state equation can be $a_0(x,y) = \theta_1(x) + \theta_2(x)\exp(y)$, with $\theta_1 \in L^p(\Omega)$, $\theta_2 \in L^\infty(\Omega)$ and $\theta_2(x) \ge 0$.

Let us state first the existence and uniqueness of the solution corresponding to the state equation (2.1).

Theorem 2.1. *Under assumption* (A2)*, equation* (2.1) *has a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar\Omega)$ for every $u \in L^2(\Omega)$. If the coefficients of A, $\{a_{ij}\}_{i,j=1}^n$, are Lipschitz functions in $\bar\Omega$, then $y_u \in H^2(\Omega)$ if Ω is convex and $y_u \in W^{2,p}(\Omega)$ if Γ is of class $C^{1,1}$ and $u \in L^p(\Omega)$.*

It is well known that, for every $u \in L^p(\Omega)$, equation (2.1) has a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar\Omega)$. A proof of this result can be obtained by the usual cut off process applied to a_0, then applying the Schauder fix point theorem combined with the monotonicity of a_0 with respect to the second variable and the L^∞ estimates for the state; *cf.* Stampacchia [20]. The continuity of y_u is proved in [12]. The $W^{2,p}(\Omega)$ and $H^2(\Omega)$ estimates can be found in Grisvard [13]. For details, the reader is referred to [1] or [2].

Now the existence of an optimal control can be proved by using standard arguments.

Theorem 2.2. *If the set of controls $\mathcal{U}_{ad}$ is not empty, then the control problem* (P) *has at least one solution.*

In the rest of the paper, $\bar u$ denotes a local minimum of (P) in the sense of the $L^\infty(\Omega)$-topology and $\bar y$ will be its associated state. The pair $(\bar y, \bar u)$ is our local reference solution. At this local solution, we will assume the linearized Slater condition:

(A5) There exists $u_0 \in \mathcal{U}_{\alpha,\beta}$ such that

$$a(x) < g(x,\bar y(x)) + \frac{\partial g}{\partial y}(x,\bar y(x)) z_0(x) < b(x) \quad \forall x \in K, \tag{2.2}$$

where $z_0 \in H_0^1(\Omega) \cap C(\bar\Omega)$ is the unique solution of

$$\left\{\begin{array}{rcll} Az + \dfrac{\partial a_0}{\partial y}(x,\bar y) z &=& u_0 - \bar u & \text{in } \Omega\\ z &=& 0 & \text{on } \Gamma.\end{array}\right. \tag{2.3}$$

Taking into account that a and b are continuous on their domains $\mathrm{Dom}(a)$ and $\mathrm{Dom}(b)$ respectively and these sets are compact subsets of K (see assumption (A4)), we deduce that (2.2) is equivalent to the existence of real $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$a(x) < \tau_1 < g(x,\bar{y}(x)) + \frac{\partial g}{\partial y}(x,\bar{y}(x))z_0(x) < \tau_2 < b(x) \quad \forall x \in K. \tag{2.4}$$

3. First and second order optimality conditions

Before deriving the first order optimality conditions satisfied by the local minimum $\bar{u}$, we recall some results about the differentiability of the mappings involved in the control problem. For the proofs, the reader is referred to Casas and Mateos [5], where a Neumann boundary condition was considered instead of a Dirichlet condition, which we consider in this paper. However, the method of proof is very similar and the changes are obvious.

Theorem 3.1. *If* (A2) *and* (A3) *hold, then the mapping* $G : L^2(\Omega) \longrightarrow C(\bar{\Omega}) \cap H_0^1(\Omega)$, *defined by* $G(u) = y_u$ *is of class* C^2. *Moreover, for all* $u, v \in L^2(\Omega)$, $z_v = G'(u)v$ *is defined as the solution of*

$$\begin{cases} Az_v + \dfrac{\partial a_0}{\partial y}(x,y_u)z_v = v & \text{in } \Omega \\ z_v = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Finally, for every $v_1, v_2 \in L^2(\Omega)$, $z_{v_1v_2} = G''(u)v_1v_2$ *is the solution of*

$$\begin{cases} Az_{v_1v_2} + \dfrac{\partial a_0}{\partial y}(x,y_u)z_{v_1v_2} + \dfrac{\partial^2 a_0}{\partial y^2}(x,y_u)z_{v_1}z_{v_2} = 0 & \text{in } \Omega \\ z_{v_1v_2} = 0 & \text{on } \Gamma, \end{cases} \tag{3.2}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

Remark 3.2. The assumption $n \le 3$ is required to make the second order optimality conditions work, because the differentiability of G from $L^2(\Omega)$ to $C(\bar{\Omega})$ is needed for the associated proof. This result holds true only for $n \le 3$.

Theorem 3.3. *Suppose that* (A2) *and* (A3) *hold. Then* $J : L^2(\Omega) \to \mathbb{R}$ *is a functional of class* C^2. *Moreover, for every* $u, v, v_1, v_2 \in L^2(\Omega)$

$$J'(u)v = \int_\Omega (\varphi_{0u} + Nu)\, v \,\mathrm{d}x \tag{3.3}$$

and

$$J''(u)v_1v_2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u)z_{v_1}z_{v_2} + Nv_1v_2 - \varphi_{0u}\frac{\partial^2 a_0}{\partial y^2}(x,y_u)z_{v_1}z_{v_2}\right] \mathrm{d}x, \tag{3.4}$$

where $y_u = G(u)$ *and* $\varphi_{0u} \in W_0^{1,s}(\Omega)$, *for all* $s < n/(n-1)$, *is the unique solution of the problem*

$$\begin{cases} A^*\varphi + \dfrac{\partial a_0}{\partial y}(x,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,y_u) & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma, \end{cases} \tag{3.5}$$

A^* *being the adjoint operator of* A *and* $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

The previous theorem and the next one follow easily from Theorem 3.1 and the chain rule.

Theorem 3.4. *If* (A2) *and* (A3) *hold, then the mapping* $F : L^2(\Omega) \to C(K)$*, defined by* $F(u) = g(\cdot, y_u(\cdot))$*, is of class* C^2*. Moreover, for every* $u, v, v_1, v_2 \in L^2(\Omega)$

$$F'(u)v = \frac{\partial g}{\partial y}(\cdot, y_u(\cdot))z_v(\cdot) \tag{3.6}$$

and

$$F''(u)v_1v_2 = \frac{\partial^2 g}{\partial y^2}(\cdot, y_u(\cdot))z_{v_1}(\cdot)z_{v_2}(\cdot) + \frac{\partial g}{\partial y}(\cdot, y_u(\cdot))z_{v_1v_2}(\cdot) \tag{3.7}$$

where $z_{v_i} = G'(u)v_i$*,* $i = 1, 2$*, and* $z_{v_1v_2} = G''(u)v_1v_2$*.*

Before stating the first order necessary optimality conditions, let us fix some notation. We denote by $M(K)$ the Banach space of all real and regular Borel measures in K, which is identified with the dual space of $C(K)$. The following result is well known, it follows from the Pontryagin principle that was derived in [7].

Theorem 3.5. *Let* $\bar{u}$ *be a local solution of* (P) *and suppose that the assumptions* (A1)–(A5) *hold. Then there exist a measure* $\bar{\mu} \in M(K)$ *and a function* $\bar{\varphi} \in W_0^{1,s}(\Omega)$*, for all* $1 \le s < n/(n-1)$*, such that*

$$\left\{\begin{array}{rcll} A^*\bar{\varphi} + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\bar{\varphi} &=& \dfrac{\partial L}{\partial y}(x, \bar{y}) + \dfrac{\partial g}{\partial y}(x, \bar{y}(x))\bar{\mu} & \textit{in } \Omega, \\ \bar{\varphi} &=& 0 & \textit{on } \Gamma, \end{array}\right. \tag{3.8}$$

$$\int_K (z(x) - g(x, \bar{y}(x)))\mathrm{d}\bar{\mu}(x) \le 0 \quad \forall z \in \mathcal{Y}_{ab}, \tag{3.9}$$

$$\int_\Omega (\bar{\varphi} + N\bar{u})(u - \bar{u})\,\mathrm{d}x \ge 0 \quad \forall u \in \mathcal{U}_{\alpha,\beta}. \tag{3.10}$$

Remark 3.6. Inequality (3.9) is equivalent with the well-known complementary slackness conditions. Along with the constraint $a(x) \le g(x, \bar{y}(x)) \le b(x)$, it implies that the support of $\bar{\mu}$ is in the set

$$K_0 = K_a \cup K_b$$

with

$$K_a = \{x \in K\colon g(x, \bar{y}(x)) = a(x)\} \text{ and } K_b = \{x \in K\colon g(x, \bar{y}(x)) = b(x)\}.$$

The Lebesgue decomposition of $\bar{\mu} = \mu_+ - \mu_-$ into the positive and negative part of the measure μ shows that $\operatorname{supp}\mu_+ \subset K_b$ and $\operatorname{supp}\mu_- \subset K_a$. Because of this property and the assumption (A5), we have that $\operatorname{supp}\bar{\mu} \cap \Gamma = \emptyset$. Notice that the continuity of a and b on their respective domains (assumption (A4)) implies that K_a and K_b are closed subsets.

Remark 3.7. From (3.10) it follows for almost all $x \in \Omega$ that

$$\bar{u}(x) = \operatorname{Proj}_{[\alpha(x),\beta(x)]}\left(-\frac{1}{N}\bar{\varphi}(x)\right) = \max\{\alpha(x), \min\{\bar{\varphi}(x), \beta(x)\}\}. \tag{3.11}$$

Let us formulate also the Lagrangian version of the optimality conditions (3.8)–(3.10). The Lagrange function $\mathcal{L} : L^2(\Omega) \times M(K) \longrightarrow \mathbb{R}$ associated with problem (P) is defined by

$$\mathcal{L}(u, \mu) = J(u) + \int_K g(x, y_u(x))\,\mathrm{d}\mu(x).$$

Using (3.3) and (3.6) we find that

$$\frac{\partial \mathcal{L}}{\partial u}(u,\mu)v = \int_\Omega (\varphi_u(x) + Nu(x))\, v(x)\,\mathrm{d}x, \tag{3.12}$$

where $\varphi_u \in W_0^{1,s}(\Omega)$, for all $1 \le s < n/(n-1)$, is the solution of the Dirichlet problem

$$\begin{cases} A^*\varphi + \dfrac{\partial a_0}{\partial y}(x,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,y_u) + \dfrac{\partial g}{\partial y}(x,y_u(x))\mu & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{3.13}$$

Now the inequality (3.10) along with (3.12) leads to

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)(u-\bar u) \ge 0 \quad \forall u \in \mathcal{U}_{\alpha,\beta}. \tag{3.14}$$

Before we set up the sufficient second order optimality conditions, we evaluate the expression of the second derivative of the Lagrangian with respect to the control. From (3.7) we get

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v_1v_2 = J''(u)v_1v_2 + \int_K \left[\frac{\partial^2 g}{\partial y^2}(x,y_u(x))z_{v_1}(x)z_{v_2}(x) + \frac{\partial g}{\partial y}(x,y_u(x))z_{v_1v_2}(x)\right]\mathrm{d}\mu(x).$$

By (3.2) and (3.4), this is equivalent to

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v_1v_2 &= \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u)z_{v_1}z_{v_2} + Nv_1v_2 - \varphi_u\frac{\partial^2 a_0}{\partial y^2}(x,y_u)z_{v_1}z_{v_2}\right]\mathrm{d}x \\ &\quad + \int_K \frac{\partial^2 g}{\partial y^2}(x,y_u(x))z_{v_1}(x)z_{v_2}(x)\,\mathrm{d}\mu(x),\end{aligned} \tag{3.15}$$

where φ_u is the solution of (3.13).

Associated with $\bar u$, we define the cone of critical directions by

$$C_{\bar u} = \{v \in L^2(\Omega)\colon v \text{ satisfies (3.16), (3.17) and (3.18) below}\},$$

$$v(x) = \begin{cases} \ge 0 \text{ if } \bar u(x) = \alpha(x), \\ \le 0 \text{ if } \bar u(x) = \beta(x), \\ = 0 \text{ if } \bar\varphi(x) + N\bar u(x) \ne 0, \end{cases} \tag{3.16}$$

$$\frac{\partial g}{\partial y}(x,\bar y(x))z_v(x) = \begin{cases} \ge 0 & \text{if } x \in K_a \\ \le 0 & \text{if } x \in K_b, \end{cases} \tag{3.17}$$

$$\int_K \frac{\partial g}{\partial y}(x,\bar y(x))z_v(x)\,\mathrm{d}\bar\mu(x) = 0, \tag{3.18}$$

where $z_v \in H_0^1(\Omega) \cap C(\bar\Omega)$ satisfies

$$\begin{cases} Az_v + \dfrac{\partial a_0}{\partial y}(x,\bar y)z_v = v & \text{in } \Omega \\ z_v = 0 & \text{on } \Gamma. \end{cases}$$

The relation (3.17) expresses the natural sign conditions, which must be fulfilled for feasible directions at active points $x \in K_a$ or K_b, respectively. On the other hand, (3.18) states that the derivative z_v must be zero whenever the corresponding Lagrange multiplier is non-vanishing. This restriction is needed for second-order sufficient conditions. Compared with the finite dimensional case, this is exactly what we can expect. Therefore the relations (3.17)–(3.18) provide a convenient extension of the usual conditions of the finite-dimensional case.

We should mention that (3.18) is new in the context of infinite-dimensional optimization problems. In earlier papers on this subject, other extensions to the infinite-dimensional case were suggested. For instance, Maurer and Zowe [16] used first-order sufficient conditions to account for the strict positivity of Lagrange multipliers. Inspired by their approach, in [8] an application to state-constrained elliptic boundary control was suggested by the authors. In terms of our problem, equation (3.18) was relaxed by

$$\int_K \frac{\partial g}{\partial y}(x,\bar y(x))z_v(x)\,\mathrm{d}\bar\mu(x) \geq -\varepsilon \int_{\{x:|\bar\varphi(x)+N\bar u(x)|\leq\tau\}} |v(x)|\,\mathrm{d}x$$

for some $\varepsilon > 0$ and $\tau > 0$, *cf.* [8], (5.15). In the next theorem, which was proven in [9], Theorem 4.3, we will see that this relaxation is not necessary. We obtain a smaller cone of critical directions that seems to be optimal. However, the reader is referred to Theorem 3.9 below, where we consider the possibility of relaxing the conditions defining the cone $C_{\bar u}$.

Theorem 3.8. *Assume that* (A1)–(A4) *hold. Let $\bar u$ be a feasible control of problem* (P)*, $\bar y$ the associated state and $(\bar\varphi,\bar\mu) \in W_0^{1,s}(\Omega)\times M(K)$, for all $1 \leq s < n/(n-1)$, satisfying* (3.8)–(3.10)*. Assume further that*

$$\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar u,\bar\mu)v^2 > 0 \quad \forall v \in C_{\bar u}\setminus\{0\}. \tag{3.19}$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that the following inequality holds:

$$J(\bar u) + \frac{\delta}{2}\|u-\bar u\|^2_{L^2(\Omega)} \leq J(u) \quad \text{if } \|u-\bar u\|_{L^2(\Omega)} \leq \varepsilon \text{ and } u \in \mathcal{U}_{ad}. \tag{3.20}$$

The condition (3.19) seems to be natural. In fact, under some regularity assumption, we can expect the inequality

$$\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar u,\bar\mu)v^2 \geq 0 \quad \forall v \in C_{\bar u}$$

to be necessary for local optimality. At least, this is the case when the state constraints are of integral type, see [5], or when K is a finite set of points, see [4]. In the general case of (P), to our best knowledge, the necessary second order optimality conditions are still open.

We finish this section by establishing an equivalent condition to (3.19) that is more convenient for the numerical analysis of problem (P). Let us introduce a cone $C^\tau_{\bar u}$ of critical directions that is bigger than $C_{\bar u}$. Given $\tau > 0$, we define

$$C^\tau_{\bar u} = \{v \in L^2(\Omega)\,|\,v \text{ satisfies (3.21)–(3.23) below}\},$$

$$v(x) = \begin{cases} \geq 0 \text{ if } \bar u(x) = \alpha(x), \\ \leq 0 \text{ if } \bar u(x) = \beta(x), \\ = 0 \text{ if } |\bar\varphi(x)+N\bar u(x)| > \tau, \end{cases} \tag{3.21}$$

$$\frac{\partial g}{\partial y}(x,\bar y(x))z_v(x) = \begin{cases} \geq -\tau\|v\|_{L^2(\Omega)} & \text{if } x \in K_a \\ \leq +\tau\|v\|_{L^2(\Omega)} & \text{if } x \in K_b, \end{cases} \tag{3.22}$$

$$\int_K \frac{\partial g}{\partial y}(x,\bar y(x))z_v(x)\,\mathrm{d}\bar\mu(x) \geq -\tau\|v\|_{L^2(\Omega)}. \tag{3.23}$$

Theorem 3.9. *Under the assumptions* (A1)–(A4)*, relation* (3.19) *holds if and only if there exist* $\tau > 0$ *and* $\rho > 0$ *such that*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})v^2 \geq \rho \|v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}. \tag{3.24}$$

Proof. Since $C_{\bar{u}} \subset C^\tau_{\bar{u}}$, it is clear that (3.24) implies (3.19). Let us prove by contradiction that (3.24) follows from (3.19). Assume that (3.19) holds but not (3.24). Then, for all positive integers k and all $\rho = \tau = 1/k$, there exists an element $v_k \in C^{1/k}_{\bar{u}}$ such that (3.24) is not satisfied, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})v_k^2 < \frac{1}{k}\|v_k\|^2_{L^2(\Omega)}. \tag{3.25}$$

Redefining v_k by $v_k/\|v_k\|_{L^2(\Omega)}$ and selecting a subsequence, if necessary, denoted in the same way, we can assume that

$$\|v_k\|_{L^2(\Omega)} = 1, \quad v_k \rightharpoonup v \text{ weakly in } L^2(\Omega) \text{ and } \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})v_k^2 < \frac{1}{k}, \tag{3.26}$$

and from (3.21)–(3.23)

$$v_k(x) = \begin{cases} \geq 0 \text{ if } \bar{u}(x) = \alpha(x), \\ \leq 0 \text{ if } \bar{u}(x) = \beta(x), \\ = 0 \text{ if } |\bar{\varphi}(x) + N\bar{u}(x)| > 1/k, \end{cases} \tag{3.27}$$

$$\frac{\partial g}{\partial y}(x, \bar{y}(x))z_{v_k}(x) = \begin{cases} \geq -1/k & \text{if } x \in K_a \\ \leq +1/k & \text{if } x \in K_b, \end{cases} \tag{3.28}$$

$$\int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_{v_k}(x)\,\mathrm{d}\bar{\mu}(x) \geq -1/k. \tag{3.29}$$

Since $z_{v_k} \to z_v$ strongly in $H^1_0(\Omega) \cap C(\bar{\Omega})$, we can pass to the limit in (3.26)–(3.29) and get that $v \in C_{\bar{u}}$ and

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})v^2 \leq 0. \tag{3.30}$$

This is only possible if $v = 0$; see (3.19). Let us note that the only delicate point to prove that $v \in C_{\bar{u}}$ is to establish (3.18). Indeed, (3.16) and (3.17) follow easily from (3.27) and (3.28). Passing to the limit in (3.29) we get

$$\int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_v(x)\,\mathrm{d}\bar{\mu}(x) \geq 0.$$

This inequality, along with (3.17) and the structure of $\bar{\mu}$, implies (3.18).

Therefore, we have that $v_k \rightharpoonup 0$ weakly in $L^2(\Omega)$ and $z_{v_k} \to 0$ strongly in $H^1_0(\Omega) \cap C(\bar{\Omega})$. Hence, using the expression (3.15) of the second derivative of the Lagrangian we get

$$N = \liminf_{k\to\infty} N\|v_k\|^2_{L^2(\Omega)} \leq \liminf_{k\to\infty} \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})v_k^2 \leq 0,$$

which is a contradiction. □

4. Regularity of the optimal control

In this section, the existence of the second derivatives of the functions involved in the control problem is not needed (*cf.* assumptions (A2)–(A4)). Let us start with the following well known regularity result for the optimal control:

Theorem 4.1. *If* $(\bar y,\bar u) \in (H_0^1(\Omega)\cap C(\bar\Omega))\times L^\infty(\Omega)$ *is a feasible pair for problem* (P) *and* $(\bar y,\bar u,\bar\varphi)$ *with* $\varphi \in \times W_0^{1,s}(\Omega)$ *satisfies the optimality system* (3.8)–(3.10)*, then* $\bar u \in W^{1,s}(\Omega)$ *for all* $s < n/(n-1)$ *and* $\bar u \in C(\bar\Omega\backslash K_0)$*.*

Since $\alpha,\ \beta,\ \bar\varphi \in W^{1,s}(\Omega)$ for every $1 \le s < n/(n-1)$, the regularity $\bar u \in W^{1,s}(\Omega)$ follows immediately from (3.11). The continuity $\bar u \in C(\bar\Omega \setminus K_0)$ is deduced in the same way. This regularity result on the control $\bar u$ can be improved if there is a finite number of points, where the state constraints are active. More precisely, let us assume that $K_0 = \{x_j\}_{j=1}^m \subset \Omega$. Then Remark 3.6 implies that

$$\bar\mu = \sum_{j=1}^m \bar\lambda_j \delta_{x_j}, \quad \text{with } \bar\lambda_j = \begin{cases} \ge 0 & \text{if } g(x_j,\bar y(x_j)) = b(x_j), \\ \le 0 & \text{if } g(x_j,\bar y(x_j)) = a(x_j), \end{cases} \tag{4.1}$$

where δ_{x_j} denotes the Dirac measure centered at x_j. If we denote by $\bar\varphi_j$, $1 \le j \le m$, and $\bar\varphi_0$ the solutions of

$$\begin{cases} A^*\bar\varphi_j + \dfrac{\partial a_0}{\partial y}(x,\bar y(x))\bar\varphi_j = \delta_{x_j} & \text{in } \Omega, \\ \bar\varphi_j = 0 & \text{on } \Gamma, \end{cases} \tag{4.2}$$

and

$$\begin{cases} A^*\bar\varphi_0 + \dfrac{\partial a_0}{\partial y}(x,\bar y(x))\bar\varphi_0 = \dfrac{\partial L}{\partial y}(x,\bar y) & \text{in } \Omega, \\ \bar\varphi_0 = 0 & \text{on } \Gamma, \end{cases} \tag{4.3}$$

then the adjoint state associated to $\bar u$ is given by

$$\bar\varphi = \bar\varphi_0 + \sum_{j=1}^m \bar\lambda_j \frac{\partial g}{\partial y}(x_j,\bar y(x_j))\bar\varphi_j. \tag{4.4}$$

Now we have the following regularity result.

Theorem 4.2. *Assume* $p > n$ *in assumption* (A2) *and* $\psi_M \in L^p(\Omega)$ *in* (A3)*. Suppose also that* $a_{i,j},\ \alpha,\ \beta \in C^{0,1}(\bar\Omega)$*,* $1 \le i,\ j \le n$ *and that* Γ *is of class* $C^{1,1}$*. Let* $(\bar y,\bar u,\bar\varphi) \in H_0^1(\Omega)\cap C(\bar\Omega)\times L^\infty(\Omega)\times W_0^{1,s}(\Omega)$*, for all* $1 \le s < n/(n-1)$*, satisfy the optimality system* (3.8)–(3.10)*. If the active set consists of finitely many points,* i.e. $K_0 = \{x_j\}_{j=1}^m \subset \Omega$*, then* $\bar u$ *belongs to* $C^{0,1}(\bar\Omega)$ *and* $\bar y$ *to* $W^{2,p}(\Omega)$*.*

Since $p > n$, it holds that $W^{2,p}(\Omega) \subset C^1(\bar\Omega)$ and therefore $\bar\varphi_0 \in C^1(\bar\Omega)$. On the other hand, $\bar\varphi_j(x) \to +\infty$ when $x \to x_j$, hence $\bar\varphi$ has singularities at the points x_j where $\bar\lambda_j \ne 0$. Consequently $\bar\varphi$ cannot be Lipschitz.

Surprisingly, this does not lower the regularity of $\bar u$: Notice that (3.11) implies that $\bar u$ is identically equal to α or β in a neighborhood of x_j, depending on the sign of $\bar\lambda_j$. This implies the desired result; see Casas [4] for the details.

Now the question arises if this Lipschitz property remains also valid for an infinite number of points where the pointwise state constraints are active. Unfortunately, the answer is negative. In fact, the optimal control can even fail to be continuous if K_0 is an infinite and numerable set. Let us present an associated

Counterexample. We set

$$\Omega = \{x \in \mathbb{R}^2 \colon \|x\| < \sqrt{2}\},\ \bar y(x) = \begin{cases} 1 & \text{if } \|x\| \le 1 \\ 1 - (\|x\|^2 - 1)^4 & \text{if } 1 < \|x\| \le \sqrt{2}, \end{cases}$$

$$K = \{x^k\}_{k=1}^\infty \cup \{x^\infty\},\ \text{where } x^k = \left(\frac{1}{k}, 0\right) \text{ and } x^\infty = (0,0),\ \bar\mu = \sum_{k=1}^\infty \frac{1}{k^2}\delta_{x^k}.$$

Now we define $\bar\varphi \in W_0^{1,s}(\Omega)$ for all $1 \le s < n/(n-1)$ as the solution of the equation

$$\left\{ \begin{array}{rcll} -\Delta\bar\varphi & = & \bar y + \bar\mu & \text{in } \Omega, \\ \bar\varphi & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{4.5}$$

The function $\bar\varphi$ can be decomposed in the following form

$$\bar\varphi(x) = \bar\psi(x) + \sum_{k=1}^{\infty} \frac{1}{k^2}[\psi_k(x) + \phi(x - x^k)],$$

where $\phi(x) = -(1/2\pi)\log\|x\|$ is the fundamental solution of $-\Delta$ and the functions $\bar\psi, \psi_k \in C^2(\bar\Omega)$ satisfy

$$\left\{ \begin{array}{rcll} -\Delta\bar\psi(x) & = & \bar y(x) & \text{in } \Omega, \\ \bar\psi(x) & = & 0 & \text{on } \Gamma, \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\Delta\psi_k(x) & = & 0 & \text{in } \Omega, \\ \psi_k(x) & = & -\phi(x - x^k) & \text{on } \Gamma. \end{array} \right.$$

Finally we set

$$\left\{ \begin{array}{l} M = \left|\bar\psi(0) + \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\psi_k(0)\right| + \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\phi(x^k) + 1, \\ \bar u(x) = \mathrm{Proj}_{[-M,+M]}(-\bar\varphi(x)) \end{array} \right. \tag{4.6}$$

and $a_0(x) = \bar u(x) + \Delta\bar y(x)$. Then $\bar u$ is the unique global solution of the control problem

$$(\mathrm{Q}) \quad \left\{ \begin{array}{l} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u^2(x) + u^2(x))\,\mathrm{d}x \\ \text{subject to } (y_u, u) \in (C(\bar\Omega) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ -M \le u(x) \le +M \quad \text{for a.e. } x \in \Omega, \\ -1 \le y_u(x) \le +1 \quad \forall x \in K, \end{array} \right.$$

where y_u is the solution of

$$\left\{ \begin{array}{rcll} -\Delta y + a_0(x) & = & u & \text{in } \Omega, \\ y & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{4.7}$$

As a first step to prove that $\bar u$ is a solution of problem, we verify that M is a real number: Since $\{\phi(x-x^k)\}_{k=1}^\infty$ is bounded in $C^2(\Gamma)$, the sequence $\{\psi_k\}_{k=1}^\infty$ is bounded in $C^2(\bar\Omega)$. Therefore, the convergence of the first series of (4.6) is obvious. The convergence of the second one is also clear,

$$\sum_{k=1}^{\infty} \frac{1}{k^2}\phi(x^k) = \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{1}{k^2}\log k < \infty.$$

Problem (Q) is strictly convex and $\bar u$ is a feasible control with associated state $\bar y$ satisfying the state constraints. Therefore, there exists a unique solution characterized by the optimality system. In other words, the first order optimality conditions are necessary and sufficient for a global minimum. Let us check that $(\bar y, \bar u, \bar\varphi, \bar\mu) \in H_0^1(\Omega) \cap C(\bar\Omega) \times L^\infty(\Omega) \times W_0^{1,s}(\Omega) \times M(K)$ satisfies the optimality system (3.8)–(3.10). First, in view of the definition of a_0, it is clear that $\bar y$ is the state associated to $\bar u$. On the other hand, $\bar\varphi$ is the solution of (4.5), which is the same as (3.8) for our example. Relation (3.10) follows directly from the definition of $\bar u$ given in (4.6). Finally, because of the definition of $\bar\mu$ and K, (3.9) can be written in the form

$$\sum_{k=1}^{\infty} \frac{1}{k^2} z(x^k) \le \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \forall z \in C(K) \text{ such that } -1 \le z(x) \le +1 \quad \forall x \in K,$$

which obviously is satisfied.

Now we prove that $\bar u$ is not continuous at $x = 0$. Notice that $\bar\varphi(x^k) = +\infty$ for every $k \in \mathbb{N}$, because $\phi(0) = +\infty$. Therefore, (4.6) implies that $\bar u(x^k) = -M$ for every k. Since $x^k \to 0$, the continuity of $\bar u$ at $x = 0$ requires that $\bar u(x) \to -M$ as $x \to 0$. However, we have for $\xi^j = (x^j + x^{j+1})/2$ that

$$\begin{aligned}\lim_{j\to\infty} \bar\varphi(\xi^j) &= \lim_{j\to\infty}\left(\bar\psi(\xi^j) + \sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(\xi^j) + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(\xi^j - x^k)\right)\\ &= \bar\psi(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(x^k),\end{aligned} \tag{4.8}$$

as we will verify below. Moreover, by the definition of M,

$$\left|\bar\psi(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(x^k)\right| \le \left|\bar\psi(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(0)\right| + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(x^k) = M - 1 < M,$$

therefore (4.8) implies $\lim_{j\to\infty}\bar\varphi(\xi^j) > -M$ and hence $\lim_{j\to\infty}\bar u(\xi^j) > -M$ by the definition (4.6) of $\bar u$, in contrary to $\bar u(x) \to -M$ as $x \to 0$.

Let us finally justify (4.8). To this end, we only have to show that

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(\xi^j) + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(\xi^j - x^k) \to \sum_{k=1}^{\infty}\frac{1}{k^2}\psi_k(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\phi(x^k),$$

as $j \to \infty$. This holds true, if the associated function series are uniformly convergent. For the first series, this is easy to see since the functions ψ_k are uniformly bounded on $\bar\Omega$, hence a multiple of $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a dominating series. Uniform convergence follows by the Weierstrass theorem. The second series is more delicate. To find a dominating series, we estimate the items as follows: We consider

$$|\phi(\xi^j - x^k)| = -\frac{1}{2\pi}\log\left|\frac{1}{2}\left[\frac{1}{j} + \frac{1}{j+1}\right] - \frac{1}{k}\right|.$$

The right-hand side can admit its maximum only at $j = k$ or $j = k - 1$ as one can easily confirm. Therefore, this maximum is certainly smaller than the sum of both values,

$$\begin{aligned}|\phi(\xi^j - x^k)| &\le -\frac{1}{2\pi}\left\{\log\left|\frac{1}{2}\left[\frac{1}{k} + \frac{1}{k+1}\right] - \frac{1}{k}\right| + \log\left|\frac{1}{2}\left[\frac{1}{k-1} + \frac{1}{k}\right] - \frac{1}{k}\right|\right\}\\ &= -\frac{1}{2\pi}\left\{\log\frac{1}{2k(k+1)} + \log\frac{1}{2k(k-1)}\right\} \le \frac{1}{\pi}\log[2k(k+1)].\end{aligned}$$

On the other hand,

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\log[2k(k+1)] = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\frac{\log[2k(k+1)]}{\sqrt{k}}$$

is obviously convergent, since $\log[2k(k+1)]/\sqrt{k} \to 0$ as $k \to \infty$.

Nevertheless, we are able to improve the regularity result of Theorem 4.1.

Theorem 4.3. *Suppose that $\bar u$ is a strict local minimum of* (P) *in the sense of the $L^2(\Omega)$ topology. We also assume that assumptions* (A1)–(A5) *hold, α, $\beta \in L^\infty(\Omega) \cap H^1(\Omega)$, $a_{ij} \in C(\bar\Omega)$ $(1 \le i, j \le n)$ and $\psi_M \in L^p(\Omega)$, $p > n/2$, in* (A3). *Then $\bar u \in H^1(\Omega)$.*

Let us remark that any global solution of (P) is a local solution of (P) in the sense of $L^2(\Omega)$, but we can expect to have more local or global solutions in the sense of $L^2(\Omega)$. Theorem 3.8 implies that $\bar u$ is at least a strict local minimum in the sense of $L^2(\Omega)$, if the sufficient second-order optimality conditions are satisfied at $\bar u$. This guarantees that $\bar u$ is the unique global solution in an $L^2(\Omega)$-neighborhood. However, the quadratic growth condition alone does not imply that $\bar u$ is an isolated minimum. The control $\bar u$ might be an accumulation point of different local minima.

Proof of Theorem 4.3. Fix $\varepsilon_{\bar u} > 0$ such that $\bar u$ is a strict global minimum of (P) in the closed ball $\bar B_{\varepsilon_{\bar u}}(\bar u) \subset L^2(\Omega)$. This implies that $\bar u$ is the unique global solution of the problem

$$(\mathrm{P}_0)\begin{cases}\min J(u)\\ \text{subject to } (y_u,u)\in (C(\bar\Omega)\cap H^1(\Omega))\times L^\infty(\Omega),\\ \alpha(x)\le u(x)\le\beta(x) \quad \text{for a.e. } x\in\Omega, \quad \|u-\bar u\|_{L^2(\Omega)}\le\varepsilon_{\bar u}\\ a(x)\le g(x,y_u(x))\le b(x) \quad \forall x\in K,\end{cases}$$

where y_u is the solution of (2.1).

Now we select a sequence $\{x_k\}_{k=1}^\infty$ being dense in $\mathrm{Dom}(a)\cup\mathrm{Dom}(b)$ and consider the family of control problems

$$(\mathrm{P}_k)\begin{cases}\min J(u)\\ \text{subject to } (y_u,u)\in (C(\bar\Omega)\cap H^1(\Omega))\times L^\infty(\Omega),\\ \alpha(x)\le u(x)\le\beta(x) \quad \text{for a.e. } x\in\Omega, \quad \|u-\bar u\|_{L^2(\Omega)}\le\varepsilon_{\bar u}\\ a(x_j)\le g(x_j,y_u(x_j))\le b(x_j), \quad 1\le j\le k.\end{cases}$$

Obviously, $\bar u$ is a feasible control for every problem (P_k). Therefore, the existence of a global minimum u_k of (P_k) follows easily by standard arguments.

The proof of the theorem is split into three steps: First, we show that the sequence $\{u_k\}_{k=1}^\infty$ converges to $\bar u$ strongly in $L^2(\Omega)$. In a second step, we will check that the linearized Slater condition corresponding to problem (P_k) holds for all sufficiently large k. Finally, we confirm the boundedness of $\{u_k\}_{k=1}^\infty$ in $H^1(\Omega)$.

Step 1: Convergence of $\{u_k\}_{k=1}^\infty$. By taking a subsequence, if necessary, we can suppose that $u_k \rightharpoonup \tilde u$ weakly in $L^2(\Omega)$. This implies that $y_k = y_{u_k}\to\tilde y = y_{\tilde u}$ strongly in $H_0^1(\Omega)\cap C(\bar\Omega)$. Because of the density of $\{x_k\}_{k=1}^\infty$ in $\mathrm{Dom}(a)\cup\mathrm{Dom}(b)$ and the fact that

$$a(x_j)\le g(x_j,\tilde y(x_j)) = \lim_{k\to\infty} g(x_j,y_k(x_j))\le b(x_j) \quad \forall j\ge 1,$$

it holds that $a(x)\le g(x,\tilde y(x))\le b(x)$ for every $x\in K$. The control constraints define a closed and convex subset of $L^2(\Omega)$, hence $\tilde u$ satisfies all the control constraints. Therefore $\tilde u$ is a feasible control for problem (P_0). Since $\bar u$ is the solution of (P_0), u_k is a solution of (P_k), and $\bar u$ is feasible for every problem (P_k), we have $J(u_k)\le J(\bar u)$ and further

$$J(\bar u)\le J(\tilde u)\le \liminf_{k\to\infty} J(u_k)\le\limsup_{k\to\infty} J(u_k)\le J(\bar u).$$

Since $\bar u$ is the unique solution of (P_0), this implies $\bar u=\tilde u$ and $J(u_k)\to J(\bar u)$, hence the strong convergence $u_k\to\bar u$ in $L^2(\Omega)$ follows from this convergence along with the uniform convergence $y_k\to\bar y$.

Step 2: The linearized Slater condition for (P_k) holds at u_k. Assumption (A5) and (2.4) ensure the existence of a number $\rho>0$ such that for every $x\in K$ the following inequalities hold

$$a(x)+\rho<\tau_1+\rho<g(x,\bar y(x))+\frac{\partial g}{\partial y}(x,\bar y(x))z_0(x)<\tau_2-\rho<b(x)-\rho. \tag{4.9}$$

Given $0 < \varepsilon < 1$, we multiply the previous inequality by ε and the inequality $a(x) \leq g(x, \bar{y}(x)) \leq b(x)$ by $1 - \varepsilon$. Next, we add both inequalities to get

$$a(x) + \varepsilon\rho < g(x, \bar{y}(x)) + \varepsilon \frac{\partial g}{\partial y}(x, \bar{y}(x)) z_0(x) < b(x) - \varepsilon\rho \quad \forall x \in K. \tag{4.10}$$

We fix

$$0 < \varepsilon < \min\left\{1, \frac{\varepsilon_{\bar{u}}}{\|u_0 - \bar{u}\|_{L^2(\Omega)}}\right\}$$

and define $u_{0,\varepsilon} = \varepsilon(u_0 - \bar{u}) + \bar{u}$. It is obvious that $u_{0,\varepsilon}$ satisfies the control constraints of problem (P_k) for any k. Consider now the solutions z_k of the boundary value problem

$$\left\{\begin{array}{rcll} Az + \dfrac{\partial a_0}{\partial y}(x, y_k) z & = & u_{0,\varepsilon} - u_k & \text{in } \Omega \\ z & = & 0 & \text{on } \Gamma. \end{array}\right.$$

In view of $u_k \to \bar{u}$ in $L^2(\Omega)$ and $y_k \to \bar{y}$ in $C(\bar{\Omega})$, we obtain the convergence $z_k \to \varepsilon z_0$ in $H_0^1(\Omega) \cap C(\bar{\Omega})$. Finally, from (4.10) we deduce the existence of $k_0 > 0$ such that

$$a(x_j) + \varepsilon\frac{\rho}{2} \leq g(x_j, y_k(x_j)) + \frac{\partial g}{\partial y}(x_j, y_k(x_j)) z_k(x_j) \leq b(x_j) - \varepsilon\frac{\rho}{2} \tag{4.11}$$

for $1 \leq j \leq k$ and every $k \geq k_0$.

Step 3: $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^1(\Omega)$. The strong convergence $u_k \to \bar{u}$ in $L^2(\Omega)$ implies that $\|u_k - \bar{u}\|_{L^2(\Omega)} < \varepsilon_{\bar{u}}$ for k large enough. Therefore, u_k does not touch the boundary of the ball $\{u \,|\, \|u - \bar{u}\|_{L^2(\Omega)} \leq \varepsilon_{\bar{u}}\}$. Consequently, the additional constraint $\|u_k - \bar{u}\|_{L^2(\Omega)} \leq \varepsilon_{\bar{u}}$ is not active, and hence u_k is a local minimum of the problem

$$(\mathrm{Q}_k) \left\{\begin{array}{l} \min J(u) \\ \text{subject to } (y_u, u) \in (C(\bar{\Omega}) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ \alpha(x) \leq u(x) \leq \beta(x) \quad \text{for a.e. } x \in \Omega, \\ a(x_j) \leq g(x_j, y_u(x_j)) \leq b(x_j) \ \ 1 \leq j \leq k. \end{array}\right.$$

Now we can apply Theorem 3.5 and deduce

$$u_k(x) = \mathrm{Proj}_{[\alpha(x),\beta(x)]}\left(-\frac{1}{N}\varphi_k(x)\right), \tag{4.12}$$

with

$$\varphi_k = \varphi_{k,0} + \sum_{j=1}^{k} \lambda_{k,j} \frac{\partial g}{\partial y}(x_j, y_k(x_j)) \varphi_{k,j}. \tag{4.13}$$

Above, $\{\lambda_{k,j}\}_{j=1}^{k}$ are the Lagrange multipliers, more precisely

$$\mu_k = \sum_{j=1}^{k} \lambda_{k,j} \delta_{x_j}, \quad \text{with } \lambda_{k,j} = \left\{\begin{array}{ll} \geq 0 & \text{if } g(x_j, y_k(x_j)) = b(x_j), \\ \leq 0 & \text{if } g(x_j, y_k(x_j)) = a(x_j). \end{array}\right. \tag{4.14}$$

Finally, $\varphi_{k,0}$ and $\{\varphi_{k,j}\}_{j=1}^k$ are given by

$$\begin{cases} A^*\varphi_{k,0} + \dfrac{\partial a_0}{\partial y}(x, y_k(x))\varphi_{k,0} = \dfrac{\partial L}{\partial y}(x, y_k) & \text{in } \Omega, \\ \varphi_{k,0} = 0 & \text{on } \Gamma, \end{cases} \tag{4.15}$$

$$\begin{cases} A^*\varphi_{k,j} + \dfrac{\partial a_0}{\partial y}(x, y_k(x))\varphi_{k,j} = \delta_{x_j} & \text{in } \Omega, \\ \varphi_{k,j} = 0 & \text{on } \Gamma. \end{cases} \tag{4.16}$$

Let us prove the following boundedness property:

$$\exists C > 0 \text{ such that } \|\mu_k\|_{M(K)} = \sum_{j=1}^k |\lambda_{k,j}| \leq C \quad \forall k. \tag{4.17}$$

Indeed, from (4.11) we get

$$\begin{cases} \lambda_{k,j} > 0 \Rightarrow g(x_j, y_k(x_j)) = b(x_j) \Rightarrow \dfrac{\partial g}{\partial y}(x_j, y_k(x_j))z_k(x_j) \leq -\varepsilon\dfrac{\rho}{2} \\ \lambda_{k,j} < 0 \Rightarrow g(x_j, y_k(x_j)) = a(x_j) \Rightarrow \dfrac{\partial g}{\partial y}(x_j, y_k(x_j))z_k(x_j) \geq +\varepsilon\dfrac{\rho}{2}. \end{cases}$$

Next, in view of (3.14) and $u_{0,\varepsilon} = \varepsilon(u_0 - \bar{u}) + \bar{u}$ we obtain

$$\begin{aligned} 0 \leq \frac{\partial \mathcal{L}}{\partial u}(u_k, \mu_k)(u_{0,\varepsilon} - u_k) &= J'(u_k)(u_{0,\varepsilon} - u_k) + \sum_{j=1}^k \lambda_{k,j}\frac{\partial g}{\partial y}(x_j, y_k(x_j))z_k(x_j) \\ &\leq J'(u_k)(u_{0,\varepsilon} - u_k) - \varepsilon\frac{\rho}{2}\sum_{j=1}^k |\lambda_{k,j}|. \end{aligned}$$

This implies that

$$\sum_{j=1}^k |\lambda_{k,j}| \leq \frac{2}{\varepsilon\rho}J'(u_k)(u_{0,\varepsilon} - u_k) \to \frac{2}{\rho}J'(\bar{u})(u_0 - \bar{u}) \text{ when } k \to \infty,$$

hence (4.17) holds. Now (4.13), (4.15) and (4.16) lead to

$$\begin{cases} A^*\varphi_k + \dfrac{\partial a_0}{\partial y}(x, y_k(x))\varphi_k = \dfrac{\partial L}{\partial y}(x, y_k) + \dfrac{\partial g}{\partial y}(x, y_k(x))\mu_k \text{ in } \Omega, \\ \varphi_k = 0 \text{ on } \Gamma. \end{cases} \tag{4.18}$$

Let us set

$$C_{\alpha,\beta} = \|\alpha\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} + 1$$

and

$$v_k(x) = \text{Proj}_{[-C_{\alpha,\beta}, +C_{\alpha,\beta}]}\left(-\frac{1}{N}\varphi_k(x)\right).$$

From the last relation and (4.12) it follows that

$$u_k(x) = \mathrm{Proj}_{[\alpha(x),\beta(x)]}(v_k(x)).$$

Notice that the trace of u_k on Γ is not necessarily zero, if $0 \notin U_{\alpha,\beta}$. Therefore it is a delicate question to multiply equation (4.18) by u_k and to integrate by parts. However, v_k vanishes on Γ and hence the previous operation can be done without difficulty and we will do it later.

The goal is to prove that $\{v_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$, which yields the boundedness of $\{u_k\}_{k=1}^\infty$ in the same space. The last claim is an immediate consequence of

$$|\nabla u_k(x)| \le |\nabla v_k(x)| + |\nabla\alpha(x)| + |\nabla\beta(x)| \quad \text{a.e. } \Omega.$$

If $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$, then $\bar u \in H^1(\Omega)$ obviously.

Let us prove the boundedness of $\{v_k\}_{k=1}^\infty$ in $H_0^1(\Omega)$. The solution of a Dirichlet problem associated with an elliptic operator with coefficients $a_{ij} \in C(\bar\Omega)$ and Lipschitz boundary Γ belongs to $W_0^{1,r}(\Omega)$, if the right hand side is in $W^{-1,r}(\Omega)$ for any $n < r < n+\varepsilon_n$, where $\varepsilon_n > 0$ depends on n and $n \in \{2,3\}$; *cf.* Jerison and Kenig [14] and Mateos [15]. If $p > n/2$, then $L^p(\Omega) \subset W^{-1,2p}(\Omega)$ and consequently $\varphi_{k,0} \in W_0^{1,r}(\Omega)$ holds for all r in the range indicated above with $r < 2p$.

In view of this, we have $\varphi_k \in W_0^{1,s}(\Omega) \cap W^{1,r}(\Omega\backslash S_k)$, where S_k is the set of points x_j such that $\lambda_{k,j}(\partial g/\partial y)(x_j, y_k(x_j)) \neq 0$. Notice that by (4.13) only these $\varphi_{k,j}$ appear in the representation of φ_k. Taking into account that v_k is constant in a neighborhood of every point $x_j \in S_k$, we deduce that $v_k \in W_0^{1,r}(\Omega) \subset C(\bar\Omega)$. Therefore, we are justified to multiply equation (4.18) by $-v_k$ and to integrate by parts. We get

$$-\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)\partial_{x_i} v_k \partial_{x_j}\varphi_k + \frac{\partial a_0}{\partial y}(x,y_k)v_k\varphi_k\right)\mathrm{d}x = -\int_\Omega \frac{\partial L}{\partial y}(x,y_k)v_k\,\mathrm{d}x - \sum_{j=1}^k \lambda_{k,j}\frac{\partial g}{\partial y}(x_j,y_k(x_j))v_k(x_j). \tag{4.19}$$

From the definition of v_k we obtain for a.a. $x \in \Omega$

$$\nabla v_k(x) = \begin{cases} -\dfrac{1}{N}\nabla\varphi_k(x) & \text{if } -C_{\alpha,\beta} \le -\dfrac{1}{N}\varphi_k(x) \le +C_{\alpha,\beta} \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Invoking this property in (4.19) along with the boundedness of $\{y_k\}_{k=1}^\infty$ in $C(\bar\Omega)$, the estimate $\|v_k\|_{L^\infty(\Omega)} \le C_{\alpha,\beta}$, and the assumptions (A3) and (A4), we get

$$\lambda_A N\int_\Omega |\nabla v_k|^2\,\mathrm{d}x \le \|\psi_M\|_{L^2(\Omega)}\|v_k\|_{L^2(\Omega)} + C\sum_{j=1}^k |\lambda_{k,j}|\,\|v_k\|_{L^\infty(\Omega)} \le C'.$$

Clearly, this implies that $\{v_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$ as required. □

5. On the uniqueness of the Lagrange multiplier $\bar\mu$

In this section, we provide a sufficient condition for the uniqueness of the Lagrange multiplier associated with the state constraints. We also analyze some situations, where these conditions are satisfied. It is known that a non-uniqueness of Lagrange multipliers may lower the efficiency of numerical methods, *e.g.* primal-dual active set methods. Moreover, some other theoretical properties of optimization problems depend on the uniqueness of multipliers. Therefore, this is desirable property.

Theorem 5.1. *Assume* (A1)–(A5) *and the existence of some* $\varepsilon > 0$ *such that*

$$T : L^2(\Omega_\varepsilon) \longrightarrow C(K_0), \ \textit{with}\ Tv = \frac{\partial g}{\partial y}(x, \bar{y}(x))z_v, \ \textit{has a dense range}, \tag{5.1}$$

where

$$\Omega_\varepsilon = \{x \in \Omega\colon \alpha(x) + \varepsilon < \bar{u}(x) < \beta(x) - \varepsilon\},$$

$z_v \in H_0^1(\Omega) \cap C(\bar{\Omega})$ *satisfies*

$$\left\{\begin{array}{rcll} Az_v + \dfrac{\partial a_0}{\partial y}(x, \bar{y})z_v & = & v & \text{in } \Omega \\ z_v & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{5.2}$$

and v *is extended by zero to the whole domain* Ω*. Then there exists a unique Lagrange multiplier* $\mu \in M(K)$ *such that* (3.8)–(3.10) *hold.*

Proof. Let us assume to the contrary that $\bar{\mu}_i$, $i = 1, 2$, are two Lagrange multipliers associated to the state constraints corresponding to the optimal control $\bar{u}$. Then (3.14) holds for $\bar{\mu} = \bar{\mu}_i$, $i = 1, 2$. Taking $v \in L^\infty(\Omega_\varepsilon)\setminus\{0\}$ arbitrarily, we have for a.e. $x \in \Omega$

$$\alpha(x) \leq u_\rho(x) = \bar{u}(x) + \rho v(x) \leq \beta(x) \quad \forall |\rho| < \frac{\varepsilon}{\|v\|_{L^\infty(\Omega_\varepsilon)}},$$

where v is extended by zero to the whole domain Ω. Inserting $u = u_\rho$ in (3.14), with positive and negative ρ and remembering that $\operatorname{supp} \bar{\mu}_i \subset K_0$ (Rem. 3.6), we deduce

$$J'(\bar{u})v + \int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_v(x)\, \mathrm{d}\bar{\mu}_i(x) = \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu}_i)v = 0, \quad i = 1, 2,$$

which leads to

$$\langle \bar{\mu}_1, Tv\rangle = -J'(\bar{u})v = \langle \bar{\mu}_2, Tv\rangle \quad \forall v \in L^\infty(\Omega_\varepsilon).$$

Since $L^\infty(\Omega_\varepsilon)$ is dense in $L^2(\Omega_\varepsilon)$ and $T(L^2(\Omega_\varepsilon))$ is dense in $C(K_0)$ we obtain from the above identity that $\bar{\mu}_1 = \bar{\mu}_2$. □

Remark 5.2. For a finite set $K = \{x_j\}_{j=1}^n$, assumption (5.1) is equivalent to the independence of the gradients $\{G_j'(\bar{u})\}_{j\in I_0}$ in $L^2(\Omega_\varepsilon)$, where the functions $G_j : L^2(\Omega_\varepsilon) \longrightarrow \mathbb{R}$ are defined by $G_j(u) = g(x_j, y_u(x_j))$ and I_0 is the set of indexes j corresponding to active constraints. It is a regularity assumption on the control problem at $\bar{u}$. This type of assumption was introduced by the authors in [6] to analyze control constrained problems with finitely many state constraints. The first author proved in [4] that, under very general hypotheses, this assumption is equivalent to the Slater condition in the case of a finite number of pointwise state constraints.

We show finally that (5.1) holds under some more explicit assumptions on $\bar{u}$ and on the set of points K_0, where the state constraint is active.

Theorem 5.3. *Assume that* (A1)–(A5) *hold and that the coefficients* a_{ij} *belong to* $C^{0,1}(\bar{\Omega})$ $(1 \leq i,\ j \leq n)$. *We also suppose the following properties:*

(1) *The Lebesgue measure of* K_0 *is zero.*
(2) *There exists* $\varepsilon > 0$ *such that, for every open connected component* $\mathcal{A}$ *of* $\Omega\setminus K_0$, *the set* $\mathcal{A} \cap \Omega_\varepsilon$ *has a nonempty interior.*
(3) $(\partial g/\partial y)(x, \bar{y}(x)) \neq 0$ *for every* $x \in K_0$.

Then the regularity assumption (5.1) *is satisfied.*

Remark 5.4. If $\alpha,\ \beta \in C(\bar{\Omega})$, then $\bar{u} \in C(\bar{\Omega}\setminus K_0)$; *cf.* Theorem 4.1. Hence property (2) of the theorem is fulfilled, if $\bar{u}$ is not identically equal to α or β in any open connected component $\mathcal{A} \subset \Omega\setminus K_0$. Indeed, since $\bar{u} \in C(\mathcal{A})$ and $\bar{u} \not\equiv \alpha$ and $\bar{u} \not\equiv \beta$ in $\mathcal{A}$, there exists $x_0 \in A$ such that $\alpha(x_0) < \bar{u}(x_0) < \beta(x_0)$. Consequently, the continuity of $\bar{u}$ implies the existence of $\varepsilon > 0$ such that $\mathcal{A} \cap \Omega_\varepsilon$ contains a ball $B_\rho(x_0)$.

Let us also mention that property (3) of the theorem is trivially satisfied if the state constraint is $a(x) \leq y(x) \leq b(x)$ for every $x \in K$.

Proof of Theorem 5.3. Fix $\varepsilon > 0$ as in property (2). We will argue by contradiction. If $\overline{\mathcal{R}(T)} \neq C(K_0)$, then there exists $\mu \in C(K_0)' = M(K_0)$, $\mu \neq 0$, such that

$$0 = \langle \mu, Tv\rangle = \int_{K_0} \frac{\partial g}{\partial y}(x, \bar{y}(x)) z_v(x)\,\mathrm{d}\mu(x) \quad \forall v \in L^2(\Omega_\varepsilon). \tag{5.3}$$

We take the function $\psi \in W_0^{1,s}(\Omega)$ for all $1 \leq s < n/(n-1)$ satisfying

$$\begin{cases} A^*\psi + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\psi = \dfrac{\partial g}{\partial y}(x, \bar{y}(x))\mu & \text{in } \Omega \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{5.4}$$

From (5.3) and (5.4), it follows for every $v \in L^2(\Omega_\varepsilon)$

$$\begin{aligned} \int_{\Omega_\varepsilon} \psi v\,\mathrm{d}x &= \int_\Omega \psi v\,\mathrm{d}x = \int_\Omega \left[Az_v + \frac{\partial a_0}{\partial y}(x, \bar{y}(x)) z_v\right]\psi\,\mathrm{d}x \\ &= \int_{K_0} \frac{\partial g}{\partial y}(x, \bar{y}(x)) z_v\,\mathrm{d}\mu = \langle \mu, Tv\rangle = 0, \end{aligned}$$

which implies that $\psi = 0$ in Ω_ε. Consider now an open connected component $\mathcal{A}$ of $\Omega\setminus K_0$. It holds $\psi = 0$ in the interior of $\mathcal{A} \cap \Omega_\varepsilon$ and

$$A^*\psi + \frac{\partial a_0}{\partial y}(x, \bar{y}(x))\psi = 0 \ \text{ in } \mathcal{A},$$

therefore $\psi = 0$ in $\mathcal{A}$; see Saut and Scheurer [19]. Thus we have that $\psi = 0$ in $\Omega\setminus K_0$, but K_0 has zero Lebesgue measure, hence $\psi = 0$ in Ω and consequently $\mu = 0$ in contrary to our previous assumption.

We conclude our paper by proving that the regularity condition (5.1) is stronger that the linearized Slater assumption (A5).

Theorem 5.5. *Under the assumptions* (A1)–(A4) *the regularity condition* (5.1) *implies the linearized Slater condition* (A5).

Proof. We define the set

$$B = \{z \in C(K_0)\colon a(x) - g(x, \bar{y}(x)) < z(x) < b(x) - g(x, \bar{y}(x))\ \forall x \in K_0\}.$$

From our assumptions it follows that B is a non empty open set of $C(K_0)$, hence condition (5.1) implies the existence of $v \in L^2(\Omega_\varepsilon)$ such that $Tv \in B$. By density of $L^\infty(\Omega_\varepsilon)$ in $L^2(\Omega_\varepsilon)$ we can assume that $v \in L^\infty(\Omega_\varepsilon)$. Outside Ω_ε, we extend v by zero. The inclusion $Tv \in B$ can be expressed by

$$a(x) < g(x,\bar{y}(x)) + \frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x) < b(x) \quad \forall x \in K_0, \tag{5.5}$$

where $z_v \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the solution of (5.2). From here, we deduce the existence of $\rho_1 > 0$ such that

$$a(x) + \rho_1 < g(x,\bar{y}(x)) + \frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x) < b(x) - \rho_1 \quad \forall x \in K_0. \tag{5.6}$$

Given $\rho \in (0,1)$ arbitrarily, multiplying the inequalities

$$a(x) \le g(x,\bar{y}(x)) \le b(x) \quad \forall x \in K \tag{5.7}$$

by $1-\rho$, (5.6) by ρ, and adding the resulting inequalities we get for every $x \in K_0$ and all $\rho \in (0,1]$

$$a(x) + \rho\rho_1 < g(x,\bar{y}(x)) + \rho\frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x) < b(x) - \rho\rho_1. \tag{5.8}$$

Define now, for any $\delta > 0$,

$$K_{0,\delta} = \{x \in K \colon \mathrm{dist}(x,K_0) < \delta\}.$$

Taking δ small enough, we get from (5.8) for all $x \in K_{0,\delta}$ and every $\rho \in (0,1]$

$$a(x) + \rho\frac{\rho_1}{2} < g(x,\bar{y}(x)) + \rho\frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x) < b(x) - \rho\frac{\rho_1}{2}\cdot \tag{5.9}$$

On the other hand, since the state constraint is not active in the compact set $K \setminus K_{0,\delta}$, we deduce the existence of $0 < \rho_2 < 1$ such that

$$a(x) + \rho_2 < g(x,\bar{y}(x)) < b(x) - \rho_2 \quad \forall x \in K \setminus K_{0,\delta}. \tag{5.10}$$

If we select a $\rho \in (0,1)$ that satisfies

$$\rho\left|\frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x)\right| < \frac{\rho_2}{2} \quad \forall x \in K,$$

we obtain from (5.10)

$$a(x) + \frac{\rho_2}{2} < g(x,\bar{y}(x)) + \rho\frac{\partial g}{\partial y}(x,\bar{y}(x))z_v(x) < b(x) - \frac{\rho_2}{2}\cdot \tag{5.11}$$

Finally, taking $0 < \rho < \varepsilon/\|v\|_{L^\infty(\Omega_\varepsilon)}$, recalling the definition of Ω_ε and that v vanishes in $\Omega \setminus \Omega_\varepsilon$, we deduce that $u_0 = \bar{u} + \rho v \in \mathcal{U}_{\alpha\beta}$. Moreover, (5.9) and (5.11) imply that (2.2) holds, which concludes the proof. □

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