

Facultad de Ciencias

MÉTODO DE ELEMENTOS FINITOS PARA DINÁMICA DE CABLES SUBACUÁTICOS (Finite element method for underwater cable dynamics)

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Resumen

Se ha desarrollado un modelo numérico para el estudio de maniobras de arrastre marinas. El modelo propuesto está basado en el estudio dinámico de un cable moviéndose entre dos cuerpos, uno con movimientos impuestos y otro libre. El modelo es una extensión de modelos previos empleados para el estudio del comportamiento de cables de anclaje, basados en el método de elementos finitos de primer orden y donde el effecto de la flexión del cable es ignorado. Las condiciones de contorno requeridas fueron estudiadas.

El método fue implementado y el código resultante fue validado con éxito, empleando para ello resultados experimentales de sistemas de arrastre presentes en la literatura. La dependencia de los resultados numéricos en el coeficiente de amortiguamiento interno y en el número de elementos se ha estudiado en este trabajo. Para ilustrar las aplicaciones de la herramienta desarrollada, se realizaron simulaciones de sistemas de arrastre en escala real, y se analizaron las distintas configuraciones de estos sistemas. Para cada configuración propuesta se analiza la variación de las tensiones en el cuerpo arrastrado y la posición de este.

Con el objetivo de desarrollar un modelo numérico que incluya los efectos de la flexión del cable, se plantea una formulación alternativa del método propuesto inicialmente. La nueva formulación permite emplear métodos de elementos finitos de orden superior, imprescindibles para resolver la ecuación del cable con el término de la fuerza de flexión. En base a esta formulación, se propone un modelo numérico con un método de elementos finitos de tercer orden. Los dos últimos modelos propuestos se implementaron y sus resultados son validados y analizados en este trabajo.

Palabras clave

Método de elementos finitos; Dinámica de cables; Sistemas de arrastre marinos; Amortiguamiento interno

Abstract

A numerical model to study marine towing maneuvers has been developed. The proposed model is based on the dynamic study of a catenary line moving between two bodies, one with imposed motions, and the other one free. The model is an extension of previous ones used to study the behavior of mooring lines, based on a first order finite element method where the bending effects were neglected. The needed boundary conditions have been studied.

The method was implemented and the resulting code was successfully validated using experimental results for towing systems found in the literature. Sensitivity analysis on the internal damping coefficient and the number of elements have been included in the present work. As an example of application of the developed tool, simulations of towing systems on real scale were analyzed for different setups. The variation of the loads at the towed body and also the position of the body are analyzed for the studied configurations.

Aiming to develop a numerical model that considers the bending effects on the lines, an alternative formulation for the numerical model was given and implemented. This allows to use higher order finite element methods, necessary to solve the line equation with the bending force term. Based on this formulation, a numerical model with a third order finite element method was proposed and implemented. Numerical results were validated and analyzed.

Keywords

Finite element method; Cable dynamics; Marine towing systems; Internal damping

1 Introduction

The rapid development of floating structures like wave energy converters, floating wind turbines and aquaculture structures have increased the use of sea waters. The new uses require new methodologies to support these activities, like the installation of new floating devices. This is usually done by a ship towing the device using a catenary line, like in Figure 1. On the other hand, submerged towing systems are also used in numerous naval engineering applications such as sonars, seabed exploration, fishing or spotting sea mines.



Figure 1: Maneuver of installation of a wind mill.

The need to guarantee that the towed body is properly controlled and that it does not get lost or damaged in the sea leads to oversizing the towing system, which implies increasing the cost of the lines and a higher fuel consumption. Therefore, reliable numerical models are needed to study this operation. Being able to accurately simulate towing cable dynamics would allow to design more efficient systems. The purpose of this project is to provide a numerical tool to analyze the towing maneuver of floating and submerged bodies.

Both maneuvers involve a cable that connect a ship with the towed device. There are different alternatives in the literature to study the mooring systems of floating bodies. These methods are based on the study of the cable dynamics using finite element method (FEM) [1, 6] and lumped mass models (LM) [4, 3]. This project is an extension of these models to the application of towing maneuvers. Most of the models are considerably recent, although the problem is a classic one. For instance, in 2010 Zhu [7] proposed a new nodal position FEM (NP-FEM) to study the towing of submerged bodies.

The numerical FEM model used to study mooring systems [1] has been modified to study the towing maneuvers. This implied the modification of the boundary conditions used in the model. In the study of mooring lines one boundary moves with the moored body and the other boundary is fixed to the seabed. In this case one boundary will be moved with enforced movements, simulating the motion of the towing ship while the other boundary is free to move the body connected to it. The presented model is validated against data published in the literature, [7, 5]. Also, a sensitivity analysis with respect to the number of elements used in the discretization of the line and the damping included in the model is performed.

The validated model have been used to study the towing of two bodies. The first example performed was the simulation of the towing of a submerged body, like a sonar, by a moving ship. Different line length and body weights have been evaluated to study the final position at which the towed body navigated. In the second example the towing body is floating, like in Figure 1. In this case, also different line lengths were tested. The effect of including an intermediate body to add extra buoyancy, or weight, to the catenary was also studied, increasing or decreasing the vertical component of the force at the towed body. Finally the influence of the vertical force was also evaluated depending on the position of the intermediate body.

This project continues with a brief description of the numerical model used to study mooring lines and the new boundary condition applied to the study of the towing maneuver. Then, the validation of the proposed model using experimental works published in the literature is presented and the proposed methodology is applied to the towing of two bodies, one submerged and one floating. Finally, a new formulation for the presented method is proposed. The nodal nature of the finite element method model in [1] getting hard to upgrade into higher order methods, and the new formulation is based on a modal point of view. It is shown that higher order methods are vital for solving the equations with bending effects terms. The new formulation is then used to present a third order method. The work ends up with a discussion of the work done and potential further work.

2 Numerical model

In the current research on towing and mooring dynamics simulation, finite element methods (FEM) are used to solve the partial differential equation (PDE) ruling cable dynamics. The equation used in the literature generally ignores bending and torsion effects and it is based on the one dimensional wave equation [1, 3, 4, 6]:

$$\rho_0 \frac{\partial^2 \mathbf{r}(t,s)}{\partial t^2} = \frac{\partial}{\partial s} \left(T(t,s) \mathbf{t}(t,s) \right) + \mathbf{f}(t,s) (1 + e(t,s)), \tag{1}$$

where $\mathbf{r}(t, s)$ denotes the position of the cable parametrized by arc length, $s \in [0, L]$ represents the variable of the parametrization of the curve of the cable, t is the time variable, ρ_0 denotes the linear density of the cable, T(s, t) is the tension of the cable, $\mathbf{t}(t, s)$ is the unitary tangential vector to the cable, $\mathbf{f}(t, s)$ is the external forces vector per unit of length and e(t, s) represents the strain of the cable.

This is the Newton's second law equation for a cable, written per unit of length. The mass times acceleration term of usual Newton's equation turns into the linear density times acceleration term. In the right hand side, the forces term of Newton's equation is divided into inner forces (tension) and external forces. The tension force vector turns into the spatial derivative of tension force vector. The external forces vector is already expressed per unit of length, but it needs to be multiplied by the linear deformation term, (1 + e(t, s)).

The external forces \mathbf{f} , are composed of the gravity and hydrostatic buoyancy force \mathbf{f}_{hg} , and the hydrodynamic forces, the normal \mathbf{f}_{dn} and tangential \mathbf{f}_{dt} drag forces and the inertia force \mathbf{f}_{mn} .

$$\mathbf{f} = \mathbf{f}_{hg} + \mathbf{f}_{dt} + \mathbf{f}_{dn} + \mathbf{f}_{mn} \tag{2}$$

These forces are defined as follows:

$$\mathbf{f}_{hg} = \rho_0 \frac{\rho_c - \rho_w}{(1+e)\rho_c} \mathbf{g}$$

$$\mathbf{f}_{dn} = -\frac{1}{2} C_{dn} d\rho_w |\mathbf{v}_n| \mathbf{v}_n$$

$$\mathbf{f}_{dt} = -\frac{1}{2} C_{dt} d\rho_w |\mathbf{v}_t| \mathbf{v}_t$$

$$\mathbf{f}_{mn} = -C_{mn} \frac{\pi d^2}{4} \rho_w \mathbf{a}_n$$
(3)

where: ρ_c is density of cable's material, ρ_w is water density, C_{dn} and C_{dt} are normal and tangential drag coefficients respectively, C_{mn} is hydrodynamic mass coefficient, d is diameter of the cable, \mathbf{v} and \mathbf{a} denote velocity and acceleration, and subindex n and t denote normal or tangential component of the vector respectively.

The gravity and buoyancy term is computed per unit of length using the Archimedes law with the linear density instead of the mass, and it is divided by (1 + e) as it does not depend on the strain of the cable. The hydrodynamic forces are based on Morison equations, and they are expressed per unit of length by substituting the cross-sectional area or the volume of the body by the corresponding length or area respectively.

In [1], first order FEM is used to solve equation (1) as follows: the PDE is transformed into the generalized problem, the Galerkin method is used and the cable is discretized in n + 1 points $\mathbf{r}(s,t) \approx (\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_n)$. Here \mathbf{r}_n is the position of the top support of the cable at the fairlead of the body, and \mathbf{r}_0 the position of the opposite end of the cable, the position of a fixed anchor in [1] and the position of the towed body in this research. This way, a system of lineal ordinary differential equations (ODE) is generated. The following approximation can be considered:

$$\dot{\mathbf{r}}_{k-1} \approx \dot{\mathbf{r}}_k \quad \dot{\mathbf{r}}_{k+1} \approx \dot{\mathbf{r}}_k \quad \ddot{\mathbf{r}}_{k-1} \approx \ddot{\mathbf{r}}_k \quad \ddot{\mathbf{r}}_{k+1} \approx \ddot{\mathbf{r}}_k \tag{4}$$

what allows the matrices describing the system of ODE to become tridiagonal and gives simplified expressions for the elementary matrices and vectors: mass elementary matrix \mathbf{M}_k , drag elementary matrix \mathbf{D}_k , stiffness elementary vector \mathbf{k}_k and external forces elementary vector \mathbf{g}_k (only gravity considered here). Although no detail on the expressions for these elementary matrices and vectors and how they are obtained in [1] is given, a similar study is provided in Section 5. Assembling of the elementary matrices results in the total system, where boundary conditions are applied by substituting identity matrices and expected acceleration vectors on the appropriate positions on the mass matrix and the total forces vector. In this work, the proposed model is implemented and the system is solved with LAPACK routines [10]. ODEPACK routines are used to obtain the time evolution of the problem: predictor-corrector Adams methods are chosen for non-stiff problems and Backwards Differentiation Formula based methods are chosen for stiff problems [11].

2.1 Damping coefficient

In this work, as it was done in [2] the tension term of equation (1) is computed by considering each element of a cable with section A_0 as a Voigt-Kelvin spring: a combination of a spring with young

modulus E and a resistance with a damping coefficient β , as it is shown in equation (5).

$$T(t,s) = EA_0\left(e + \beta \frac{\partial e}{\partial t}\right) \tag{5}$$

In other words, the term $EA_0\beta \frac{\partial e}{\partial t}$ is added to the Hook's law equation $(T(t,s) = EA_0e)$. The exact and the numerical expressions of e(s,t) are given by equation (6). For the numerical expression the strain will be considered constant on each element.

$$e(s,t) = \left| \frac{\partial \mathbf{r}(s,t)}{\partial s} \right| - 1 \approx \frac{1}{l} |\mathbf{r}_k - \mathbf{r}_{k-1}| - 1 = e_k(t)$$
(6)

where l is the length of each element that is considered in the cable, $s \in ((k-1) \cdot l, k \cdot l)$ and $k \in \{1, ..., n\}$. Considering that $\mathbf{r}_k = (x_k(t), y_k(t), z_k(t))$ equation (6) can be rewritten as:

$$e_k(t) = \frac{1}{l} \cdot \sqrt{\left(x_k(t) - x_{k-1}(t)\right)^2 + \left(y_k(t) - y_{k-1}(t)\right)^2 + \left(z_k(t) - z_{k-1}(t)\right)^2} - 1 \tag{7}$$

Taking the derivative, considering that: $\frac{\partial \mathbf{r}_k}{\partial t} = \dot{\mathbf{r}}_k = (\dot{x}_k, \dot{y}_k, \dot{z}_k)$; a numerical expression for the time derivative of the cable strain is obtained.

$$\dot{e}_{k} = \frac{1}{l} \frac{1}{|\mathbf{r}_{k} - \mathbf{r}_{k-1}|} \left[(x_{k} - x_{k-1})(\dot{x}_{k} - \dot{x}_{k-1}) + (y_{k} - y_{k-1})(\dot{y}_{k} - \dot{y}_{k-1}) + (z_{k} - z_{k-1})(\dot{z}_{k} - \dot{z}_{k-1}) \right]$$
(8)

For the tension term, Hook's law is used in [1]. When the Voigt-Kelvin model is considered instead, the only expression that changes is the one for the elementary stiff matrix:

$$\mathbf{k}_{k} = \frac{EA}{l} \left[\left(\frac{\epsilon_{k} - l}{\epsilon_{k}} \cdot \mathbf{l}_{k} - \frac{\epsilon_{k+1} - l}{\epsilon_{k+1}} \cdot \mathbf{l}_{k+1} \right) + \beta \cdot \left(\frac{\dot{e}_{k}}{\epsilon_{k}} \cdot \mathbf{l}_{k} - \frac{\dot{e}_{k+1}}{\epsilon_{k+1}} \cdot \mathbf{l}_{k+1} \right) \right]$$
(9)

In equation (9) the notation used in [1] is taken, where $\mathbf{l}_k = \mathbf{r}_k - \mathbf{r}_{k-1}$ and $\epsilon_k = |\mathbf{l}_k|$.

2.2 Towing boundary condition

In order to implement the boundary conditions of a towing problem, the expected acceleration at the towed body, a sphere in this work, needs to be calculated. First, it is necessary to compute the total force applied on the body. In this work this is done using the Morison equation, which, for an object submerged in a fluid with a certain flow, gives the force parallel to the flow applied in the body. If the fluid density is ρ , the flow velocity is \vec{u} and the body has volume V, cross-sectional area perpendicular to the flow A, drag coefficient C_d and added mass coefficient C_a ($C_m = 1 + C_a$), the Morison equation (10) is:

$$\vec{F}_M = \rho C_m V \cdot \dot{\vec{u}} + \frac{1}{2} \rho C_d A \cdot \vec{u} |\vec{u}| \tag{10}$$

Also, other forces should be considered, like gravity, buoyancy and tension on the sphere. For the gravity and buoyancy, if the body has a mass m and g is the gravity acceleration on Earth surface, the force can be written as:

$$\vec{F}_{gb} = (V_u \cdot \rho - m) \cdot g \cdot \mathbf{e}_z \tag{11}$$

where V_u is the volume under water, studied in detail later for the spherical case. For the tension force, since the body will be placed at the first node:

$$\vec{F}_T = EA_0 \cdot (e_1 + \beta \cdot \dot{e}_1) \cdot \frac{\mathbf{r}_1 - \mathbf{r}_0}{|\mathbf{r}_1 - \mathbf{r}_0|}$$
(12)

Now, considering equations (10), (11) and (12), when the body is a sphere of radius R and mass m located on the first node, the acceleration on the body is:

$$\mathbf{a}_{sphere} = \frac{\vec{F}_M + \vec{F}_{gb} + \vec{F}_T}{m + C_a} \tag{13}$$

where the flow velocity or acceleration are taken to be the velocity or acceleration of the flow at the point where the first node is minus the velocity or acceleration of the first node, and:

• $A^{sphere} = \pi R^2$ (cross-sectional area)

•
$$V^{sphere} = \frac{4}{3}\pi R^3$$

•
$$V_u^{sphere} = \frac{\pi h^2}{3}(3R-h)$$

- $h = min\{2R, max\{0, R z_{sphere}\}\}$
- $C_m^{sphere} = 0.5$
- $C_d^{sphere} = 0.47$

Volume and cross-sectional area of the sphere are well known and the drag and mass coefficients expressions are given in literature [8, 9]. The expression for V_u^{sphere} comes from the volume of a spherical cap where h is the height of the cap. The expression for h gives the height of the spherical cap that is under water where z_{sphere} is the z coordinate of the center of the sphere, as shown in Figure 2. The min and max in the h equation allow to consider the different possibilities for the sphere's relative position with the water level: completely submerged, most of it submerged, most of it emerged and completely emerged.



Figure 2: Explicative diagram of the under water volume of the sphere.

For a sphere located in a inner node i of the cable, the acceleration at that node would be replaced by the acceleration in equation (13), but using the inner or intermediate node's position, velocity and acceleration. Then, \vec{F}_T in equation (12) is replaced by:

$$\vec{F}_T = EA_0 \cdot \left[(e_i + \beta \cdot \dot{e}_i) \cdot \frac{\mathbf{r}_{i-1} - \mathbf{r}_i}{|\mathbf{r}_{i-1} - \mathbf{r}_i|} + (e_{i+1} + \beta \cdot \dot{e}_{i+1}) \cdot \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{|\mathbf{r}_{i+1} - \mathbf{r}_i|} \right]$$
(14)

2.3 Snapping free cable condition

For the case where nothing is hanging from the towing cable, the boundary condition was approximated as the boundary condition for a sphere with the cable's density and diameter.

3 Validation

In order to validate our numerical results two distinct cases are studied: a sphere hanging with a cable from a boat that oscillates vertically and a cable swinging after being released of one of the two supports that were holding both of its ends at the same height. The first one uses the experimental and numerical results shown in Zhu's paper [7] and the second one uses the results from Koh's one [5].

3.1 Zhu's experiment

This experiment considers a sphere hanging with a cable from a boat that oscillates vertically, as shown in Figure 3, and it studies the vertical tension at the top of the cable. This is done for two different frequencies (0.807 Hz and 1.27 Hz) and with a 78mm amplitude of oscillation. In this project, the optimal results of the finite element method with a damping coefficient are displayed for both frequencies and the dependence of the numerical results on the number of nodes and the internal damping coefficient are studied for the higher frequency of vertical oscillation.



Figure 3: Zhu's experiment layout.

3.1.1 Optimal simulation

Optimal simulation uses 40 nodes, a 0.01s time-step and an internal damping coefficient $\beta = 10^{-4}$. For the lower frequency, the results are shown in Figure 4. It is observed that the new method's numerical results show great accuracy at predicting the peak tension, improving Zhu results as those predict a lower tension. This does not happen for the minimum tension, where Zhu results are better and our results predict a higher tension. This means that the presented method may be more reliable at predicting peak tensions, what is desirable when designing mooring or towing systems, as it lets the researcher know if the cable is going to break or not. Higher peak tensions are studied for the high frequency experiment, when the cable starts "snapping" in watter.



Figure 4: Optimal validation with Zhu's low frequency experiment results.

For the higher frequency results, displayed in Figure 5, the presented method shows lower noise than Zhu's, a constant peak tension and an excellent agreement with the maximum peak tension registered in the experimental results. This backs the method as a good predictor of the peak tension, as it was seen for low frequency. The fact that for most of the peaks seen in experimental results the new method predicts a higher peak tension than the measured one could be a problem, but as the maximum tension registered in the experiment is accurately predicted numerically, it can be considered that these differences are due to horizontal dispersion of energy or "chaotic" damping effects caused by turbulence not registered by the model, which don't take place on every oscillation in reality. On the one hand this is starting to show the limitations of the model, but on the other hand it is not needed to consider so many physical phenomena to predict maximum peak tensions in the cable, and, as it was previously said, this is probably the most important information that can be obtained when simulating mooring or towing systems.



Figure 5: Optimal validation with Zhu's high frequency experiment results.

3.1.2 Dependence on the internal damping coefficient

The study of the dependence on the internal damping coefficient is done in Figure 6 and shows that for zero internal damping, a great amount of noise is produced, similar to Zhu's numerical results. This was expected as introducing the internal damping coefficient is one of the main differences of this method with Zhu's model. For a high internal damping coefficient, the snap tension is not observed anymore. Then it can be assured than an appropriate election of the internal damping coefficient is essential to accurately predict the maximum peak tensions. The computational times for the $\beta = 0$, $\beta = 10^{-4}$, and $\beta = 0.1$ internal damping coefficients simulations were 30 s, 1.2 s and 1.9 s for each second of simulation respectively, which shows that introducing the β coefficient implies lower computational time. Lower noise is highly related with lower computational time: the lower the internal oscillations, the faster the convergence of the ODE solvers is.



Figure 6: Numerical results for different β values compared with experimental results.

3.1.3 Dependence on the number of nodes

Furthermore, the analysis of the dependence on the number of nodes is also performed. Figures 7 and 8 show low dependence on the number of nodes. The detailed figure shows better agreement with the maximum peak tension and lower noise for the higher number of nodes, as it was expected. The computational times for the 5, 10, 20 and 40 nodes simulations are 0.7 s, 1.2 s, 4.6 s and 18.2 s for each second of simulation respectively. This means that using a low number of elements the results well be slightly conservative (larger prediction for peaks). Also, the computational cost is smaller for lower number of nodes.



Figure 7: Numerical results for different number of nodes compared with experimental results.



Figure 8: Detailed numerical results for different number of nodes compared with experimental results.

3.2 Koh's experiment

This experiment measures the shape of a cable and its tension at the top support when it swings. The cable is initially hanging as in the initial condition shown in Figure 10a, the swing is produced after the cable is released from one of the two supports that were holding both ends of the cable at the same height. Further detail on the experiment can be found at Koh's paper [5]. In this project the new method's numerical results are compared for the shape of the cable and the tension with Koh's numerical and experimental results . Also the dependence of the tension results with the internal damping coefficient and the number of elements are studied.

3.2.1 Optimal simulation

The optimal simulation uses 80 nodes, a 10^{-3} s time-step and an internal damping coefficient $\beta = 6.2 \cdot 10^{-2}$, as Koh does in his paper. In Figure 9 it can be observed that the new method's results for the tension are really similar to Koh's numerical results. The tension oscillation for the first 0.5 seconds is lower for the presented method, showing an slightly better agreement with the experimental results than Koh's. For the peak tension at 0.6 s, Koh is slightly closer to the experimental results, but the presented method has better accuracy when the snap tension shows. From 0.6s to 1.4s, both numerical results are almost the same but fail to predict accurately a minimum on the tension at 0.8s. At 1.7s, experimental results show one more snap tension peak, that the presented method predicts more accurately. Overall the presented method is at least as reliable as Koh's method.



Figure 9: Optimal validation with Koh's experiment results.

Comparing Figures 10a and 10b shows that the shape of the cable swing is almost the same as the measured in the experiment, and no remarkable differences can be observed.



Figure 10: Results for a cable swing.

3.2.2 Dependence on the internal damping coefficient

A similar behavior was observed in relation to the dependence of the results on the damping coefficient value, as before. Figure 11 presents that high internal damping smooths all the tension oscillations, zero internal damping results implies too much noise and the optimal internal damping gives good results. The computational times for $\beta = 0$, $\beta = 0.062$, and $\beta = 1$ internal damping coefficients simulations are 59.0 s, 13.6 s and 15.7 s, respectively. This reinforces the conclusion observed before relating the computational cost and the oscillation of the results.



Figure 11: Numerical results for different β values compared with experimental results.

3.2.3 Dependence on the number of nodes

Again, a study of the dependence on the number of nodes is developed. Figure 12 shows that the lower the number of nodes is, the smoother the oscillations are and the phase differences become bigger. The computational times for the 10, 20, 40 and 80 elements simulations are 0.27 s, 0.71 s, 2.25 s and 13.6 s, respectively. There is no significant difference among the results for 40 and 80 elements, but the computational times for 80 nodes are six times higher. It makes no sense to keep increasing the number of nodes to reach higher agreement with the experimental results. The lack of agreement here is probably due to the physical phenomena that is not considered in the equation that is used: the forces caused by the bending of the cable as it swings may be contributing to the movement of the cable and the tension at the top support.



Figure 12: Numerical results for different number of nodes compared with experimental results.

4 Analysis of different towing systems

In this section different towing systems will be studied as the towed body can be floating or submerged. It is also interesting to study cases with intermediate bodies that can help to optimize the towing systems. To accomplish this goal, the cable is set on a certain boundary condition (vertical as in Zhu's experiment or hanging horizontally as in Koh's one) and the boundary condition at the top support is imposed to be a horizontal displacement: it starts moving after 5 seconds of simulation (so the system gets stable before the top support starts moving at a constant speed). In general the final shape of the cable is studied as well as the time dependence of some parameters, for example the depth of the towed body.

4.1 Towing of a submerged body

First, a submerged body towing system is studied by choosing a sphere with density higher than water. The setup of the towing is shown in Figure 13. The initial condition is hanging vertically as in Zhu's experiment. Two different cable configurations have been used. The characteristics of the system are shown in Table 1.



Figure 13: Submerged body towing system set up.

	Cables	Towed body		
Material	Nylon	Polyethylene	D	4.8 m
λ	4.8 kg/m	2.9 kg/m	М	$7.74 \cdot 10^4 \text{ kg}$
EA	$3.87 \cdot 10^8$ N·m	$1.02 \cdot 10^8 \text{ N} \cdot \text{m}$	C_d	0.5
L	55 m	55 m	C_m	0.18
d	88 mm	72 mm		
C_{dt}	0.01	0.01		
C_{dn}	1.2	1.2		
C_{mn}	0.5	0.5		

Table 1: Towing a submerged body data.

Figures 14a and 14b show the results once they reach equilibrium and the time evolution of the depth of the sphere respectively for three different towing systems. These systems use the same spheres and the same cables, but three different lengths for the cables. The cable used was the Nylon one and the speed of the boat was 3.4 m/s. Table 2 shows the computed cable tension at the boat support.



(b) Time evolution of the variation of the submerged bodies' vertical coordinate respect to the initial value.

Figure 14: Towing systems that differ in the cable length.

Cable length (m)	T_x (N)	T_z (N)	T (N)
50	$3.052 \cdot 10^4$	$1.736 \cdot 10^{5}$	$1.763 \cdot 10^{5}$
55	$3.346 \cdot 10^4$	$1.730 \cdot 10^{5}$	$1.762 \cdot 10^{5}$
60	$3.638 \cdot 10^4$	$1.724 \cdot 10^{5}$	$1.761 \cdot 10^{5}$

Table 2: Results at equilibrium of the tension on the boat's end of the cable for floating body towing systems with different cable lengths.

As it was expected, the longer the cable the deeper and the further away the towed body stays. Figure 14b also shows that as the length increases, the higher the towed body rises due to the speed of the boat, and that the time evolution of the depth of the body shows the same behavior disregarding the length of the cable. The tension has larger horizontal component and lower vertical component for longer cables, although the total tension does not depend so much on the length. This gives useful information on how to choose the length of a cable for this type of towing systems within the length ranges that may be considered on a design, for example: in terms of fuel efficiency it could be more desirable that the cable tension pulls the boat horizontally rather than vertically, and longer cables would lead to better efficiency.

Figures 15a and 15b show the same results as the two previous figures, but here, the towing systems have two different cable material instead of three different cable lengths. The cable length was kept constant at 55 m and the speed of the boat was 4.6 m/s. Table 3 shows the cable tension at the boat support.



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Cable material	T_x (N)	T_z (N)	T (N)
Polyethylene	$4.909 \cdot 10^4$	$1.693 \cdot 10^{5}$	$1.763 \cdot 10^{5}$
Nylon	$5.882 \cdot 10^4$	$1.661 \cdot 10^5$	$1.762 \cdot 10^5$

Figure 15: Submerged towed bodies towed with cables for two different materials.

Table 3: Results at equilibrium of the tension on the boat's end of the cable for submerged body towing systems with different cable materials.

It is observed that the nylon cable rises more and keeps a longer distance with the boat than the polyethylene one, although nylon's density is higher. This is due to higher tangential drag forces on the cable due to its thicker diameter. This has a great effect on the tension at the boat support, as it is shown at Table 3, and it can be deduced that the thicker the cable, the more horizontal its shape is, and the bigger the horizontal component of the tension gets. This should also be considered with the cable length, previously studied, when designing a towing system.

Figures 16a and 16b show the results for the shape of the cables once they reach equilibrium and the time evolution of the depth of the sphere respectively for three different cases. This system uses the same spheres and the same cable (the 55 m Nylon cable) for the three cases, but different ship speeds. Table 4 shows the computed cable tension at the boat support.

Speed (m/s)	T_x (N)	T_z (N)	T (N)
3.4	$3.346 \cdot 10^4$	$1.730 \cdot 10^{5}$	$1.762 \cdot 10^{5}$
4.6	$5.882 \cdot 10^4$	$1.661 \cdot 10^{5}$	$1.762 \cdot 10^{5}$
6.7	$1.059 \cdot 10^{5}$	$1.411 \cdot 10^5$	$1.764 \cdot 10^{5}$

Table 4: Results at equilibrium of the tension on the boat's end of the cable for floating body towing systems with different ship speeds.



Figure 16: Submerged towed bodies towed at two different speeds.

Figures show that the higher the boat speed is, the lower the depth of the towed body is. A surprising result here is the low dependence of the total tension on the speed observed in Table 4. This is because drag forces for the considered speeds are low compared to the weight of the body.

4.2 Towing of a floating body

Finally, a floating body towing system is studied. To do so, a sphere with density lower than water is chosen, and the initial condition sets the cable hanging horizontally as in Koh's experiment [5]. The setup of the towing is shown in Figure 17. Similarly to the previous section, the dependence of the system on the length of the cable is studied. Figure 18a displays the shapes of the cables after they reach equilibrium and Figure 18b the time evolution of the z-coordinate of the deepest point of the cable and the time evolution of the horizontal span between the top support and the towed body. The initial horizontal span is the same for the three systems, the boat speed is 3.4 m/s and the characteristics of the towing systems are shown in Table 5. Table 6 shows the computed cable tension at the towed body.



Figure 17: Floating body towing system set up. Simple towing (black line), towing with intermediate body (red lines).

Ca	ble	Towed	l body	Intermediate body	
λ	20.68 kg/m	D	8.5 m	D	2.5 m
EA	$2.45 \cdot 10^8$ N m	М	$3.11 \cdot 10^5 \text{ kg}$	М	$3.11 \cdot 10^2 \text{ kg}$
L	60 m	C_d	0.5	C_d	0.5
d	80 mm	C_m	0.18	C_m	0.18
C_{dt}	0.01				
C_{dn}	1.2				
C_{mn}	0.5				

Table 5: Towing a floating body data.



(a) Cables shapes at equilibrium.

(b) Time evolution of the horizontal span and the minimum vertical coordinate of the cable.

Figure 18:	Floating	bodies	towed	with	cables	OI	ainerent	lengths.	

Cable length (m)	T_x (N)	T_z (N)	T (N)
55	7.54	$5.482 \cdot 10^3$	$5.482 \cdot 10^{3}$
60	7.54	$5.999 \cdot 10^3$	$6.000 \cdot 10^3$
65	7.54	$6.531 \cdot 10^3$	$6.532 \cdot 10^{3}$

Table 6: Results at equilibrium of the tension on the towed body's end of the cable for floating body towing systems with different cable length

In Figure 18a it is observed that, when the system is stable, the depth of the cable and horizontal span grows with cable length. In Figure 18b, it can be seen that both the horizontal span, X_{span} , and the cable depth, Z_{min} , are decreasing for the first 20 seconds. This is because when the top support is not moving, the weight of the cable pulls the sphere towards the top support and then the cable gets deeper. Once the top support, ship, starts moving, it is seen how both X_{span} and Z_{min} , increase rapidly. For minimum cable depth, in Figure 18b, a big difference can be observed among the three systems, as the initial horizontal span is set to be the same, so the longer cable hang deeper. There is a smaller difference among the three systems at final equilibrium, as the horizontal span is not the same anymore. For horizontal span, it is observed that, as expected, the initial span is the same, but as the system reaches equilibrium, the longer cables let the sphere hold more distance with the top support.

In terms of the tension at the towed body, the length difference leads to a constant horizontal tension, which is opposing the drag forces on the body, while the vertical tension increases as the longer cables are significantly heavier.

Finally, the inclusion of an intermediate body in the towing systems is studied. To do so, the intermediate body boundary conditions described in Section 2.2 are imposed. This can be done for any internal node of the cable, and for different sizes and masses of the spheres chosen to be placed as an intermediate body. Figure 19a displays the final state of three towing systems of a floating body: again, one with no intermediate body, and two with a floating intermediate body, one on the center and the other one closer to the towed body (at the nodes 50 and 80 respectively, where the total number of nodes was 100). Figure 19b shows the time evolution of the horizontal span, X_{span} , and the deepest point of the cable, Z_{min} , and Table 7 shows the computed cable tension at the towed body. Apart from the intermediate body difference, all the remaining parameters are kept the same as in Table 5.



Figure 19: Floating towed bodies towed with cables with different intermediate body arrangements.

Intermediate body	T_x (N)	T_z (N)	T (N)
No intermediate body	7.603	$5.999 \cdot 10^{3}$	$6.000 \cdot 10^{3}$
Central floating body	7.603	$2.696 \cdot 10^3$	$2.697 \cdot 10^{3}$
Lateral floating body	7.603	$0.7785 \cdot 10^{3}$	$0.7792 \cdot 10^{3}$

Table 7: Results at equilibrium of the tension on the towed body's end of the cable for floating body towing systems with different intermediate body arrangements.

In Figure 19, it is observed that the horizontal span is higher for the system with the intermediate body in the middle of the cable, then for the system with the intermediate body closer to the towed body and finally for the system with no intermediate body. The same order is valid for increasing depth of the lowest point of the cable. The interesting result here is the low vertical tension over the towed body for the systems with intermediate bodies, specially for the system with the floater closer to the towed body. This shows that using hydrodynamic floaters attached to the cable near the towed body can lower the vertical tension on the towed structure. When this happens, the submerged volume of the structure decreases slightly and consequently the drag forces are reduced, what might increase efficiency.

5 Higher order approach

5.1 Motivation

On the validation of the numerical results given by the method in the previous sections with the experimental results in Koh's paper [5], it was seen that the agreement was low. Then, it was explained that bending and torsion effects could imply significant effects on the forces on the cable, not considered in equation (1). Equation (15), as it appears in [12], considers these effects on the cable:

$$-(EI\mathbf{r}'')'' + [(T - EI\kappa^2)\mathbf{r}']' + [GJ\tau(\mathbf{r}' \times \mathbf{r}'')]' + \mathbf{q} = \left(\frac{1}{4}\pi d\rho_c \mathbf{I}\right)\ddot{\mathbf{r}}; \quad (GJ\tau)' = 0$$
(15)

A brief look at these equations, without paying attention to their meaning, is enough to see that there are fourth order spatial derivatives in the equation unknown, \mathbf{r} . This implies that a first order FEM would not success at solving precisely this equation, as even for the generalized problem, it would be necessary to take the third derivative of the linear functions of the basis chosen when using the Galerkin method. Those derivatives go to zero and information is lost in the process. Also, even if the higher order methods where not necessary to solve equation (15), they are known to be faster and more precise than first order methods.

In this project, the FEM used in [1] was not deeply explained. That was because although the IHAC had an implementation of that method, on the first part of this project the objective was improving that implementation with the internal damping coefficient and adding the towed body boundary condition to the code and at the same time, understand the principles of the finite element method and the equation studied. It was also noticed that due to the approximations taken in that method, and its nodal set up, increasing the order of the method was a hard task to achieve, so it was decided to develop a new first order FEM without approximations, and with a set up focused on the elements, and not in the nodes, as we will see in the following section, in some way that increasing the order of the method was simple. Considering this, it was more interesting to explain this new finite element method in detail, rather than the method in [1].

5.2 Alternate fist order FEM

Let us recall equation (1), rewriting it slightly:

$$\rho_0 \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{\partial}{\partial s} \left(E A_0 \frac{e + \beta \frac{\partial e}{\partial t}}{1 + e} \frac{\partial \mathbf{r}}{\partial s} \right) + \mathbf{f} (1 + e), \tag{16}$$

by taking:

$$\begin{split} \boldsymbol{t}(t,s) &= \frac{1}{1+e} \frac{\partial \mathbf{r}}{\partial s}, \\ T(t,s) &= EA_0 \left(e + \beta \frac{\partial e}{\partial t} \right), \\ e &= \left| \frac{\partial \mathbf{r}}{\partial s} \right| - 1. \end{split}$$

And, for simplicity, let's use the notation $T(t,s) = EA_0 \left(e + \beta \frac{\partial e}{\partial t}\right) \frac{1}{1+e}$:

$$\rho_0 \frac{\partial^2 \mathbf{r}}{\partial t^2} - \frac{\partial}{\partial s} \left(T(t, s) \frac{\partial \mathbf{r}}{\partial s} \right) - \mathbf{f}(1+e) = 0 \tag{17}$$

Weak formulation of the problem needs to be obtained in order to use the finite element method, and to do so, equation (17) is multiplied by a continuous test function, $\mathbf{w}(s,t)$ belonging to V, the space of solution functions, that must meet $\mathbf{w}(0,t) = \mathbf{w}(L,t) = 0$, and the integral is taken on the whole domain, [0, L], where L is the length of the cable:

$$\int_{0}^{L} \left(\rho_{0} \frac{\partial^{2} \mathbf{r}}{\partial t^{2}} - \frac{\partial}{\partial s} \left(T(t, s) \frac{\partial \mathbf{r}}{\partial s} \right) - \mathbf{f}(1+e) \right) \mathbf{w} ds = 0$$
(18)

Using integration by parts on the second term:

$$\int_{0}^{L} \left(\rho_{0} \frac{\partial^{2} \mathbf{r}}{\partial t^{2}} \mathbf{w} + \left(T(t,s) \frac{\partial \mathbf{r}}{\partial s} \right) \frac{\partial \mathbf{w}}{\partial s} - \mathbf{f}(1+e) \mathbf{w} \right) ds - \left[\left(T(t,s) \frac{\partial \mathbf{r}}{\partial s} \right) \mathbf{w} \right]_{0}^{L} = 0$$
(19)

Given the conditions over \mathbf{w} , the term outside of the integral vanishes, and the weak formulation is obtained.

$$\int_{0}^{L} \left(\rho_0 \frac{\partial^2 \mathbf{r}}{\partial t^2} \mathbf{w} + \left(T(t, s) \frac{\partial \mathbf{r}}{\partial s} \right) \frac{\partial \mathbf{w}}{\partial s} - \mathbf{f}(1+e) \mathbf{w} \right) ds = 0$$
(20)

Choosing a basis

Now, a finite dimension subset V_h of V is considered, and a basis is taken $\{\varphi_i\}_{i=0}^N$. The solution of the equation can be written as the linear combination of the functions on the basis.

$$\mathbf{r}(s,t) = \sum_{i=0}^{N} r_i(t)\varphi_i(s).$$
(21)

And the spatial derivatives:

$$\frac{\partial \mathbf{r}}{\partial s}(s,t) = \sum_{i=0}^{N} r_i(t) \frac{\partial \varphi_i}{\partial s}(s).$$
(22)

Taking $\mathbf{w} = \varphi_k$ in equation (20) with $k = 0, 1, \dots, N$, and using the two previous equations:

$$\int_{0}^{L} \left(\sum_{i=0}^{N} \rho_0 \frac{\partial^2 r_i}{\partial t^2} \varphi_i \varphi_k + \sum_{i=0}^{N} \left(T(t,s) r_i \frac{\partial \varphi_i}{\partial s} \right) \frac{\partial \varphi_k}{\partial s} - \mathbf{f}(1+e) \varphi_k \right) ds = 0 \quad \forall k$$
(23)

Taking N nodes in the domain, $\{s_1, \ldots, s_N\}$, two of them in both ends of the cable, we define two functions for each element:

$$\begin{cases} \varphi_i(s) = \frac{s_{i+1} - s_i}{s_{i+1} - s_i} \\ \varphi_{i+1}(s) = \frac{s - s_i}{s_{i+1} - s_i} \end{cases} \quad if \ s \in [s_i, s_{i+1}]; \quad \begin{cases} \varphi_i(s) = 0 \\ \varphi_{i+1}(s) = 0 \end{cases} \quad if \ s \notin [s_i, s_{i+1}] \end{cases}$$
(24)

From PDE to a ODE system

On the element $[s_i, s_{i+1}]$, there are two non-zero functions, φ_i and φ_{i+1} , using linearity of the integral and taking k = i, equation (23) is turned into:

$$\rho_{0} \frac{\partial^{2} r_{i}}{\partial t^{2}} \int_{s_{i}}^{s_{i+1}} \varphi_{i} \varphi_{i} ds + \rho_{0} \frac{\partial^{2} r_{i+1}}{\partial t^{2}} \int_{s_{i}}^{s_{i+1}} \varphi_{i+1} \varphi_{i} ds + \int_{s_{i}}^{s_{i+1}} T(t,s) r_{i} \frac{\partial \varphi_{i}}{\partial s} \frac{\partial \varphi_{i}}{\partial s} ds + \int_{s_{i}}^{s_{i+1}} T(t,s) r_{i+1} \frac{\partial \varphi_{i+1}}{\partial s} \frac{\partial \varphi_{i}}{\partial s} ds - \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e) \varphi_{i} ds = 0 \quad (25)$$

Now, taking k = i + 1:

$$\rho_{0} \frac{\partial^{2} r_{i}}{\partial t^{2}} \int_{s_{i}}^{s_{i+1}} \varphi_{i} \varphi_{i+1} ds + \rho_{0} \frac{\partial^{2} r_{i+1}}{\partial t^{2}} \int_{s_{i}}^{s_{i+1}} \varphi_{i+1} \varphi_{i+1} ds + \int_{s_{i}}^{s_{i+1}} T(t,s) r_{i+1} \frac{\partial \varphi_{i+1}}{\partial s} \frac{\partial \varphi_{i+1}}{\partial s} ds + \int_{s_{i}}^{s_{i+1}} T(t,s) r_{i+1} \frac{\partial \varphi_{i+1}}{\partial s} \frac{\partial \varphi_{i+1}}{\partial s} ds - \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e) \varphi_{i+1} ds = 0 \quad (26)$$

Both equations can be written together in a matrix expression:

$$\rho_{0} \begin{bmatrix}
\int_{s_{i}}^{s_{i+1}} \varphi_{i}\varphi_{i}ds & \int_{s_{i}}^{s_{i+1}} \varphi_{i}\varphi_{i+1}ds \\
\int_{s_{i}}^{s_{i+1}} \varphi_{i+1}\varphi_{i}ds & \int_{s_{i}}^{s_{i+1}} \varphi_{i+1}\varphi_{i+1}ds
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^{2}r_{i}}{\partial t^{2}} \\
\frac{\partial^{2}r_{i+1}}{\partial t^{2}}
\end{bmatrix}
+
\begin{bmatrix}
\int_{s_{i}}^{s_{i+1}} T(t,s)\frac{\partial\varphi_{i}}{\partial s}\frac{\partial\varphi_{i}}{\partial s}ds & \int_{s_{i}}^{s_{i+1}} T(t,s)\frac{\partial\varphi_{i}}{\partial s}\frac{\partial\varphi_{i+1}}{\partial s}ds \\
\int_{s_{i}}^{s_{i+1}} T(t,s)\frac{\partial\varphi_{i}}{\partial s}\frac{\partial\varphi_{i+1}}{\partial s}ds & \int_{s_{i}}^{s_{i+1}} T(t,s)\frac{\partial\varphi_{i+1}}{\partial s}\frac{\partial\varphi_{i+1}}{\partial s}ds
\end{bmatrix}
\begin{bmatrix}
r_{i} \\
r_{i+1}
\end{bmatrix}
=
\begin{bmatrix}
\int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i}ds \\
\int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i+1}ds
\end{bmatrix}$$
(27)

Recalling the definition of T(t, s), and as EA_0 is constant for every element:

$$\int_{s_i}^{s_{i+1}} T(t,s) \frac{\partial \varphi_i}{\partial s} \frac{\partial \varphi_i}{\partial s} ds = EA_0 \int_{s_i}^{s_{i+1}} \left(e + \beta \frac{\partial e}{\partial t} \right) \frac{1}{1+e} \frac{\partial \varphi_i}{\partial s} \frac{\partial \varphi_i}{\partial s} ds \tag{28}$$

That leads to:

$$\rho_{0} \begin{bmatrix} \int_{s_{i}}^{s_{i+1}} \varphi_{i}\varphi_{i}ds & \int_{s_{i}}^{s_{i+1}} \varphi_{i}\varphi_{i+1}ds \\ \int_{s_{i}}^{s_{i+1}} \varphi_{i+1}\varphi_{i}ds & \int_{s_{i}}^{s_{i+1}} \varphi_{i+1}\varphi_{i+1}ds \end{bmatrix} \begin{bmatrix} \frac{\partial^{2}r_{i}}{\partial t^{2}} \\ \frac{\partial^{2}r_{i+1}}{\partial t^{2}} \end{bmatrix}$$
$$+EA_{0} \begin{bmatrix} \int_{s_{i}}^{s_{i+1}} \left(e+\beta\frac{\partial e}{\partial t}\right)\frac{1}{1+e}\frac{\partial \varphi_{i}}{\partial s}\frac{\partial \varphi_{i}}{\partial s}ds & \int_{s_{i}}^{s_{i+1}} \left(e+\beta\frac{\partial e}{\partial t}\right)\frac{1}{1+e}\frac{\partial \varphi_{i}}{\partial s}\frac{\partial \varphi_{i+1}}{\partial s}ds \\ \int_{s_{i}}^{s_{i+1}} \left(e+\beta\frac{\partial e}{\partial t}\right)\frac{1}{1+e}\frac{\partial \varphi_{i}}{\partial s}\frac{\partial \varphi_{i+1}}{\partial s}ds & \int_{s_{i}}^{s_{i+1}} \left(e+\beta\frac{\partial e}{\partial t}\right)\frac{1}{1+e}\frac{\partial \varphi_{i+1}}{\partial s}\frac{\partial \varphi_{i+1}}{\partial s}ds \end{bmatrix} \begin{bmatrix} r_{i} \\ r_{i+1} \end{bmatrix}$$
$$= \begin{bmatrix} \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i}ds \\ \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i}ds \end{bmatrix}$$
(29)

Computing the integrals

In order to solve these integrals, a variable change will be taken, moving from the interval $[s_i, s_{i+1}]$ to [0, 1]. To do so, the following function is defined::

$$F_i: \begin{bmatrix} 0,1 \end{bmatrix} \longrightarrow \begin{bmatrix} s_i, s_{i+1} \end{bmatrix} \\ x \longrightarrow x(s_{i+1} - s_i) + s_i$$
(30)

To be able to use the Change of Variables theorem¹, the Jacobian of this function must be computed:

$$J(F_i) = s_{i+1} - s_i = l_i \tag{31}$$

obtaining the length of the i-th element. Using this in the basis functions that are non-zero on the i-th element we have:

$$\phi_k = \varphi_{i-1+k} \circ F_i^{-1}, \ k = 1, 2.$$
(32)

and then,

$$\phi_1(x) = x, \ y \ \phi_2(x) = 1 - x \tag{33}$$

By the Change of Variables theorem:

¹Change of Variables theorem $g: A \to B$ and $f: B \to \mathbb{R}$ then $\int_B f = \int_A f \circ g |J(g)|$

$$\int_{s_i}^{s_{i+1}} \varphi_i(s)\varphi_i(s)ds = \frac{1}{l_i} \int_0^1 (1-x)^2 dx = \frac{1}{l_i} \frac{1}{3}$$
(34)

$$\int_{s_i}^{s_{i+1}} \varphi_{i+1}(s)\varphi_{i+2}(s)ds = \frac{1}{l_i} \int_0^1 x^2 dx = \frac{1}{l_i} \frac{1}{3}$$
(35)

$$\int_{s_i}^{s_{i+1}} \varphi_i(s)\varphi_{i+1}(s)ds = \frac{1}{l_i} \int_0^1 x(1-x)dx = \frac{1}{l_i} \frac{1}{6}$$
(36)

For the space derivatives:

$$\int_{s_i}^{s_{i+1}} \varphi_i'(s)\varphi_i'(s)ds = \frac{1}{l_i} \int_0^1 (-1)^2 dx = \frac{1}{l_i}$$
(37)

$$\int_{s_i}^{s_{i+1}} \varphi'_{i+1}(s)\varphi'_{i+1}(s)ds = \frac{1}{l_i} \int_0^1 1^2 dx = \frac{1}{l_i}$$
(38)

$$\int_{s_i}^{s_{i+1}} \varphi_i'(s) \varphi_{i+1}'(s) ds = \frac{1}{l_i} \int_0^1 1(-1) dx = -\frac{1}{l_i}$$
(39)

For equation (29), considering $\left(e + \beta \frac{\partial e}{\partial t}\right) \frac{1}{1+e}$ constant on each element, the following is obtained:

$$\rho_{0} \frac{1}{6} \frac{1}{l_{i}} \begin{bmatrix} 2Id_{3} & Id_{3} \\ Id_{3} & 2Id_{3} \end{bmatrix} \begin{bmatrix} \frac{\partial^{2}r_{i}}{\partial t^{2}} \\ \frac{\partial^{2}r_{i+1}}{\partial t^{2}} \end{bmatrix} + EA_{0} \left(e + \beta \frac{\partial e}{\partial t}\right) \frac{1}{1+e} \frac{1}{l_{i}} \begin{bmatrix} Id_{3} & -Id_{3} \\ -Id_{3} & Id_{3} \end{bmatrix} \begin{bmatrix} r_{i} \\ r_{i+1} \end{bmatrix}$$
$$= \begin{bmatrix} \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i}ds \\ \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i+1}ds \end{bmatrix}$$
(40)

where Id_3 is the 3×3 identity matrix.

Using the Simpson rule on the external forces integrals:

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right],\tag{41}$$

that is:

$$\begin{bmatrix} \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i}ds \\ \int_{s_{i}}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i+1}ds \end{bmatrix} \approx (1+e)\frac{s_{i+1}-s_{i}}{6} \begin{bmatrix} \mathbf{f}(s_{i})\varphi_{i}(s_{i}) + 4\mathbf{f}\left(\frac{s_{i}+s_{i+1}}{2}\right)\varphi_{i}\left(\frac{s_{i}+s_{i+1}}{2}\right) + \mathbf{f}(s_{i+1})\varphi_{i}(s_{i+1}) \\ \mathbf{f}(s_{i})\varphi_{i+1}(s_{i}) + 4\mathbf{f}\left(\frac{s_{i}+s_{i+1}}{2}\right)\varphi_{i+1}\left(\frac{s_{i}+s_{i+1}}{2}\right) + \mathbf{f}(s_{i+1})\varphi_{i+1}(s_{i+1}) \end{bmatrix}$$
(42)

Recalling that:

$$\varphi_i(s_i) = 1; \ \varphi_i\left(\frac{s_i + s_{i+1}}{2}\right) = \frac{1}{2}; \ \varphi_i(s_{i+1}) = 0$$
(43)

$$\varphi_{i+1}(s_i) = 0; \ \varphi_{i+1}\left(\frac{s_i + s_{i+1}}{2}\right) = \frac{1}{2}; \ \varphi_{i+1}(s_{i+1}) = 1$$
(44)

We get:

$$\begin{bmatrix} \int_{s_i}^{s_{i+1}} \mathbf{f}(1+e)\varphi_i ds \\ \int_{s_i}^{s_{i+1}} \mathbf{f}(1+e)\varphi_{i+1} ds \end{bmatrix} \approx (1+e)\frac{s_{i+1}-s_i}{6} \begin{bmatrix} \mathbf{f}(s_i) + 4\mathbf{f}\left(\frac{s_i+s_{i+1}}{2}\right)\frac{1}{2} \\ 4\mathbf{f}\left(\frac{s_i+s_{i+1}}{2}\right)\frac{1}{2} + \mathbf{f}(s_{i+1}) \end{bmatrix}.$$
 (45)

Building the system

Assembling the matrices that were found in the previous section properly, leads to a system of second order ordinary differential equations (as shown in [13]):

$$\mathbf{M} \cdot \ddot{\mathbf{r}}(t) + \mathbf{K}(t) \cdot \mathbf{r}(t) = \mathbf{F}(t), \tag{46}$$

where **M** is the mass matrix, **K** the stiff matrix and **F** the external forces vector. Vector **r** denotes the positions of each node and $\ddot{\mathbf{r}}(t) = \frac{\partial^2 \mathbf{r}}{\partial t^2}(t)$.

On every time step, the acceleration is computed as:

$$\ddot{\mathbf{r}}(t+\delta t) = \mathbf{M}^{-1} \cdot (\mathbf{F}(t) - \mathbf{K}(t) \cdot \mathbf{r}(t))$$
(47)

But the mass matrix is never inverted, a numerical method as LU or Cholesky is used.

First, let's write again the elementary matrices. For each element k, the corresponding elementary mass matrix is:

$$M_k = \frac{\rho_0}{l_k} \frac{1}{6} \begin{bmatrix} 2Id & Id \\ Id & 2Id \end{bmatrix}.$$
 (48)

where Id is the 3×3 identity matrix.

It is a matrix constant in time, and there is a matrix constant for every element, M_0 , that multiplied by the scalar ρ_0/l_k gives the specific information of the element.

$$M_0 = \frac{1}{6} \begin{bmatrix} 2Id & Id \\ Id & 2Id \end{bmatrix}.$$
 (49)

The elementary stiffness matrix for the element k is:

$$K_k(t) = \frac{EA_0}{l_k} \frac{1}{1 + e_k(t)} \left[e_k(t) + \beta \dot{e}_k(t) \right] \left[\begin{array}{cc} Id & -Id \\ -Id & Id \end{array} \right]$$
(50)

In this case, the matrix is not constant in time, but a matrix constant in time can be taken as:

$$K_0 = EA_0 \begin{bmatrix} Id & -Id \\ -Id & Id \end{bmatrix},$$
(51)

For every time step this matrix will be multiplied by a scalar, to obtain stiffness elementary matrix:

$$K_k(t) = \frac{1}{1 + e_k(t)} \frac{1}{l_k} \left[e_k(t) + \beta \dot{e}_k(t) \right] K_0.$$
(52)

where e_k and \dot{e}_k are computed as it was done in equations (7) and (8).

The external forces elementary vector is:

$$F_{k} = (1 + e_{k}) \frac{s_{k+1} - s_{k}}{6} \begin{bmatrix} \mathbf{f}(s_{k}) + 4\mathbf{f}\left(\frac{s_{k} + s_{k+1}}{2}\right)\frac{1}{2} \\ 4\mathbf{f}\left(\frac{s_{k} + s_{k+1}}{2}\right)\frac{1}{2} + \mathbf{f}(s_{k+1}) \end{bmatrix}.$$
(53)

where the external forces follow equation (2).

To assemble the matrices, two $(3 \cdot N \times 3 \cdot N)$ zero matrices **M** and **K** and a $(3 \cdot N)$ zero column vector **F** are first built. Then a simple algorithm is followed, for k = 1 to k = N:

$$\mathbf{M}(6(k-1)+1:6k, 6(k-1)+1:6k) = \mathbf{M}(6(k-1)+1:6k, 6(k-1)+1:6k) + M_k$$
$$\mathbf{K}(6(k-1)+1:6k, 6(k-1)+1:6k) = \mathbf{K}(6(k-1)+1:6k, 6(k-1)+1:6k) + K_k$$
$$\mathbf{F}(6(k-1)+1:6k) = \mathbf{F}(6(k-1)+1:6k) + F_k$$

Boundary and initial conditions

Let $\mathbf{F}_{total} = \mathbf{F} - \mathbf{K} \cdot \mathbf{r}$ be the total forces vector. In general the accelerations at the ends of the cable are known, $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_N$. Then, for every time step, after computing \mathbf{M} and \mathbf{F}_{total} , in order to impose the boundary conditions:

$$\mathbf{M}(1:3,1:3) = Id_3, \quad \mathbf{M}(N-2:N,N-2:N) = Id_3$$
$$\mathbf{M}(1:3,4:N) = 0, \quad \mathbf{M}(N-2:N,1:N-3) = 0$$
$$\mathbf{F}_{total}(1:3) = \ddot{\mathbf{r}}_1, \quad \mathbf{F}_{total}(N-2:N) = \ddot{\mathbf{r}}_N$$

Also, if the speed or position for some of the ends of the cable are known, their values are updated with the expected one for every time step.

For the initial condition, the shape of the catenary is computed analytically and the velocity and acceleration are supposed to be zero.

Time evolution

Finally, the time evolution of the system is obtained using subroutines of the Fortran library ODEPACK, as it was done in Section 2.

5.3 Implementation

The alternative first order FEM described in the previous section was also implemented using Fortran90. Figure 20 shows a scheme of the implementation. Three files are used to write the code (in black rectangles in the figure) and other three files contain the information used by the code that the user must provide (red rectangles). The main program is represented with a blue background and the subroutines used have an orange background. Blue arrows denote subroutine calls and red arrows denote data files reading.

At the IHAC, there is a code, NuevoFEM, with the main structure, and more than 2000 lines. This code did not work properly. For that reason, I created new subroutines, changed how the linear system was solved to a banded matrix method, implemented different boundary conditions and tried different ODE solvers, until the code could be validated. I also made several MatLab and Python scripts to analyze and paint the code results, and I learned how to use the gfortran debugging tools, and Ubuntu operative system. For the first order code in the previous sections, I had to develop a similar task. On the other hand there was not an existing code at the IHAC for the third order method proposed in Section 5.5, and I had to rewrite NuevoFEM to implement it.



Figure 20: Implementation scheme.

The main program begins reading the data given by the user, either reading it directly in the main program or calling a subroutine that does it. The data obtained is saved on some variables and used to allocate other. These variables were either previously declared on the main program or subroutine, or on a data structure created on a separated file. Variables saved on the data structure can be used on every subroutine of the code.

Following this scheme, time is set to be zero and the initial condition is imposed. To do this, a subroutine that uses the data on the position of the boat, given by the user and previously read, is called. This subroutine, GetPosicionCuerpo, saves the position, velocity and acceleration of the fairlead on the data structure for the current time-step using interpolation and numerical derivation. Subroutine FEM_InitLine is then called, it uses the position of the fairlead, the position of the opposite end of the cable, given by the user and saved on the data structure previously, and some subroutines based on the analytic expressions for the shape of a catenary to set the initial condition and save it on the variable of the data structure that contains the positions of the nodes on the cable, a $3 \cdot N$ vector, where N is the number of nodes. Velocity and acceleration of the cable is set to zero.

The next step is to start the temporary loop. Time is set to be dt. Then, the time loop begins calling GetPositionCuerpo again, and continues calling the ODEPACK subroutine that solves the time evolution of the system, imposes boundary conditions on the position and velocity substituting the appropriate values on the vectors containing this information, writes the results on the output files, and updates time t = t + dt. The ODEPACK routine requires as an input a subroutine, called FEM_TimeEvolutionOdepack in this project, that for a given time, position and speed, returns the acceleration. Using this, numerical methods for solving ODE systems are used on the ODEPACK subroutine, as it was said before, predictor-corrector Adams methods are chosen for non-stiff problems and Backwards Differentiation Formula based methods are chosen for stiff problems. In order to assure the correct solution of the system, the relative and absolute tolerances are chosen to be 10^{-7} and 10^{-9} respectively.

All that is left to see is how FEM_TimeEvolutionOdepack works. First, for the current position, velocity and acceleration of the cable, three subroutines that assemble the elementary matrices and vectors generated by three other subroutines. The assembled mass matrix \mathbf{M} , stiff matrix \mathbf{K} and external forces vector \mathbf{F} are saved on the data structure this way. Using the stiff matrix and the position of the cable \mathbf{r} , the tension at both ends of the cable is computed and written in the corresponding output files. After this, the total forces vector is computed as $\mathbf{F}_{total} = \mathbf{F} - \mathbf{K} \cdot \mathbf{r}$, and boundary conditions are imposed on \mathbf{M} and \mathbf{F}_{total} . Finally the system $\mathbf{M} \cdot \ddot{\mathbf{r}} = \mathbf{F}_{total}$ is solved using LAPACK subroutines, where $\ddot{\mathbf{r}}$ is the vector containing the accelerations on the cable. To do this two extra subroutines not showed in the figure are used, one computes the external forces on each node given the position, speed and acceleration of that node, and this subroutine is called by an other one that computes the elementary forces vectors. The other extra subroutine gets water velocity and acceleration on any required point using potential wave theory, to use the appropriate value of the speed and acceleration of the flow with respect to the cable while computing the external forces.

5.4 New first order FEM results

The alternative first order method was used for mooring system simulation. This choice was due to the great agreement of Aamo [1] based simulations with experimental results for this kind of systems. The purpose of reformulating the first order was to take a step forward on the process of increasing the order to be able to solve the cable equation with the bending term. This is why towing systems simulations are not used here; the objective is not to see how the numerical results are for towing systems, but to see if the cable equation is properly solved with the new formulation.

To simulate mooring systems, a new term of interaction with the seabed was added to the external forces. The boundary condition for a mooring is really simple: imposing the position of anchor at the end of the cable, and zero velocity and acceleration. The experiment consisted on a boat moored to the seabed with a cable oscillating horizontally, and the horizontal tension of the cable at the tension was studied. Figure 21 shows a comparison of Aamo [1] based FEM and the new first order FEM results.



Figure 21: New first order FEM results.

We observe the excellent agreement in Figure 21. The new method shows lower noise and a smaller tension peak, but overall it can be concluded that the cable equation was successfully solved with the alternative formulation. The computational times were found to be very similar for both methods, one second per second of simulation approximately. The next step is to use this formulation to increase the order of the FEM method.

5.5 Third order approach

In this case, the formulation theory is exactly the same, just changing the election of the function basis.

Choosing a basis

For first order, degree one polynomials were chosen for the basis. Now, degree three polynomials are chosen, and four different polynomials will be defined on each element. To do so the polynomials are first defined in the [0, 1] interval. The conditions imposed on these polynomials are their values and the derivatives values at the nodes, each polynomial will have only one of those values equal to one, an the rest of them zero, as shown in equation (54). This way, the coefficients of the solution expansion in terms of functions of the basis, won't have position meaning exclusively, but also space derivative meaning.Let $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ be the four polynomials, then:

$$\begin{aligned}
\phi_1(0) &= 1, \phi_1'(0) = 0, \phi_1(1) = 0, \phi_1'(1) = 0 \\
\phi_2(0) &= 0, \phi_2'(0) = 1, \phi_2(1) = 0, \phi_2'(1) = 0 \\
\phi_3(0) &= 0, \phi_3'(0) = 0, \phi_3(1) = 1, \phi_3'(1) = 0 \\
\phi_4(0) &= 0, \phi_4'(0) = 0, \phi_4(1) = 0, \phi_4'(1) = 1
\end{aligned}$$
(54)

Using divided differences interpolation technique:

$$\begin{cases} \phi_1(x) = 2x^3 - 3x^2 + 1\\ \phi_2(x) = x^3 - 2x^2 + x\\ \phi_3(x) = -2x^3 + 3x^2\\ \phi_4(x) = x^3 - x^2 \end{cases} \quad x \in [0, 1]$$

$$(55)$$

And the first derivative:

$$\begin{cases} \phi_1'(x) = 6x^2 - 6x \\ \phi_2'(x) = 3x^2 - 4x + 1 \\ \phi_3'(x) = -6x^2 + 6x \\ \phi_4'(x) = 3x^2 - 2x \end{cases}$$
(56)

Choosing these functions guarantees the continuity of the derivative of the solution. In order to define the actual functions of the basis, the same change of variable function is used and the following functions are obtained.:

$$\begin{cases} \varphi_{4(k-1)+1}(s) = \phi_1(F_k^{-1}(s)) \\ \varphi_{4(k-1)+2}(s) = l_k \cdot \phi_2(F_k^{-1}(s)) \\ \varphi_{4(k-1)+3}(s) = \phi_3(F_k^{-1}(s)) \\ \varphi_{4(k-1)+4}(s) = l_k \cdot \phi_4(F_k^{-1}(s)) \end{cases} s \in [s_k, s_{k+1}]$$
(57)

And their derivatives, using the chain derivation rule:

$$\begin{pmatrix} \varphi'_{4(k-1)+1}(s) = \frac{1}{l_k} \cdot \phi'_1(F_k^{-1}(s)) \\ \varphi'_{4(k-1)+2}(s) = \phi'_2(F_k^{-1}(s)) \\ \varphi'_{4(k-1)+3}(s) = \frac{1}{l_k} \cdot \phi'_3(F_k^{-1}(s)) \\ \varphi'_{4(k-1)+4}(s) = \phi'_4(F_k^{-1}(s))
\end{cases} (58)$$

In equation (58) it is clear that the terms l_k where introduced at ϕ_2 and ϕ_4 on equation (57), to have that $\varphi'_{4(k-1)+2}(0) = 1$ and $\varphi'_{4(k-1)+4}(1) = 1$. This is necessary for the coefficients to have the physical meaning of spatial derivative.

From PDE to ODE system

Considering the element $[s_k, s_{k+1}]$, we have four non-zero functions defined over it, φ_{n+1} , φ_{n+2} , φ_{n+3} , φ_{n+4} and φ_{n+5} where $n = 4 \cdot (k-1)$, using these functions in equation (23):

$$\rho_{0} \int_{s_{k}}^{s_{k+1}} \left[\left(\frac{\partial^{2} r_{k}}{\partial t^{2}} \varphi_{n+1} + \frac{\partial^{2} r_{k}'}{\partial t^{2}} \varphi_{n+2} + \frac{\partial^{2} r_{k+1}}{\partial t^{2}} \varphi_{n+3} + \frac{\partial^{2} r_{k+1}'}{\partial t^{2}} \varphi_{n+4} \right) \varphi_{j} + \left(T(t,s) r_{k} \frac{\partial \varphi_{n+1}}{\partial s} + T(t,s) r_{k}' \frac{\partial \varphi_{n+2}}{\partial s} + T(t,s) r_{k+1} \frac{\partial \varphi_{n+3}}{\partial s} + T(t,s) r_{k+1}' \frac{\partial \varphi_{n+4}}{\partial s} \right) \frac{\partial \varphi_{j}}{\partial s} - \mathbf{f}(1+e) \varphi_{j} \right] ds = 0 \quad (59)$$

where $j \in \{n + 1, n + 2, n + 3, n + 4\}$. Using integral linearity:

$$\rho_{0} \frac{\partial^{2} r_{k}}{\partial t^{2}} \int_{s_{k}}^{s_{k+1}} \varphi_{n+1} \varphi_{j} ds + \rho_{0} \frac{\partial^{2} r_{k}'}{\partial t^{2}} \int_{s_{k}}^{s_{k+1}} \varphi_{n+2} \varphi_{j} ds + \\ + \rho_{0} \frac{\partial^{2} r_{k+1}}{\partial t^{2}} \int_{s_{k}}^{s_{k+1}} \varphi_{n+3} \varphi_{j} ds + \rho_{0} \frac{\partial^{2} r_{k+1}'}{\partial t^{2}} \int_{s_{k}}^{s_{k+1}} \varphi_{n+4} \varphi_{j} ds + \\ + r_{k} \int_{s_{k}}^{s_{k+1}} T(t,s) \frac{\partial \varphi_{n+1}}{\partial s} \frac{\partial \varphi_{j}}{\partial s} ds + r_{k}' \int_{s_{k}}^{s_{k+1}} T(t,s) \frac{\partial \varphi_{n+2}}{\partial s} \frac{\partial \varphi_{j}}{\partial s} ds + \\ + r_{k+1} \int_{s_{k}}^{s_{k+1}} T(t,s) \frac{\partial \varphi_{n+3}}{\partial s} \frac{\partial \varphi_{j}}{\partial s} ds + r_{k+1}' \int_{s_{k}}^{s_{k+1}} T(t,s) \frac{\partial \varphi_{n+4}}{\partial s} \frac{\partial \varphi_{j}}{\partial s} ds - \\ - \int_{s_{k}}^{s_{k+1}} \mathbf{f}(1+e) \varphi_{j} ds = 0 \quad (60)$$

Using the four possible values for j, four equations are obtained that can be expressed with a matrix notation. Proceeding as it was done for first order:

$$\rho_0 \cdot \mathbf{M}_{0,k} \cdot \frac{\partial^2}{\partial t} \mathbf{r}_k + E A_0 \left(e + \beta \frac{\partial e}{\partial t} \right) \frac{1}{1+e} \cdot \mathbf{K}_{0,k} \cdot \mathbf{r}_k = \mathbf{F}_k \tag{61}$$

where $\mathbf{M}_{0,k}$ and $\mathbf{K}_{0,k}$ are (12×12) matrices and $\frac{\partial^2}{\partial t}\mathbf{r}$, \mathbf{r} and \mathbf{F}_k are column vectors with dimension 12. The elements of these matrices and vectors are defined as follows:

(62)

- $\mathbf{M}_{0,k}^{ij} = Id \cdot \int_{s_k}^{s_{k+1}} \varphi_i \varphi_j ds$ • $\frac{\partial^2}{\partial t^2} \mathbf{r}_k = \left(\frac{\partial^2 r_k}{\partial t^2}, \frac{\partial^2 r'_k}{\partial t^2}, \frac{\partial^2 r_{k+1}}{\partial t^2}, \frac{\partial^2 r'_{k+1}}{\partial t^2}\right)^T$
- $\mathbf{K}_{0,k}^{ij} = Id \cdot \int_{s_k}^{s_{k+1}} \varphi_i' \varphi_j' ds$
- $\mathbf{r}_k = (r_k, r'_k, r_{k+1}, r'_{k+1})^T$
- $\mathbf{F}_k = [\mathbf{F}_k^1, \mathbf{F}_k^2, \dots, \mathbf{F}_k^N]^T$
- $\mathbf{F}_k^i = (1+e) \int_{s_k}^{s_{k+1}} \mathbf{f} \varphi_i ds$

Here it should be considered that i and j go from $4 \cdot (k-1) + 1$ to $4 \cdot (k-1) + 4$ (four possible values), Id is dimension 3 identity matrix and $r_k = (x, y, z)$ and $\mathbf{f} = (f_x, f_y, f_z)$ (dimension three) in order to understand that the dimension of these matrices and vectors is 12 (=4.3).

This way a general system is obtained for an element, linked with the systems on the neighbor elements. All of them should be assembled to get the global ODE system: $\mathbf{M} \cdot \frac{\partial^2}{\partial t} \mathbf{r} + \mathbf{K} \cdot \mathbf{r} = \mathbf{F}$.

Computing the integrals

Using the same procedure as in first order, the following expressions are obtained for the integrals (i and j go from 1 to 4):

$$\int_{s_k}^{s_{k+1}} \varphi_{4(k-1)+i}(s)\varphi_{4(k-1)+j}(s)ds = \begin{cases} l_k \int_0^1 \phi_i(x)\phi_j(x)dx \text{ if } i, j \text{ odd} \\ l_k^2 \int_0^1 \phi_i(x)\phi_j(x)dx \text{ if } i \text{ even, } j \text{ odd} \\ l_k^3 \int_0^1 \phi_i(x)\phi_j(x)dx \text{ if } i, j \text{ even} \end{cases}$$
(63)

For the derivatives:

$$\int_{s_k}^{s_{k+1}} \varphi'_{4(k-1)+i}(s) \varphi'_{4(k-1)+j}(s) ds = \begin{cases} \frac{1}{l_k} \int_0^1 \phi'_i(x) \phi'_j(x) dx \text{ if } i, j \text{ odd} \\ \int_0^1 \phi'_i(x) \phi'_j(x) dx \text{ if } i \text{ even, } j \text{ odd} \\ l_k \int_0^1 \phi'_i(x) \phi'_j(x) dx \text{ if } i, j \text{ even} \end{cases}$$
(64)

These integrals are easy to compute, for example, using the symbolic calculus tool of MatLab.

Computing the elementary vector integrals is different for third order with respect to fist order. Considering **f** in equation (2), and the forces that build it up, it is seen that the force is only known at the nodes, so linear interpolation is used. This linear term multiplied by the linear basis functions on first order lead to a second order term, so Simpson rule was appropriate. For third order, a fourth order term is obtained and Simpson is not exact. To compute these integrals, let \mathbf{f}_k be the force at the node k. Then the force for the integral over the element k will be taken as $\mathbf{f} = \frac{s_{k+1}-s}{s_{k+1}-s_k} \cdot \mathbf{f}_k + \frac{s-s_k}{s_{k+1}-s_k} \cdot \mathbf{f}_{k+1}$ and, for integrals over (0, 1) the following is considered: $\mathbf{f} = (1-x) \cdot \mathbf{f}_k + x \cdot \mathbf{f}_{k+1}$.

Building the system

The elementary matrices and vectors can be rewritten:

- $\mathbf{M}_k = \rho_0 \cdot l_k \mathbf{M}_0^k$
- $\mathbf{K}_k = \frac{EA_0}{l_k} \left(e_k + \beta_k \frac{\partial e_k}{\partial t} \right) \frac{1}{1+e_k} \mathbf{K}_0^k$ • $\mathbf{F}_k = [\mathbf{F}_k^1, \mathbf{F}_k^2, \mathbf{F}_k^3, \mathbf{F}_k^4]^T$

•
$$\mathbf{F}_{k}^{1,3} = (1+e_{k}) \cdot l_{k} \int_{0}^{1} \mathbf{f} \phi_{i} dx = (1+e_{k}) \cdot l_{k} \left[\mathbf{f}_{k} \cdot \int_{0}^{1} (1-x) \phi_{i} dx + \mathbf{f}_{k+1} \cdot \int_{0}^{1} x \phi_{i} dx \right]$$

• $\mathbf{F}_{k}^{2,4} = (1+e_{k}) \cdot l_{k}^{2} \left[\mathbf{f}_{k} \cdot \int_{0}^{1} (1-x) \phi_{i} dx + \mathbf{f}_{k+1} \cdot \int_{0}^{1} x \phi_{i} dx \right]$

Where:

•
$$\mathbf{M}_{0}^{k} = \begin{pmatrix} Id \cdot \int_{0}^{1} \phi_{1}\phi_{1}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{1}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{1}\phi_{3}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{1}\phi_{4}dx \\ Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{2}\phi_{1}dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{2}\phi_{2}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{2}\phi_{3}dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{2}\phi_{4}dx \\ Id \cdot \int_{0}^{1} \phi_{3}\phi_{1}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{3}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{3}\phi_{3}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{3}\phi_{4}dx \\ Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{4}\phi_{1}dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{4}\phi_{2}dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{4}\phi_{3}dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{4}\phi_{4}dx \end{pmatrix}$$
•
$$\mathbf{K}_{0}^{k} = \begin{pmatrix} Id \cdot \int_{0}^{1} \phi_{1}'\phi_{1}'dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{1}'\phi_{2}'dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{1}'\phi_{3}'dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{1}'\phi_{4}'dx \end{pmatrix} \\ Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{2}'\phi_{1}'dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{2}'\phi_{2}'dx & Id \cdot l_{k} \cdot \int_{0}^{1} \phi_{2}'\phi_{3}'dx & Id \cdot l_{k}^{2} \cdot \int_{0}^{1} \phi_{2}'\phi_{4}'dx \end{pmatrix}$$

These matrices can be obtained as the product term by term of \mathbf{M}_0 with \mathbf{L}_k and \mathbf{K}_0 with \mathbf{L}_k respectively. As \mathbf{M}_0 and \mathbf{K}_0 are the same for all elements, it is only needed to compute them once, and only compute \mathbf{L}_k for each element:

•
$$\mathbf{M}_{0} = \begin{pmatrix} Id \cdot \int_{0}^{1} \phi_{1}\phi_{1}dx & Id \cdot \int_{0}^{1} \phi_{1}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{1}\phi_{3}dx & Id \cdot \int_{0}^{1} \phi_{1}\phi_{4}dx \\ Id \cdot \int_{0}^{1} \phi_{2}\phi_{1}dx & Id \cdot \int_{0}^{1} \phi_{2}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{2}\phi_{3}dx & Id \cdot \int_{0}^{1} \phi_{2}\phi_{4}dx \\ Id \cdot \int_{0}^{1} \phi_{3}\phi_{1}dx & Id \cdot \int_{0}^{1} \phi_{3}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{3}\phi_{3}dx & Id \cdot \int_{0}^{1} \phi_{3}\phi_{4}dx \\ Id \cdot \int_{0}^{1} \phi_{4}\phi_{1}dx & Id \cdot \int_{0}^{1} \phi_{4}\phi_{2}dx & Id \cdot \int_{0}^{1} \phi_{4}\phi_{3}dx & Id \cdot \int_{0}^{1} \phi_{4}\phi_{4}dx \end{pmatrix}$$

•
$$\mathbf{K}_{0} = \begin{pmatrix} Id \cdot \int_{0}^{1} \phi_{1}'\phi_{1}'dx & Id \cdot \int_{0}^{1} \phi_{1}'\phi_{2}'dx & Id \cdot \int_{0}^{1} \phi_{1}'\phi_{3}'dx & Id \cdot \int_{0}^{1} \phi_{1}'\phi_{4}'dx \\ Id \cdot \int_{0}^{1} \phi_{2}'\phi_{1}'dx & Id \cdot \int_{0}^{1} \phi_{2}'\phi_{2}'dx & Id \cdot \int_{0}^{1} \phi_{2}'\phi_{3}'dx & Id \cdot \int_{0}^{1} \phi_{2}'\phi_{4}'dx \\ Id \cdot \int_{0}^{1} \phi_{3}'\phi_{1}'dx & Id \cdot \int_{0}^{1} \phi_{3}'\phi_{2}'dx & Id \cdot \int_{0}^{1} \phi_{3}'\phi_{3}'dx & Id \cdot \int_{0}^{1} \phi_{3}'\phi_{4}'dx \\ Id \cdot \int_{0}^{1} \phi_{4}'\phi_{1}'dx & Id \cdot \int_{0}^{1} \phi_{4}'\phi_{2}'dx & Id \cdot \int_{0}^{1} \phi_{4}'\phi_{3}'dx & Id \cdot \int_{0}^{1} \phi_{4}'\phi_{4}'dx \end{pmatrix}$$

•
$$\mathbf{L}_{k} = \begin{pmatrix} Id & Id \cdot l_{k} & Id & Id \cdot l_{k} \\ Id \cdot l_{k} & Id \cdot l_{k}^{2} & Id \cdot l_{k} & Id \cdot l_{k}^{2} \\ Id \cdot & Id \cdot l_{k} & Id & Id \cdot l_{k} \\ Id \cdot l_{k} & Id \cdot l_{k}^{2} & Id \cdot l_{k} & Id \cdot l_{k}^{2} \end{pmatrix}$$

Assembling the matrices and vectors is analogous to what was done for first order.

Boundary and initial conditions

Boundary and initial conditions are imposed using the same technique as in first order. For the initial condition, there is no problem because the analytic expressions for the shape of the catenary also give information about its derivative. The problem comes at giving boundary conditions for the second time derivatives of the first space derivative of the position at the ends of the cable, that is $\frac{\partial^2 r'_1}{\partial t^2}$ and $\frac{\partial^2 r'_N}{\partial t^2}$. Those values are also unknown, in the current research several options were chosen: not imposing any boundary condition for these values, setting the values to be zero or limiting the modulus of those vectors, but any of those where good to solve the stability problem.

5.6 Stability problem for high order

The third order method is implemented in a very similar way as for first order. This was possible because the theory behind both methods followed the same structure. During the research, even fifth order was implemented. But for both methods there were stability problems, as it can be seen in Figure 22, where the results of the simulations for the experiment seen in section 5.4 by Aamo [1] based FEM and the new third order FEM are displayed.



Figure 22: Stability problem for higher order.

During the first 4 seconds of simulation, the third order method agrees with the fist order Aamo based method. But after that, the noise starts increasing until for second 5 the simulation stops after NaN value is obtained. This is a stiff problem behavior, but even for the Backwards Differentiation Formula based methods, recommended for stiff problems, the method remains unstable. A big amount of time was spent looking for an error on the code, trying different boundary and initial conditions for the derivatives without success in solving this problem. Techniques as limiting the maximum values of the spatial derivatives where considered, what made the method stable, but the agreement with the validated results was lost. For simpler problems, with smaller amplitude for the boat oscillations and without ground effects, the method was stable.

The main idea to explain this behavior is that the spatial derivatives are not controlled. When the distance among the nodes gets larger, given the way that the strain is computed, the tension force also increases, what gets the nodes back together, bounding the values for the positions of the nodes. On the other hand, the values of the spatial derivatives on the nodes are not used when computing any forces, so if these values start increasing, the forces will not act on the cable limiting that increase. Not having a mechanism to bound the spatial derivatives this way, allows them to increase without control until a NaN value is obtained. Bad election of boundary or initial conditions over the spatial derivatives may also be contributing to the problem. Two solutions are proposed to solve this problem.

The first solution would be studying how to introduce the numerical values of the spatial derivatives on the nodes in the forces equations or in the strain calculations, in a way that the forces act like recovery forces for these values. The second solution is more drastic and it would imply changing the basis functions, choosing the four polynomials on the [0, 1] interval verifying:

$$\begin{cases} \phi_1(0) = 1, \phi_1(1/3) = 0, \phi_1(2/3) = 0, \phi_1(1) = 0 \\ \phi_2(0) = 0, \phi_2(1/3) = 1, \phi_2(2/3) = 0, \phi_2(1) = 0 \\ \phi_3(0) = 0, \phi_3(1/3) = 0, \phi_3(2/3) = 1, \phi_3(1) = 0 \\ \phi_4(0) = 0, \phi_4(1/3) = 0, \phi_4(2/3) = 0, \phi_4(1) = 1 \end{cases}$$
(65)

This way, the meaning of all the coefficients used is position of a point of the catenary and it is easier to implement the initial and boundary conditions, and the external forces would act like recovery forces over all the coefficients.

Developing these solutions and implementing the resulting methods is left as further research.

6 Conclusions and further research

The numerical model introduced in [1] has been extended with the implementation of different boundary conditions. These extensions opened the door to the study of towing maneuvers. Also the Voigt-Kelvin model for springs was used instead of the Hook's law, as it was done in [2], an internal damping coefficient was introduced in the equations.

The results obtained with the presented model were validated using experimental results published in literature, in particular, Zhu's [7] and Koh's [5] experiments were considered for validation. The presented method showed good accuracy at predicting tension peaks for snapping cables when the bending effects where not important. On the other hand, it failed at predicting properly the tension on a cable exposed to larger bending effects, as the model ignore such effects. Bending effects are negligible for most of towing systems used in modern days naval procedures, this being a reason to consider the proposed method a potential tool to design towing maneuvers.

The sensitivity analysis of the results to the internal damping coefficient and the number of nodes was studied. It was shown that an appropriate election of the damping coefficient was necessary to predict accurately the peak tension of the cable. The method used to choose the internal damping coefficient was the calibration of different values selecting the values that gave a good agreement with the experimental results. It was also found that introducing internal damping lead to lower computational times. This is due to its effect of reducing the vibrations on the cable, what allows faster convergence of the ODE solvers. For the number of nodes dependence, results showed that for simple problems it was enough with a low number of nodes, approximately 10. More complex problems, as the cable swing, required a higher number, at least 20. In both cases increasing the number of nodes further from the minimum requirements did not increase accuracy significantly, but it did increase computational time, and it can be concluded that choosing an appropriate number of nodes, using calibration techniques, is essential to guarantee a correct result and a low computational time.

Application cases of towing systems were studied. The method provides important results for the tension of the cable at both ends, boat and towed body, and for the position of the cable and the towed body. The information provided is a step forward in the tools used for the design of towing maneuvers.

Finally, the necessity of implementing higher order finite element methods for solving the cable equation that considers the bending effects was shown. The presented method was not easy to modify into a higher order, so a new first order finite element method was proposed and implemented. This method could be easily modified into a higher order method, and the code was successfully validated. Then the method was modified into third order, but the resulting method was found unstable.

Further research would involve solving the stability problems of the third order method. A proposed strategy to do this is changing the basis function, so the meaning of all the coefficients is position of the cable, instead of position and spatial derivative of the cable, and the current boundary and initial conditions and the computation of external forces guarantees a good control of all the coefficients of the solution. Other proposed solution is modifying the way that forces are computed, and/or the boundary and initial conditions, so for the current basis functions all the coefficients are well controlled. Once this problem is solved implementing the same method for the cable equation that considers bending would be the next step.

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