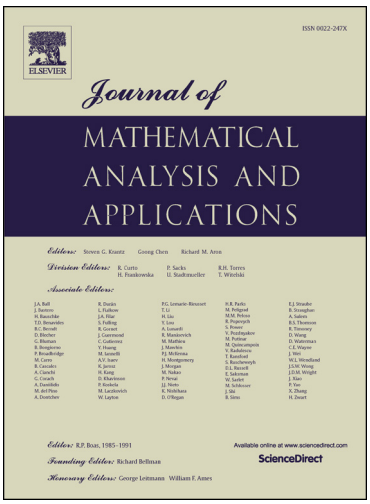
Accepted Manuscript



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PII: S0022-247X(17)30499-7
DOI: http://dx.doi.org/10.1016/j.jmaa.2017.05.039
Reference: YJMAA 21400

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 28 April 2017

Please cite this article in press as: J. Araujo, Nonmaximal ideals and the Berkovich space of the algebra of bounded analytic functions, *J. Math. Anal. Appl.* (2017), http://dx.doi.org/10.1016/j.jmaa.2017.05.039

NONMAXIMAL IDEALS AND THE BERKOVICH SPACE OF THE ALGEBRA OF BOUNDED ANALYTIC FUNCTIONS

JESÚS ARAUJO

Abstract. We prove that the Berkovich space (or multiplicative spectrum) of the algebra of bounded analytic functions on the open unit disk of an algebraically closed nonarchimedean field contains multiplicative seminorms that are not norms and whose kernel is not a maximal ideal. We also prove that in general these seminorms are not univocally determined by their kernels, and provide a method for obtaining families of different seminorms sharing the same kernel. The relation with the Berkovich space of the Tate algebra is also given.

1. Introduction

Throughout $\mathbb{K}$ is an algebraically closed field complete with respect to a (nontrivial) nonarchimedean absolute value $|\cdot|$ and H^∞ denotes the space of ($\mathbb{K}$-valued) bounded analytic functions on the open disk $\mathbb{D} := \{z \in \mathbb{K} : |z| < 1\}$, that is, the space of bounded power series on $\mathbb{D}$. When endowed with the Gauss norm (which coincides with the sup norm $\|\cdot\|$), the space H^∞ becomes a Banach algebra. We remark that, given a nonzero $f(z) = \sum_0^\infty a_n z^n \in H^\infty$, the value

$$\|f\| = \sup_{n\geq 0} |a_n| = \sup_{z\in\mathbb{D}} |f(z)|$$

does not necessarily belong to the value group $|\mathbb{K}^\times| := \{|z| : z \in \mathbb{K} \setminus \{0\}\}$.

A remarkable difference with respect to the complex case is that in a Banach algebra over $\mathbb{K}$ there can be maximal ideals that are not the kernel of any multiplicative linear functional. For this reason, the classical definition of spectrum (or maximal ideal space) of a complex Banach algebra does not carry over to the ultrametric setting. Nevertheless, the standard definition of Berkovich space (or multiplicative spectrum) yields the usual spectrum when adapted to the complex context (see Definition 1.1 and Remark 1.1).

Not much is known about the Berkovich space $\mathfrak{M}$ of H^∞. Points in $\mathfrak{M}$ are *seminorms*, and theoretically they can be divided into four types, namely:

I. Points whose kernel is a maximal ideal of codimension 1,

Key words and phrases. Ultrametric Banach algebras; multiplicative seminorms; Berkovich space; multiplicative spectrum; nonarchimedean analytic functions.

Research partially supported by the Spanish Government (MINECO, Grant MTM2016-77143-P).

II. Points whose kernel is a maximal ideal of codimension different from 1,
III. Points whose kernel is trivial, that is, equal to $\{0\}$.
IV. Points whose kernel is a nonzero nonmaximal *prime* ideal.

Points of type I can be identified with those in $\mathbb{D}$ (see [8]), as each of them is the absolute value evaluation δ_z at a point z of $\mathbb{D}$ (that is, $\delta_z(f) = |f(z)|$ for every $f \in H^\infty$).

Points of type II can be obtained by the use of ultrafilters and, in particular, *regular* sequences (a sequence (z_n) in $\mathbb{D}$ is said to be regular if $\inf_{n\in\mathbb{N}} \prod_{m\neq n} |z_n - z_m| > 0$). The key point in studying regular sequences consists of identifying each of them with a bounded sequence in $\mathbb{K}$ via the map $\mathbf{i} : H^\infty \to \ell^\infty$, $f \in H^\infty \mapsto (f(z_n)) \in \ell^\infty$. Given a regular sequence (z_n), every maximal ideal containing the ideal $\mathfrak{I}$ of all functions $f \in H^\infty$ vanishing at every z_n can be identified with an ultrafilter in $\mathbb{N}$, that is, a point in the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ (see [13, Corollary 4.7]). Thus, given a regular sequence $\mathbf{z} = (z_n)$ and a nonprincipal ultrafilter $\mathfrak{u}$ in $\mathbb{N}$ (that is, a point $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$), the seminorm

$$\delta_{\mathbf{z},\mathfrak{u}} := \lim_{\mathfrak{u}} \delta_{z_n}$$

is a point of type II. In this paper, we say that a sequence (z_n) in $\mathbb{D}$ is regular with respect to a nonprincipal ultrafilter $\mathfrak{u}$ in $\mathbb{N}$ if there exists $C \in \mathfrak{u}$ such that $(z_n)_{n\in C}$ is regular, that is,

$$\inf_{n\in C} \prod_{\substack{m\in C\\ m\neq n}} |z_n - z_m| > 0.$$

Points of type III are obviously given by multiplicative norms. The simplest case of a multiplicative norm is of the form ζ_D, for any nontrivial disk D contained in $\mathbb{D}$, where

$$\zeta_D(f) := \sup_{z\in D} |f(z)|$$

for all $f \in H^\infty$.

Our goal in this paper is to prove that the set of points of type IV is nonempty, and to study some of its features. The fact that there exist points of type IV disproves a conjecture raised in [8]. On the other hand, we also prove that there exist kernels shared by infinitely many different points of type IV. This is in sharp contrast with the situation known so far, where each maximal kernel univocally determines a seminorm.

Note that the existence of a nonzero nonmaximal closed prime ideal does not necessarily imply the existence of points of type IV. The question of the existence of such an ideal in H^∞, raised in [13], remained unknown for many years, until it was finally solved (in the positive) in [6] when $\mathbb{K}$ is of characteristic 0. Of course our result here gives a positive answer for any $\mathbb{K}$, and we can even grant the existence of infinite chains of *closed* prime ideals (see [13, Problem after Lemma 4.10]).

Definition 1.1. Let A be a unital commutative Banach algebra over $\mathbb{K}$. A map $\varphi : A \to [0, +\infty)$ is a continuous multiplicative *ring* seminorm on A if the following conditions hold:

(1) $\varphi(0_A) = 0$ and $\varphi(1_A) = 1$.
(2) $\varphi(ab) = \varphi(a)\ \varphi(b)$ for all $a, b \in A$.
(3) $\varphi(a + b) \leq \varphi(a) + \varphi(b)$ for all $a, b \in A$.
(4) $\varphi(a) \leq \|a\|$ for all $a \in A$.

Remark 1.1. We assume that $\|1_A\| = 1$. It is straightforward to show (see for instance [4, Lemma 1.7]) that every continuous multiplicative ring seminorm is also an *ultrametric algebra* seminorm on A, that is, it further satisfies:

(5) $\varphi(\lambda a) = |\lambda|\, \varphi(a)$ for all $\lambda \in \mathbb{K}$ and $a \in A$.
(6) $\varphi(a + b) \leq \max\{\varphi(a), \varphi(b)\}$ for all $a, b \in A$.

The Berkovich space (or multiplicative spectrum) $\mathscr{M}(A)$ of A is the set of all continuous multiplicative (in any of the equivalent senses of Definition 1.1 and Remark 1.1) seminorms endowed with the topology of simple convergence, that is, a net $(\zeta_\lambda)_{\lambda\in\Lambda}$ in $\mathscr{M}(A)$ converges to $\zeta_0 \in \mathscr{M}(A)$ if $(\zeta_\lambda(a))_{\lambda\in\Lambda}$ converges to $\zeta_0(a)$ for all $a \in A$. It is well known that $\mathscr{M}(A)$ is Hausdorff and compact (see for instance [1, Theorem 1.2.1] or [4, Theorem 1.11]). Indeed, the multiplicative spectrum of some algebras is a compactification of $\mathbb{D}$ (see [1, 2, 4, 5, 11, 17]). Nevertheless, in our case, it is unknown if $\mathbb{D}$ is dense in $\mathfrak{M} = \mathscr{M}(H^\infty)$, which is a nonarchimedean version of the Corona problem (a related problem was solved in [13]). In fact, what is now known is that $\mathbb{D}$ is dense in the subset of all seminorms whose kernel is a maximal ideal (see [7]). In this paper, all seminorms we deal with belong to the closure of $\mathbb{D}$ (see Theorem 2.4).

It is easy to check that the kernel $\ker\zeta := \{f \in A : \zeta(f) = 0\}$ of every element $\zeta \in \mathscr{M}(A)$ is a closed prime ideal of A. When we say that a seminorm has *maximal kernel* or *nonzero nonmaximal kernel*, we mean that its kernel is a *maximal ideal* or a *nonzero nonmaximal ideal*, respectively.

We see that if D is a (closed or open) disk, then ζ_D belongs to $\mathfrak{M}$. Also, since $|\mathbb{K}^\times|$ is dense in $\mathbb{R}^+$, $\zeta_{D^+(z,r)} = \zeta_{D^-(z,r)}$ for $z \in \mathbb{D}$ and $r \in (0, 1)$ (where $D^+(z, r)$ and $D^-(z, r)$ are the closed and open disks with center z and radius r, respectively).

Recall that, given $f \in H^\infty$ and $z_0 \in \mathbb{D}$, f can be written by $f(z) = \sum_{n=1} a_n(z - z_0)^n$ for every $z \in \mathbb{D}$ (see for instance [15, Theorem 25.1]), and that z_0 is a zero of f of multiplicity $m \geq 1$ if there is $g \in H^\infty$ with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^m g(z)$ for all z. For $E \subset \mathbb{D}$, we denote by $\mathtt{Z}(f, E)$ the number of zeros of f in E (by this we will always mean *taking into account multiplicities*).

For $r > 0$, $C(0, r)$ will be the set of all z with $|z| = r$. If $D^+(z, r) \subset C(0, |z|)$ and $w_1, \ldots, w_n$ are the zeros of $f \in H^\infty$ with absolute value $|z|$,

then we define

$$\xi_{D^+(z,r)}(f) := \begin{cases} r^{\mathtt{Z}(f,D^+(z,r))} \prod_{|z-w_i|>r} |z-w_i| & \text{if } \mathtt{Z}(f,C(0,|z|)) \neq 0, \\ 1 & \text{if } \mathtt{Z}(f,C(0,|z|)) = 0, \end{cases}$$

where we understand that $\prod_{|z-w_i|>r} |z-w_i| = 1$ if $|z-w_i| \leq r$ for all $i = 1, \ldots, n$.

In this paper we mainly study the set $\mathfrak{M}_0$ of all seminorms of the form $\varphi := \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$, where $\mathfrak{u}$ is any nonprincipal ultrafilter in $\mathbb{N}$, (z_n) is any sequence in $\mathbb{D}$ with $\lim_{n\to\infty} |z_n| = 1$, and (r_n) is any sequence in $(0,1)$. Obviously, in many cases $\varphi := \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \|\cdot\|$. This happens for instance when the set of all $n \in \mathbb{N}$ such that $|z_n| \leq r_n$ belongs to $\mathfrak{u}$. But even in this case we can also write $\|\cdot\| = \lim_{\mathfrak{u}} \zeta_{D^+\left(z_n,|z_n|^2\right)}$, so we can assume that $r_n < |z_n|$ for all n. It is clear that, if $\lim_{\mathfrak{u}} r_n > 0$, then there exist $r \in (0,1)$ and a sequence $\mathbf{k} = (k_n)$ in $\mathbb{N}$ (not necessarily unique) such that $0 < r = \lim_{\mathfrak{u}} {r_n}^{k_n} < 1$. We will see in Corollary 6.2 that

$$\varphi = \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} := \lim_{\mathfrak{u}} \zeta_{D^+\left(z_n,\sqrt[k_n]{r}\right)}.$$

This means that, when $\lim_{\mathfrak{u}} r_n > 0$, we can restrict ourselves to seminorms of the special form $\varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$. On the other hand, it is very easy to see that, if $\lim_{\mathfrak{u}} r_n = 0$, then $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \delta_{\mathbf{z},\mathfrak{u}} := \lim_{\mathfrak{u}} \delta_{z_n}$. We also prove that, in fact, all points in $\mathfrak{M}_0$ can be written in the form $\delta_{\mathbf{z},\mathfrak{u}}$ (see Theorem 2.4).

We can say more. Given $\varphi \in \mathfrak{M}_0$, there exist a sequence (w_n) in $\mathbb{D}$ with $\lim_{n\to\infty} |w_n| = 1$ and a sequence (s_n) in $(0,1)$ such that the disks $D^+(w_n, s_n)$ are pairwise disjoint and $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(w_n,s_n)}$ (see Corollaries 6.3 and 6.4).

We also deal here with two subsets of $\mathfrak{M}_0$: $\mathfrak{M}_0'$ and $\mathfrak{M}_1$. The set $\mathfrak{M}_0'$ consists of all the limits of the above form $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$, where (z_n) is *regular with respect to* $\mathfrak{u}$ and all the disks $D^+(z_n, r_n)$, $n \in C$, are pairwise disjoint for some $C \in \mathfrak{u}$. If we drop the requirement that (z_n) be regular with respect to $\mathfrak{u}$, then the results we obtain are quite different (see Proposition 6.7; see also Corollary 6.4).

As for the second set, $\mathfrak{M}_1$, it has the remarkable property that no seminorm in it is determined by its kernel, that is, there are many other seminorms having the same kernel. For the description of $\mathfrak{M}_1$, we generalize the notion of regular sequence as follows: Given a sequence $\mathbf{z} = (z_n)$ in $\mathbb{D}$ and a nonprincipal ultrafilter $\mathfrak{u}$ in $\mathbb{N}$, we denote by $\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$ the set of all sequences $\mathbf{k} = (k_n)$ in $\mathbb{N}$ for which there exists $C_{\mathbf{k}} \in \mathfrak{u}$ such that

$$\inf_{n\in C_{\mathbf{k}}} \prod_{\substack{m\in C_{\mathbf{k}} \\ m\neq n}} |z_n - z_m|^{k_m} > 0.$$

Now, for a nonprincipal ultrafilter $\mathfrak{u}$ of $\mathbb{N}$, $\mathbf{k} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$ and $r \in (0,1)$, we set $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} := \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$, that is,

$$\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} := \lim_{\mathfrak{u}} \zeta_{D^+\left(z_n,\sqrt[k_n]{r}\right)},$$

and

$$\left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right) := \left\{\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} : r \in (0,1)\right\}.$$

We put, for $\mathbf{z}$ and $\mathfrak{u}$ fixed, $\mathfrak{M}_{\mathbf{z},\mathfrak{u}} := \bigcup_{\mathbf{k}\in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})} \left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right)$, and more in general $\mathfrak{M}_{\mathbf{z}} := \bigcup_{\mathfrak{u}\in\beta\mathbb{N}\setminus\mathbb{N}} \mathfrak{M}_{\mathbf{z},\mathfrak{u}}$. Finally, we set $\mathfrak{M}_1 := \bigcup_{\mathbf{z}} \mathfrak{M}_{\mathbf{z}}$.

Note that, in principle, a seminorm $\varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}'_0$ cannot be written as $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$ because $\mathbf{k}$ does not necessarily belong to $\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$ (nevertheless, in general it does, as can be seen in Theorem 2.10). On the other hand, $\mathfrak{M}_1$ is indeed a subset of $\mathfrak{M}'_0$ (see Remark 6.2). But, of course, the fact that a seminorm $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}_1$ belongs to $\mathfrak{M}'_0$ does not necessarily imply that there exists $C \in \mathfrak{u}$ such that all disks $D^+\left(z_n, \sqrt[k_n]{r}\right)$ are pairwise disjoint for $n \in C$. Nevertheless, we have the following remark that will be used later.

Remark 1.2. If there exists $C \in \mathfrak{u}$ with $M := \inf_{n\in C} \prod_{\substack{m\in C\\ m\neq n}} |z_n - z_m|^{k_m} > 0$ and $0 < r_0 < M$, then all the disks $D^+\left(z_n, \sqrt[k_n]{r_0}\right)$, $n \in C$, are pairwise disjoint.

By $\mathbf{1}$, we denote the sequence constantly equal to 1. In general, $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$ are used, respectively, for sequences (k_n), (l_n) and (m_n) in $\mathbb{N}$. Also $\mathbf{z}$, $\mathbf{w}$, and $\mathbf{v}$ denote, respectively, sequences (z_n), (w_n) and (v_n) in $\mathbb{D}$.

As usual, given a topological space A and a subset B of A, $\mathrm{cl}_A B$ denotes the closure of B in A.

The paper is organized as follows. In Section 2 we state the main results. In Section 3, we give some technical results that are used through the paper. In Section 4, we show that the Berkovich space of the Tate algebra T_1 (without one point) can be homeomorphically embedded as an open subset of $\mathfrak{M}$ (Theorem 2.12). In Section 5, we study the existence of bounded analytic functions with a prescribed number of zeros, paying attention to their norms. In Section 6, we study how the same seminorm can be expressed in different forms, and we prove in particular Theorem 2.4. Section 7 is devoted to proving most of the results stated in Section 3 (and some others concerning $\mathfrak{M}_1$).

2. Main results

Theorem 2.1. *Let $\varphi \in \mathfrak{M}_0$ have nonzero kernel. Given $f \in \ker\varphi$ with $f \neq 0$ and $r \in (0, \|f\|)$, there exists $\psi \in \mathfrak{M}'_0$ with nonzero nonmaximal kernel such that $\varphi \leq \psi$ and $\psi(f) = r$.*

In particular all kernels of seminorms $\delta_{\mathbf{z},\mathfrak{u}}$, with $\mathbf{z}$ regular with respect to $\mathfrak{u}$, strictly contain nontrivial kernels. Therefore, Theorem 2.1 provides a positive answer to the question of the existence of seminorms with nonzero nonmaximal kernel. We easily deduce the following.

Corollary 2.2. *H^∞ contains infinite chains of nonzero closed prime ideals.*

Theorem 2.3. *Let* $\mathbf{z}$ *be regular with respect to a nonprincipal ultrafilter* $\mathfrak{u}$ *in* $\mathbb{N}$*. Then there exists a linearly ordered compact and connected set* $A^{\mathfrak{u}}_{\mathbf{z}} \subset \mathfrak{M}$ *with* $\delta_{\mathbf{z},\mathfrak{u}} = \min A^{\mathfrak{u}}_{\mathbf{z}}$ *and* $\|\ \| = \max A^{\mathfrak{u}}_{\mathbf{z}}$ *such that* $\ker\varphi$ *is nonzero and nonmaximal for all* $\varphi \in A^{\mathfrak{u}}_{\mathbf{z}} \setminus \{\delta_{\mathbf{z},\mathfrak{u}}, \|\ \|\}$*.*

Points in $\mathfrak{M}_0$ can in fact be written in the form $\delta_{\mathbf{w},\mathfrak{v}} := \lim_{\mathfrak{v}} \delta_{w_m}$, where $\mathbf{w}$ may be *not* regular with respect to $\mathfrak{v}$.

Theorem 2.4. $\mathfrak{M}_0 = \{\delta_{\mathbf{w},\mathfrak{v}} : \lim_{n\to\infty} |w_n| = 1, \mathfrak{v} \in \beta\mathbb{N} \setminus \mathbb{N}\}$.

Theorem 2.5. *Let* $\mathbf{z}$ *be a regular sequence with respect to* $\mathfrak{u} \in \beta\mathbb{N}\setminus\mathbb{N}$*. Then, for each* $\mathbf{k} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$*, the maps* $\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} := \lim_{r\to 0} \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$ *and* $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} := \lim_{r\to 1} \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$ *exist and belong to* $\mathfrak{M}_0$*, and*

$$\mathrm{cl}_{\mathfrak{M}}\left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right) = \left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right) \cup \left\{\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right\}.$$

Moreover $\mathrm{cl}_{\mathfrak{M}}\left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right)$ *is homeomorphic to the interval* $[0,1]$*, through a homeomorphism sending* $\left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right)$ *onto* $(0,1)$*.*

The following result says that many seminorms share the same nonzero nonmaximal kernels.

Corollary 2.6. *Let* $\mathbf{z}$ *be a regular sequence with respect to* $\mathfrak{u} \in \beta\mathbb{N}\setminus\mathbb{N}$*. Then, for each* $\mathbf{k} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$*, all seminorms in* $\left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right] := \left(\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}, \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right) \cup \left\{\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}\right\}$ *have the same kernel.*

We can compare Corollary 2.6 with Theorem 1 in [7], where it is proven that each maximal ideal is the kernel of a *unique* seminorm.

In view of Corollary 2.6, we can consider kernels of seminorms in $\mathfrak{M}_{\mathbf{z},\mathfrak{u}}$ given by different sequences $\mathbf{k}$ and $\mathbf{l}$. It is very easy to deduce that they coincide when $\lim_{\mathfrak{u}} l_n/k_n \in (0,+\infty)$. In any other case, we have the following corollary.

Corollary 2.7. *Let* $\mathbf{z}$ *be a regular sequence with respect to* $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$*. Let* $\mathbf{k},\mathbf{l} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$*. If* $\lim_{\mathfrak{u}} l_n/k_n = 0$*, then* $\ker\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} \subsetneq \ker\zeta^{\mathbf{l},1}_{\mathbf{z},\mathfrak{u}}$*.*

Corollary 2.8. *The kernel of every point in* $\mathfrak{M}_1$ *is nonzero and nonmaximal.*

We easily deduce that $\ker\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}$ is always nonzero and nonmaximal, and that $\ker\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$ is nonzero. Moreover, if $\lim_{\mathfrak{u}} k_n < +\infty$, then $\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} = \delta_{\mathbf{z},\mathfrak{u}}$, so its kernel is maximal. Now, we see that the converse also holds.

Corollary 2.9. *Let* $\mathbf{z}$ *be a regular sequence with respect to* $\mathfrak{u} \in \beta\mathbb{N}\setminus\mathbb{N}$*. Then, for each* $\mathbf{k} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$*,* $\ker\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$ *is nonmaximal if and only if* $\lim_{\mathfrak{u}} k_n = +\infty$*.*

Next, if $\varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}, \varphi^{\mathbf{l},s}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}'_0$ do not belong to $\mathfrak{M}_1$, then $\varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} = \varphi^{\mathbf{l},s}_{\mathbf{z},\mathfrak{u}}$. That is, all points in $\mathfrak{M}'_0$ (with nonmaximal kernel) belong to $\mathfrak{M}_1$ but at most one:

Theorem 2.10. *Given $\varphi = \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}'_0$, either $\varphi \in \mathfrak{M}_{\mathbf{z},\mathfrak{u}}$ or*

$$\varphi = \sup_{\mathbf{m}\in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})} \zeta^{\mathbf{m},1}_{\mathbf{z},\mathfrak{u}}.$$

Corollary 2.11. *Let $\varphi = \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}'_0$, where $(|z_n|)$ is strictly increasing. Then either $\varphi \in \mathfrak{M}_1$ or $\varphi = \| \ \|$.*

We finish our list of main results with a theorem linking the Berkovich space of the Tate algebra T_1 with $\mathfrak{M}$. Recall that T_1 is the Banach algebra of analytic functions on the closed unit disk $D^+(0,1)$, that is, the space of all power series with coefficients in $\mathbb{K}$ converging on $D^+(0,1)$. It coincides with the subspace of H^∞ consisting of all power series $\sum_{n=0}^\infty a_n z^n$ with $\lim_{n\to\infty} |a_n| = 0$, and contains the polynomial algebra $\mathbb{K}[z]$ as a dense subset.

The Berkovich space $\mathscr{M}(T_1)$ is well known (see for instance [1, 1.4.4]). Each $\varphi \in \mathscr{M}(T_1)$ can be written in terms of (a limit of) seminorms $\zeta_{D^+(a,r)}$, in such a way that there is a natural extension of each φ to a $\mathbf{i}(\varphi) \in \mathfrak{M}$ defined in the same terms. We put $\mathscr{M}^* := \mathscr{M}(T_1) \setminus \{\| \ \|\}$.

Theorem 2.12. *The canonical map*

$$\mathbf{i} : \mathscr{M}^* \to \mathbf{i}(\mathscr{M}^*) \subset \mathfrak{M}$$

is a homeomorphism. Moreover $\mathbf{i}(\mathscr{M}^*)$ is open in $\mathfrak{M}$, and $\mathscr{M}(T_1)$ is homeomorphic to a quotient of $\mathfrak{M}$.

3. Some technical results

We begin this section by giving some well known results concerning the zeros of analytic functions. Suppose that $f(z) = 1 + \sum_{n=1}^\infty a_n z^n \in H^\infty$. For each $r \in [0,1)$, let $M_r(f) := \max_{n\geq 0} |a_n| r^n$. We say that $r \in (0,1)$ is a critical radius for f if there are at least two distinct indices m, k such that

$$M_r(f) = |a_m| r^m = |a_k| r^k.$$

It turns out that r is a critical radius for f if and only if $C(0,r)$ contains a zero of f. Indeed, the number of zeros (taking into account multiplicities) located in $C(0,r)$ coincides with the number

$$\mathtt{Z}(f, C(0,r)) = \nu_r(f) - \mu_r(f)$$

where $\nu_r(f)$ and $\mu_r(f)$ are defined, respectively, as the greatest and the smallest n such that $|a_n| r^n = M_r(f)$ (see for instance [14, Section 2.2, Theorem 1] for a proof when $\mathbb{K}$ is an algebraically closed extension of $\mathbb{Q}_p$ but valid also for our $\mathbb{K}$). It is clear from the definition that, if $r < s$, then $\nu_r(f) \leq \mu_s(f)$. In fact, the critical radii form an increasing (finite or infinite) sequence (R_n) satisfying $\mu_{R_n}(f) = \nu_{R_{n-1}}(f)$ for all $n \geq 2$ that, when infinite, has 1 as its only accumulation point.

Hence, if $r \in (0,1)$ is not a critical radius, then there exists only one $n_r \in \mathbb{N}$ with $|a_{n_r}| r^{n_r} = M_r(f)$ and $|f(z)| = |a_{n_r}| r^{n_r}$ for all z with $|z| = r$. It turns out that $n_r = \nu_{R_i}(f)$, where R_i is the greatest critical radius strictly less than r, if there is any, and $n_r = \mu_{R_1}(f) = 0$ otherwise.

On the other hand, writing $\nu_n = \nu_{R_n}(f)$ for short, we see that $|f(0)| = 1 = |a_{\nu_1}| {R_1}^{\nu_1}$ and $|a_{\nu_1}| = 1/{R_1}^{\nu_1}$. Also, $|a_{\nu_1}| {R_2}^{\nu_1} = |a_{\nu_2}| {R_2}^{\nu_2}$, giving $|a_{\nu_2}| = 1/\left({R_1}^{\nu_1} {R_2}^{\nu_2-\nu_1}\right) = 1/\prod_{i=1}^{2} {R_i}^{\mathrm{Z}(f,C(0,R_i))}$. For all n, this process leads to $|a_{\nu_n}| = 1/\prod_{i=1}^{n} {R_i}^{\mathrm{Z}(f,C(0,R_i))}$. We finally remark that

$$\|f\| = \sup_n |a_{\nu_n}| = \frac{1}{\prod_{i=1}^{\infty} {R_i}^{\mathrm{Z}(f,C(0,R_i))}}.$$

We continue with the results of this section. The proof of the following lemma is easy.

Lemma 3.1. *Suppose that $f \in H^\infty$ has exactly k zeros $w_1, \ldots, w_k$ of absolute value $R \in (0,1)$. Then $f(z) = g(z)\prod_{i=1}^{k}(z - w_i)$, where $g \in H^\infty$ has no zeros of absolute value R. Also, $\|f\| = \|g\|$.*

Lemma 3.2. *Let $f \in H^\infty$ be such that $f(0) = 1$, and suppose that its critical radii are $R_1 < R_2 < \cdots < 1$. Suppose also that for each $i \in \mathbb{N}$, f has exactly m_i zeros $w_1^i, \ldots, w_{m_i}^i$ in $C(0, R_i)$. Then, given $z \in \mathbb{D}$ with $|z| = R_k$,*

$$|f(z)| = \frac{{R_k}^{m_1+\cdots+m_{k-1}} \prod_{j=1}^{m_k} \left|z - w_j^k\right|}{\prod_{i=1}^{k} {R_i}^{m_i}}.$$

Similarly, if $R_k < R := |z| < R_{k+1}$, then

$$|f(z)| = \frac{R^{m_1+\cdots+m_k}}{\prod_{i=1}^{k} {R_i}^{m_i}}.$$

Proof. By Lemma 3.1, $f(z) = g(z)\prod_{i=1}^{k}\prod_{j=1}^{m_i}\left(z - w_j^i\right)$, where $g \in H^\infty$ has m_i zeros in each $C(0, R_i)$ for every $i > k$, and no other zeros. This implies that the critical radii of g are the R_i for $i > k$ and that $|g(z)|$ is constantly equal to $|g(0)|$ on $D^-(0, R_{k+1})$, that is, when $|z| < R_{k+1}$,

$$|g(z)| = |g(0)| = 1/{R_1}^{m_1} \cdots {R_k}^{m_k}.$$

Now, the result follows easily. □

Corollary 3.3. *Suppose that $f \in H^\infty$ has no zeros in $D^-(z_0, r)$, where $0 < r \leq |z_0|$. Then $|f(z)| = |f(z_0)|$ for every $z \in D^-(z_0, r)$, and $\zeta_{D^+(z_0,r)}(f) = |f(z_0)|$.*

Corollary 3.4 will be very useful.

Corollary 3.4. *Let $\mathbf{z}$ be a sequence in $\mathbb{D}$ with $(|z_n|)$ increasing and converging to 1, and let (r_n) be a sequence in $(0,1)$ with $D^+(z_n, r_n) \subset C(0, |z_n|)$ for all n. Given a nonprincipal ultrafilter $\mathfrak{u}$ in $\mathbb{N}$,*

$$\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}(f) = \|f\| \lim_{\mathfrak{u}} \xi_{D^+(z_n,r_n)}(f)$$

for every $f \in H^\infty$.

Proof. Since each $\xi_{D^+(z_n,r_n)}$ is multiplicative, it is enough to prove it for $f \in H^\infty$ with $f(0) = 1$. Also, the result is obvious if f has a finite number of zeros in $\mathbb{D}$, so we assume that the sequence (w_k) of its zeros satisfies that $(|w_k|)$ is increasing and convergent to 1.

For each $n \in \mathbb{N}$, take k_n as the largest k with $|w_k| \leq |z_n|$. If $|w_{k_n}| < |z_n|$, then $\xi_{D^+(z_n,r_n)}(f) = 1$ and, by Lemma 3.2,

$$\zeta_{D^+(z_n,r_n)}(f) = \frac{|z_n|^{k_n}}{\prod_{i=1}^{k_n} |w_i|} = \frac{|z_n|^{k_n}}{\prod_{i=1}^{k_n} |w_i|}\, \xi_{D^+(z_n,r_n)}(f).$$

Similarly, if $|w_{k_n}| = |z_n|$ and $M_{k_n} = \operatorname{card}\{m : |w_m| = |w_{k_n}|\}$, then

$$\zeta_{D^+(z_n,r_n)}(f) = \frac{1}{|z_n|^{M_{k_n}}} \frac{|z_n|^{k_n}}{\prod_{i=1}^{k_n} |w_i|}\, \xi_{D^+(z_n,r_n)}(f).$$

Now, recall that, if (a_n) is a decreasing sequence in $\mathbb{R}$ with $\sum_{n=1}^{\infty} a_n < +\infty$, then $\lim_{n\to\infty} n a_n = 0$. Equivalently, since $(|w_k|)$ is increasing and $\prod_{k=1}^{\infty} |w_k| > 0$, $\lim_{n\to\infty} |w_{k_n}|^{k_n} = 1$, so $\lim_{n\to\infty} |z_n|^{k_n} = 1$ and, consequently, $\lim_{n\to\infty} |z_n|^{M_{k_n}} = 1$. On the other hand, since $\|f\| = 1/\prod_{k=1}^{\infty} |w_k|$, we easily conclude the result. □

We give a final lemma that will be used later.

Lemma 3.5. *Let $z \in \mathbb{D}$, $z \neq 0$, and suppose that $0 < s < r < |z|$. If $f \in H^\infty$, then*

$$\left(\frac{s}{r}\right)^{\mathtt{Z}\left(f,D^-(z,r)\right)} \xi_{D^+(z,r)}(f) \leq \xi_{D^+(z,s)}(f) \leq \xi_{D^+(z,r)}(f).$$

Proof. It is clear that $\xi_{D^+(z,s)}(f) \leq \xi_{D^+(z,r)}(f)$. On the other hand, if $w_1, \ldots, w_n$ are the zeros of f in $D^-(z,r) \setminus D^+(z,s)$, and $z_1, \ldots, z_m$ are the zeros of f in $C(0,|z|) \setminus D^-(z,r)$, then

$$\begin{aligned}
\xi_{D^+(z,s)}(f) &= s^{\mathtt{Z}\left(f,D^+(z,s)\right)} \prod_{i=1}^{n} |z - w_i| \prod_{j=1}^{m} |z - z_j| \\
&\geq \left(\frac{s}{r}\right)^{\mathtt{Z}\left(f,D^-(z,r)\right)} r^{\mathtt{Z}\left(f,D^-(z,r)\right)} \prod_{j=1}^{m} |z - z_j| \\
&= \left(\frac{s}{r}\right)^{\mathtt{Z}\left(f,D^-(z,r)\right)} \xi_{D^+(z,r)}(f),
\end{aligned}$$

and we are done. □

4. $\mathfrak{M}$ and $\mathscr{M}^*$

Proposition 4.1 is given in [6]. For the sake of completeness, we provide a (different) proof.

Proposition 4.1. *Suppose that $\varphi \in \mathfrak{M}$ satisfies $\varphi = \psi \in \mathscr{M}^*$ on $\mathbb{K}[z]$. Then $\varphi = \mathbf{i}(\psi)$.*

Proof. To see that $\varphi = \mathbf{i}(\psi)$, it is enough to prove the equality at any $f \in H^\infty$ satisfying $f(0) = 1$ and having infinitely many critical radii R_j. Since $\psi \neq \|\ \|$, we can find $r \in (0,1)$ with $\psi \leq \zeta_{D^+(0,r)}$, and we may assume that f has m_j zeros in each $C(0,R_j)$, and that $r < R_1 < R_2 < \cdots$. For each $R \in (R_1, 1)$, we write $f = P_R f_R$, where $P_R \in \mathbb{K}[z]$ is the product $P_R(z) := \prod_{i=1}^{n}(z - z_i)$, being the z_i all the zeros of f in $D^+(0,R)$, and $f_R \in H^\infty$ has no zeros in $D^+(0,R)$.

Claim. *The limit* $[\varphi](f) := \lim_{R\to 1}\varphi(f_R)$ *exists, and*

$$\varphi(f) = \frac{\mathbf{i}(\psi)(f)}{\|f\|}\,[\varphi](f).$$

For $R \in (R_1, 1)$ fixed, let N be the largest integer with $R_N \leq R$, so that $R_1, \ldots, R_N$ are the critical radii of P_R. Obviously $|P_R|$ and $|f|$ are constant in $D^+(0,r)$, so $\psi(P_R) = |P_R(0)| = \prod_{j=1}^{N} {R_j}^{m_j}$, and $\mathbf{i}(\psi)(f) = |f(0)| = 1$. Since $\|f\| = 1/\prod_{n=1}^{\infty} {R_n}^{m_n}$, $\lim_{R\to 1}\psi(P_R)\|f\| = 1 = \mathbf{i}(\psi)(f)$. Also $\varphi(f) = \psi(P_R)\varphi(f_R)$ for all R, so by taking limits we prove the claim. ■

Also, since $\|P_R\| = 1$, $\|f_R\| = \|f\|$ for all R, and consequently $[\varphi](f) \leq \|f\|$ and $\varphi(f) \leq \mathbf{i}(\psi)(f)$. We easily conclude that $\varphi(g) \leq \mathbf{i}(\psi)(g)$ whenever $g \in H^\infty$ has constant absolute value on $D^+(0,r)$.

Suppose next that $\varphi(f) < \mathbf{i}(\psi)(f)$, that is, $\varphi(f) < 1$. Note that $f(z) := 1 + \sum_{n=1}^{\infty} a_n z^n$ and, since there are no critical radii $R \leq r$, $M := \sup_{n\in\mathbb{N}} a_n r^n < 1$, so the function $h(z) := f(z) - 1$ satisfies $|h(z)| \leq M$ for all $z \in D^+(0,r)$ and $\mathbf{i}(\psi)(h) \leq \zeta_{D^+(0,r)}(h) < 1$. We can write $h = Pg$, where $P \in \mathbb{K}[z]$ and $g \in H^\infty$ has constant absolute value in $D^+(0,r)$, which implies that $\varphi(g) \leq \mathbf{i}(\psi)(g)$. Obviously, $\psi(P)\mathbf{i}(\psi)(g) = \mathbf{i}(\psi)(h) < 1$, whereas $\psi(P)\varphi(g) = \varphi(h) = 1$ (because $\varphi(f) < 1$ and $\varphi(1) = 1$), implying that $\mathbf{i}(\psi)(g) < \varphi(g)$. Since this is impossible, we conclude that $\varphi(f) = \mathbf{i}(\psi)(f)$. □

Proof of Theorem 2.12. It is obvious that $\mathbf{i}$ is injective and that $\mathbf{i}^{-1} : \mathbf{i}(\mathscr{M}^*) \to \mathscr{M}^*$ is continuous. Next, suppose that $(\zeta_\lambda)_{\lambda\in\Lambda}$ is a net in $\mathscr{M}^*$ convergent to $\zeta_{\lambda_0} \in \mathscr{M}^*$. By the definition of convergence of a net, since $\zeta_{\lambda_0} \neq \|\ \|$, there exist $r \in (0,1)$ and $\lambda_1 \in \Lambda$ such that $\zeta_\lambda \leq \zeta_{D^+(0,r)}$ for all $\lambda \geq \lambda_1$, and $\zeta_{\lambda_0} \leq \zeta_{D^+(0,r)}$. This implies in particular that, for $g \in H^\infty$, if $|g|$ is constant in $D^+(0,r)$, then $\mathbf{i}(\zeta_{\lambda_0})(g) = |g(0)| = \mathbf{i}(\zeta_\lambda)(g)$ for all $\lambda \geq \lambda_1$.

Now consider $f \in H^\infty$. Obviously $f = Pg$ where P is a polynomial with all its zeros in $D^+(0,r)$ and $g \in H^\infty$ has no zeros in $D^+(0,r)$. Then, taking into account that $P \in \mathbb{K}[z]$, for $\lambda \geq \lambda_1$ and $\lambda = \lambda_0$, $\mathbf{i}(\zeta_\lambda)(f) = \zeta_\lambda(P)\,|g(0)|$. Consequently $(\mathbf{i}(\zeta_\lambda)(f))_{\lambda\in\Lambda}$ converges to $\mathbf{i}(\zeta_{\lambda_0})(f)$. The fact that $\mathbf{i}$ is continuous follows easily.

We next see that $\mathbf{i}(\mathscr{M}^*)$ is open in $\mathfrak{M}$. Given $\varphi \in \mathscr{M}^*$, there exists $r < 1$ such that $\varphi \leq \zeta_{D^+(0,r)}$ and a polynomial $P \in \mathbb{K}[z]$ with all its zeros in $D^+(0,r)$ such that $\zeta_{D^+(0,r)}(P) < \|P\|/2$. Now if $\psi \in \mathfrak{M}$ satisfies $|\psi(P) - \mathbf{i}(\varphi)(P)| < \|P\|/2$, then $\psi(P) < \|P\|$, so the restriction of ψ to $\mathbb{K}[z]$ is not equal to $\|\ \|$. By Proposition 4.1, ψ belongs to $\mathbf{i}(\mathscr{M}^*)$.

Finally, the map $T : \mathfrak{M} \to \mathscr{M}(T_1)$ that coincides with $\mathbf{i}^{-1}$ on $\mathbf{i}(\mathscr{M}^*)$ and sends $\mathfrak{M} \setminus \mathbf{i}(\mathscr{M}^*)$ to $\| \ \|$ is easily seen to be continuous and closed. The result now follows from [3, Proposition 2.4.3]. □

5. Sequences of zeros

It is well known that, in complex analysis, under some natural conditions, a bounded analytic function can be constructed to have zeros precisely at a given sequence (z_n) of complex numbers in the open unit disk, each with a prescribed multiplicity (see [10, Theorem II.2.2]). A similar result does not hold for nonarchimedean fields, in particular when they are not spherically complete, as it is the case of the p-adic complex fields $\mathbb{C}_p$ (see [12]). Nevertheless, in the nonarchimedean context, an analytic function (not necessarily bounded) can be found having as zeros the points of the sequence (z_n) when it satisfies a natural condition, but with multiplicities larger (and not necessarily equal) than those prescribed (see [9], and [4, Theorem 25.5] for a detailed proof).

Roughly speaking, here we are interested in finding $f \in H^\infty$ having zeros not at points of a given sequence (z_n), but *close* to them, and paying attention instead to the the fact that any of those zeros is simple and that $\|f\|$ belongs to $|\mathbb{K}^\times|$.

We begin with a well known result (see for instance [16, p. 15]).

Lemma 5.1. *Let $\gamma_1, \ldots, \gamma_n \in \mathbb{K}$ be pairwise different. Then the rank of the Vandermonde matrix*

$$\begin{pmatrix} 1 & \gamma_1 & {\gamma_1}^2 & \cdots & {\gamma_1}^{n-1} \\ 1 & \gamma_2 & {\gamma_2}^2 & \cdots & {\gamma_2}^{n-1} \\ 1 & \gamma_3 & {\gamma_3}^2 & \cdots & {\gamma_3}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_n & {\gamma_n}^2 & \cdots & {\gamma_n}^{n-1} \end{pmatrix}$$

is n.

Next, and throughout this section, we use the notation and basic properties of critical radii and zeros of analytic functions given at the beginning of Section 3.

Lemma 5.2. *Let $P(z) := a_0 + a_1 z + \cdots + z^n \in \mathbb{K}[z]$. If $P(z) = \prod_{i=1}^n (z - z_i)$ with $z_1, \ldots, z_n \in \mathbb{D}$, then $|a_i| \leq 1$ for all $i \in \{0, \ldots, n-1\}$.*

Lemma 5.3. *Let $P_1(z), Q(z) \in \mathbb{K}[z]$, where the degree of $P_1(z)$ is $n > 0$, and let*

$$P_2(z) := P_1(z) + z^{n+1} Q(z).$$

Suppose that R_1 is a critical radius of $P_2(z)$ satisfying $\mu_{R_1}(P_2) > n$ and that $C(0, R_1)$ contains exactly k zeros of $P_2(z)$, $k > 0$. Then it also contains exactly k zeros of $Q(z)$.

Proof. We write $P_1(z) := a_0 + a_1 z + \ldots + a_n z^n$ and $Q(z) := a_{n+1} + a_{n+2} z + \ldots + a_{n+m+1} z^m$, so that $P_2(z) = \sum_{i=0}^{n+m+1} a_i z^i$. By definition, $|a_i| {R_1}^i < M_{R_1}(P_2)$ for all $i \notin \{\mu_{R_1}(P_2), \ldots, \nu_{R_1}(P_2)\}$. Also

$$M_{R_1}(P_2) = \left|a_{\mu_{R_1}(P_2)}\right| {R_1}^{\mu_{R_1}(P_2)} = \left|a_{\nu_{R_1}(P_2)}\right| {R_1}^{\nu_{R_1}(P_2)},$$

and $|a_i| {R_1}^i \leq M_{R_1}(P_2)$ for $i \in \{\mu_{R_1}(P_2), \ldots, \nu_{R_1}(P_2)\}$. Taking into account that $\mu_{R_1}(P_2) \geq n+1$, we easily see that $\mu_{R_1}(Q) = \mu_{R_1}(P_2) - n - 1$ and $\nu_{R_1}(Q) = \nu_{R_1}(P_2) - n - 1$. Since, $\nu_{R_1}(P_2) = k + \mu_{R_1}(P_2)$, the conclusion follows easily. □

Lemma 5.4. *Let $M_1, M_2, M_3 \in \mathbb{N}$. Let $P_1(z) = p_1(z)q_1(z) = 1 + \sum_{n=1}^{M_1+M_2} a_n z^n$, where $p_1(z), q_1(z) \in \mathbb{K}[z]$ have degrees M_1 and M_2, respectively. Let A_1 and A_2 be the sets of zeros of $p_1(z)$ and $q_1(z)$, respectively, and suppose that each zero of $q_1(z)$ is simple and $\max_{z\in A_1} |z| < \min_{z\in A_2} |z|$.*

Suppose that $S \in |\mathbb{K}^\times|$ and that $A_3 \subset \mathbb{K}$ has M_3 points and satisfy $\max_{z\in A_2} |z| < S < \min_{z\in A_3} |z|$.

Then there exists $Q(z) \in \mathbb{K}[z]$ of degree $M_2 + M_3$ such that $P_2(z) := P_1(z) + z^{M_1+M_2+1} Q(z)$ can be written as

$$P_2(z) = p_2(z)q_2(z),$$

with $p_2(z) = r_2(z)s_2(z)$, where M_1, M_2+1 and M_2+M_3 are the degrees of $r_2(z)$, $s_2(z)$ and $q_2(z)$, respectively, and

- *each $z \in A_2 \cup A_3$ is a simple zero of $q_2(z)$;*
- *$r_2(z)$ has the same critical radii as $p_1(z)$, and the same number of zeros in each critical radius;*
- *all $M_2 + 1$ zeros of $s_2(z)$ are contained in $C(0, S)$.*

Proof. We suppose that

$$\begin{aligned} \{|z| : z \in A_1\} &= \{R_1, \ldots, R_{N_1}\} \\ \{|z| : z \in A_2\} &= \{R_{N_1+1}, \ldots, R_{N_2}\} \\ \{|z| : z \in A_3\} &= \{R_{N_2+1}, \ldots, R_{N_3}\}, \end{aligned}$$

with $R_1 < \cdots < R_{N_3}$, and that for each $j \in \{N_1 + 1, \ldots, N_3\}$, there are k_j (pairwise different) points $z \in A_2 \cup A_3$ with $|z| = R_j$. Also, for each $j \in \{1, \ldots, N_1\}$, there are k_j zeros in A_1 with absolute value R_j.

Fix $w_1 \in \mathbb{K}$ with $|w_1| = S$, so $R_{N_2} < |w_1| < R_{N_2+1}$. According to Lemma 5.1, there exist M_2+M_3+1 coefficients $b_0, \ldots, b_{M_2+M_3} \in \mathbb{K}$ such that $b_0 + b_1 z + \cdots + b_{M_2+M_3} z^{M_2+M_3} = -P_1(z)/z^{M_1+M_2+1}$ for all $z \in A_2 \cup A_3 \cup \{w_1\}$, that is,

$$P_2(z) := P_1(z) + z^{M_1+M_2+1} \left(\sum_{n=0}^{M_2+M_3} b_n z^n \right) = 0.$$

Since R_{N_2+1} is bigger than $|w_1|$ and $\mu_{|w_1|}(P_1) = \nu_{|w_1|}(P_1) = M_1 + M_2$,

$$\mu_{R_{N_2+1}}(P_2) > \mu_{|w_1|}(P_2) \geq \mu_{|w_1|}(P_1) = M_1 + M_2,$$

and we conclude from Lemma 5.3 that $Q_0(z) := \sum_{n=0}^{M_2+M_3} b_n z^n$ has exactly k_i zeros in each $C(0, R_i)$ for $i = N_2+1, \ldots, N_3$. On the other hand, each $z \in A_2$ is a zero of $Q_0(z)$, and the degree of $Q_0(z)$ is $M_2 + M_3$, so $Q_0(z)$ has exactly k_i zeros in each $C(0, R_i)$ for $i = N_1 + 1, \ldots, N_3$. Since it has no other zeros, again by Lemma 5.3, if $S_1 > R_{N_2}$ with $S_1 \neq R_{N_2+1}, \ldots, R_{N_3}$ is a critical radius for P_2, then $\mu_{S_1}(P_2) \leq M_1 + M_2$, implying that $\nu_{R_{N_2}}(P_2) \leq M_1 + M_2$. On the other hand, since $\nu_{R_{N_2}}(P_2) \geq \nu_{R_{N_2}}(P_1)$, we deduce that $\nu_{R_{N_2}}(P_2) = M_1 + M_2 = \mu_{S_1}(P_2)$. We conclude that $|w_1|$ is the only critical radius for P_2 bigger than R_{N_2} and different from all other R_i, which necessarily gives $\mu_{|w_1|}(P_2) = M_1 + M_2$ and $\nu_{|w_1|}(P_2) = \mu_{R_{N_2+1}}(P_2) = M_1 + 2M_2 + 1$. This implies that $P_2(z)$ has $M_2 + 1$ zeros in $C(0, |w_1|)$.

Finally, since $\nu_{R_{N_2}}(P_2) = M_1 + M_2$ we have that $|a_{M_1+M_2}|\, {R_{N_2}}^{M_1+M_2} > |b_n|\, {R_{N_2}}^{M_1+M_2+n+1}$ for all $n \geq 0$, which implies that

$$|a_{M_1+M_2}|\, R^{M_1+M_2} > |b_n|\, R^{M_1+M_2+n+1}$$

whenever $0 < R < R_{N_2}$. Consequently, critical radii of $P_2(z)$ and $P_1(z)$ in $(0, R_{N_2}]$ coincide, as well as the number of zeros in each critical radius. This means that each $C(0, R_i)$ contains exactly k_i zeros of $P_2(z)$, for $i = 1, \ldots, N_2$. □

Proposition 5.5. *Let* $\mathbf{z}$ *be a sequence in* $\mathbb{D}$ *with* $c := \prod_{n=1}^{\infty} |z_n| > 0$. *Suppose that the disks* $D^+(z_n, \epsilon_n)$, $n \in \mathbb{N}$, *are pairwise disjoint. Then there exists* $f \in H^\infty$ *with* $f(0) = 1$ *and* $\|f\| \in |\mathbb{K}^\times|$ *having exactly a single zero in each* $D^+(z_n, \epsilon_n)$ *and such that, for any other zero* z *of* f, $|z| \neq |z_n|$ *for every* $n \in \mathbb{N}$.

Proof. Let $\{R_i : i \in \mathbb{N}\} = \{|z_n| : n \in \mathbb{N}\}$, and suppose that, for each i, $R_i < R_{i+1}$ and $z_1^i, \ldots, z_{k_i}^i$ are those z_n of absolute value R_i. We select $\delta_i \in |\mathbb{K}^\times|$, $\delta_i \leq \min\{\epsilon_n : |z_n| = R_i\}$, and assume also that $\delta_1 < R_1$.

Pick any $N_1 \in \mathbb{N}$ and define $M_1 := \sum_{i=1}^{N_1} k_i$. Then take $N_2 > N_1$ such that, for $M_2 := \sum_{i=N_1+1}^{N_2} k_i$, ${R_{N_1}}^{M_2} < c \min_{1 \leq i \leq N_1} {\delta_i}^{k_i}$.

Inductively, for any other $n \in \mathbb{N}$, pick $N_{n+1} > N_n$ such that

$${R_{N_n}}^{M_{n+1}} \leq c \min_{N_{n-1}+1 \leq i \leq N_n} {\delta_i}^{k_i},$$

where $M_{n+1} := \sum_{i=N_n+1}^{N_{n+1}} k_i$.

Based on the sequence (R_{N_n}), we fix a new sequence $(S_n)_{n \geq 2}$ in $|\mathbb{K}^\times|$ with

$$R_{N_n} < S_n < R_{N_n+1}$$

for all $n \geq 2$. Next call $N_0 := 0$ and, for $n \geq 1$,

$$B_n := \{R_i : N_{n-1} + 1 \leq i \leq N_n\}$$

and

$$A_n := \left\{z_1^i, \ldots, z_{k_i}^i : N_{n-1} + 1 \leq i \leq N_n\right\}.$$

Clearly, the polynomial

$$P_1(z) := \prod_{i=1}^{N_2}\left(\prod_{j=1}^{k_i}\left(1-\frac{z}{z_j^i}\right)\right)$$

has degree M_1+M_2 and its (simple) zeros are the points in $A_1\cup A_2$.

We write $P_1(z) := 1+a_1z+\cdots+a_{M_1+M_2}z^{M_1+M_2}$. Next, by Lemma 5.4, we can inductively construct a sequence (P_n) in $\mathbb{K}[z]$ such that, for all $n\geq 2$ and $L_n := M_1+2M_2+\cdots+2M_n+M_{n+1}+n-1$,

$$P_n(z) := 1+a_1z+\cdots+a_{L_n}z^{L_n}$$

can be written as

$$P_n(z) = p_n(z)q_n(z) = r_n(z)s_n(z)q_n(z),$$

for some polynomials

$$q_n(z) := \prod_{i\in B_n\cup B_{n+1}}\left(\prod_{j=1}^{k_i}\left(1-\frac{z}{z_j^i}\right)\right)$$

of degree M_n+M_{n+1} having all points in $A_n\cup A_{n+1}$ as simple zeros,

$$r_n(z) := \prod_{i\in B_1\cup\cdots\cup B_{n-1}}\left(\prod_{j=1}^{k_i}\left(1-\frac{z}{w_j^i}\right)\right)$$

of degree $M_1+M_2+\cdots+M_{n-1}$ and with critical radii R_i for all $i\leq N_{i-1}$, having k_i (not necessarily simple) zeros w_j^i in each $C(0,R_i)$, and

$$s_n(z) := \prod_{i=2}^{n}\left(\prod_{j=1}^{M_i+1}\left(1-\frac{z}{u_j^i}\right)\right)$$

of degree $M_2+M_3+\cdots+M_n+n-1$ and with critical radii S_i $(i\leq n)$, having M_i+1 (not necessarily simple) zeros u_j^i in each $C(0,S_i)$.

Hence, for $l\in B_n$ and $|z|=R_l$,

$$\begin{aligned}|q_n(z)| &= \prod_{i=N_{n-1}+1}^{l}\left(\prod_{j=1}^{k_i}\left|1-\frac{z}{z_j^i}\right|\right)\\ &= \frac{{R_l}^{k_{N_{n-1}+1}+\cdots+k_{l-1}}}{{R_{N_{n-1}+1}}^{k_{N_{n-1}+1}}\cdots{R_{l-1}}^{k_{l-1}}}\prod_{j=1}^{k_l}\left|\frac{z-z_j^l}{z_j^l}\right|,\end{aligned}$$

$$|r_n(z)| = \frac{{R_l}^{M_1+\cdots+M_{n-1}}}{{R_1}^{k_1}\cdots{R_{N_{n-1}}}^{k_{N_{n-1}}}}$$

and

$$|s_n(z)| = \prod_{i=2}^{n-1}\left(\prod_{j=1}^{M_i+1}\left|1-\frac{z}{u_j^i}\right|\right) = \frac{{R_l}^{M_2+\cdots+M_{n-1}+n-2}}{{S_2}^{M_2+1}\cdots{S_{n-1}}^{M_{n-1}+1}}.$$

This implies that

$$|P_n(z)| = |s_n(z)|\, T_l \prod_{j=1}^{k_l} \left|z - z_j^l\right|, \tag{5.1}$$

where

$$T_l := \frac{{R_l}^{k_1+\cdots+k_{l-1}}}{{R_1}^{k_1}\cdots{R_l}^{k_l}}.$$

On the other hand, if in addition $\left|z - z_j^l\right| \geq \delta_l$ for all j, then $\prod_{j=1}^{k_l}\left|z - z_j^l\right| \geq {\delta_l}^{k_l}$. Taking into account that

$$|a_{L_n}| = \frac{1}{{S_2}^{M_2+1}\ldots{S_n}^{M_n+1}{R_1}^{k_1}\cdots{R_{N_n+1}}^{k_{N_{n+1}}}},$$

we easily obtain

$$\begin{aligned}
|a_{L_n}|\,{R_l}^{L_n}\frac{1}{|s_n(z)|}\frac{1}{T_l} &= \frac{{R_l}^{M_1+M_2+\cdots+M_{n-1}+2M_n+M_{n+1}+1}}{{S_n}^{M_n+1}{R_{l+1}}^{k_{l+1}}\cdots{R_{N_{n+1}}}^{k_{N_{n+1}}}{R_l}^{k_1+\cdots+k_{l-1}}} \\
&< \frac{{R_l}^{k_l+\cdots+k_{N_{n+1}}}}{{R_{l+1}}^{k_{l+1}}\cdots{R_{N_{n+1}}}^{k_{N_{n+1}}}} \\
&< \frac{{R_{N_n}}^{M_{n+1}}}{c} \\
&\leq {\delta_l}^{k_l} \\
&\leq \prod_{j=1}^{k_l}\left|z - z_j^l\right|.
\end{aligned}$$

This implies, by Equality 5.1,

$$|a_{L_n}|\,{R_l}^{L_n} < |P_n(z)|. \tag{5.2}$$

Next, using Lemma 5.2 it is easy to see that, since each $P_n(z)$ has all its zeros contained in $\mathbb{D}$ and $P_n(0) = 1$, all its coefficients satisfy

$$|a_i| \leq \frac{1}{\prod_{j=1}^{\infty}{R_j}^{k_j}\prod_{j=2}^{\infty}{S_j}^{M_j+1}} \leq \frac{1}{c^3},$$

which implies that $f(z) := 1 + \sum_{m=1}^{\infty} a_m z^m$ is bounded and, consequently, belongs to H^∞. Also, the critical radii for f are the R_i and the S_i, and it has exactly k_i zeros in each $C(0, R_i)$ and $M_i + 1$ zeros in each $C(0, S_i)$. We now define, for $n \in \mathbb{N}$,

$$g_n(z) := f(z) - P_n(z) = \sum_{m=L_n+1}^{\infty} a_m z^m.$$

Note that, since $|a_{L_n}|\,{R_{N_{n+1}}}^{L_n} > |a_m|\,{R_{N_{n+1}}}^{m}$ for all $m > L_n$,

$$|a_{L_n}|\,{R_l}^{L_n} > |a_m|\,{R_l}^{m}$$

for $l \in B_n$, and consequently, if $|z| = R_l$, then

$$|g_n(z)| < |a_{L_n}| \, {R_l}^{L_n}. \tag{5.3}$$

We deduce from Inequalities 5.2 and 5.3 that $|a_{L_n}| \, {R_l}^{L_n} < |f(z)|$ whenever $|z| = R_l$ and $\left|z - z_j^l\right| \geq \delta_l$ for all $j \in \{1, \ldots, k_l\}$. In particular, if we fix $j \in \{1, \ldots, k_l\}$ and take $w \in \mathbb{D}$ with $\left|w - z_j^l\right| = \delta_l$, then $|f(w)| > |a_{L_n}| \, {R_l}^{L_n} > \left|g_n(z_j^l)\right|$. On the other hand, $P_n\left(z_j^l\right) = 0$, so $\left|f\left(z_j^l\right)\right| = \left|g_n\left(z_j^l\right)\right|$. This means that, if we define $h(z) := f\left(z + z_j^l\right)$, then $|h(0)| < |h(z)|$ whenever $|z| = \delta_l$. We conclude that either $h(0) = 0$ or there is a critical radius for h between 0 and δ_l, and consequently there is a zero of f in $D^-\left(z_j^l, \delta_l\right)$. Since f has exactly k_l zeros in $C(0, R_l)$, we are done.

It just remains to prove that the above f can be taken to satisfy $\|f\| \in |\mathbb{K}^\times|$. Note that apart from the R_n, the critical radii of the function f are certain $S_n \in |\mathbb{K}^\times| \cap (R_{N_n}, R_{N_n+1})$, $n \geq 2$, chosen at will. Let us next see that these can be selected in such a way that $\|f\| = 1/\left(\prod_{n=1}^\infty {R_n}^{k_n} \prod_{n=2}^\infty {S_n}^{M_n+1}\right)$ belongs to $|\mathbb{K}^\times|$. Clearly, it is enough to show that every value in the interval $\left(\prod_{n=2}^\infty {R_{N_n}}^{M_n+1}, \prod_{n=2}^\infty {R_{N_{n+1}}}^{M_n+1}\right)$ is attained by products of the form $\prod_{n=2}^\infty {S_n}^{M_n+1}$. It is easy to see that this is equivalent to proving that, given a set D dense in $(0, +\infty)$, if (a_n) and (b_n) are sequences in $(0, +\infty)$ with $\sum_{n=1}^\infty b_n < \infty$ and $0 < a_n < b_n$ for all n, then every $T \in (\sum_{n=1}^\infty a_n, \sum_{n=1}^\infty b_n)$ can be written in the form $T = \sum_{n=1}^\infty q_n(t_n)$ with all $q(t_n) \in D$, where $q_n(s) := sa_n + (1-s)b_n$ for every $s \in [0,1]$ and $n \in \mathbb{N}$.

First, it is clear that there exists $s_1 \in (0,1)$ with $T = \sum_{n=1}^\infty q_n(s_1)$. We fix $\epsilon > 0$ and pick $t_1 \in [0,1]$ with $q_1(t_1) \in D$ and $q_1(t_1) + q_2(0) < q_1(s_1) + q_2(s_1) < q_1(t_1) + q_2(1)$ such that $|q_1(t_1) - q_1(s_1)| < \epsilon$. Then there exists $s_2 \in (0,1)$ with $q_1(t_1) + q_2(s_2) = q_1(s_1) + q_2(s_1)$, that is, $q_1(t_1) + q_2(s_2) + q_3(0) < \sum_{n=1}^3 q_n(s_1) < q_1(t_1) + q_2(s_2) + q_3(1)$. Consequently, there exists $t_2 \in (0,1)$ with $q_2(t_2) \in D$ such that

$$q_1(t_1) + q_2(t_2) + q_3(0) < \sum_{n=1}^3 q_n(s_1) < q_1(t_1) + q_2(t_2) + q_3(1)$$

and $\left|q_1(t_1) + q_2(t_2) - \sum_{n=1}^2 q_n(s_1)\right| < \epsilon/2$.

Clearly, we inductively find a sequence (t_n) in $(0,1)$ with $q_n(t_n) \in D$ for each n, and such that $\left|\sum_{n=1}^k q_n(t_n) - \sum_{n=1}^k q_n(s_1)\right| < \epsilon/k$ for all k. Thus $T = \sum_{n=1}^\infty q_n(t_n)$, and we are done. □

Remark 5.1. Note that, for a sequence (T_n) in $(0,1)$, the function f in Proposition 5.5 can be taken so that no T_n is a critical radius for f.

6. Sequences determining the same seminorms

In this section we first show that a seminorm is determined by the behaviour of the radii of seminorms along an ultrafilter.

Lemma 6.1. *Let $z_0 \in \mathbb{D} \setminus \{0\}$ and $s, r \in (0,1)$ satisfy $s \leq r < |z_0|$. Then, for every $f \in H^\infty \setminus \{0\}$,*

$$0 \leq \zeta_{D^+(z_0,r)}(f) - \zeta_{D^+(z_0,s)}(f) \leq \left(r^{\mathtt{z}(f,D^+(z_0,r))} - s^{\mathtt{z}(f,D^+(z_0,r))}\right) \|f\|.$$

Proof. We write $f(z) = g(z)\prod_{i=1}^m (z - w_i)$, where $w_1, \ldots, w_m$ are the zeros of f in $C(0,|z_0|)$. Taking into account Corollary 3.3, it is easy to see that $\zeta_{D^+(z_0,r)}(g) = \zeta_{D^+(0,s)}(g)$. Consequently,

$$\zeta_{D^+(z_0,r)}(f) = \zeta_{D^+(z_0,r)}(g)\ r^{\mathtt{z}(f,D^+(z_0,r))} \prod_{w_i \notin D^+(z_0,r)} |z_0 - w_i|$$

and that

$$\zeta_{D^+(z_0,s)}(f) \geq \zeta_{D^+(z_0,r)}(g)\ s^{\mathtt{z}(f,D^+(z_0,r))} \prod_{w_i \notin D^+(z_0,r)} |z_0 - w_i|.$$

Since $\|f\| = \|g\|$, the conclusion follows easily. □

Corollary 6.2. *Let $\mathbf{k}$ be a sequence in $\mathbb{N}$. Suppose that $\mathfrak{u}$ is a nonprincipal ultrafilter in $\mathbb{N}$ and that (r_n) and (s_n) are sequences in $(0,1)$ such that $\lim_{\mathfrak{u}} {r_n}^{k_n} = \lim_{\mathfrak{u}} {s_n}^{k_n} \neq 0, 1$. If $\mathbf{z}$ is a sequence in $\mathbb{D}$ with $r_n, s_n < |z_n|$ for all n, then $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,s_n)}$.*

Proof. We can assume that $A \in \mathfrak{u}$ satisfies $s_n \leq r_n$ for every $n \in A$.

Let $f \in H^\infty$, $f \neq 0$. By Lemma 6.1, for $n \in A$,

$$0 \leq \zeta_{D^+(z_n,r_n)}(f) - \zeta_{D^+(z_n,s_n)}(f) \leq \left({r_n}^{\mathtt{z}(f,D^+(z_n,r_n))} - {s_n}^{\mathtt{z}(f,D^+(z_n,r_n))}\right) \|f\|.$$

Obviously, from the hypothesis we deduce that $\lim_{\mathfrak{u}} {r_n}^{t_n} = \lim_{\mathfrak{u}} {s_n}^{t_n}$ for every sequence (t_n) of natural numbers, and the conclusion follows. □

Remark 6.1. In the case when $\mathbb{K}$ is not spherically complete, a natural question is whether the limit of norms based on filters in $\mathbb{D}$ with no center allows us to define new seminorms. We will see that this is not the case. Suppose that, for each $n \in \mathbb{N}$, $\|\cdot\|_n = \lim_{m\to\infty} \zeta_{D^+(z_m^n, s_m^n)}$, where

$$C(0,|z_1^n|) \supset D^+(z_1^n, s_1^n) \supset D^+(z_2^n, s_2^n) \supset \cdots$$

and $\bigcap_{m=1}^\infty D^+(z_m^n, s_m^n) = \emptyset$. Suppose also that $\lim_{n\to\infty} |z_1^n| = 1$. Take a nonprincipal ultrafilter $\mathfrak{u}$ in $\mathbb{N}$ and define the seminorm $\psi := \lim_{\mathfrak{u}} \|\cdot\|_n \in \mathfrak{M}$. It is clear that $s_n := \lim_{m\to\infty} s_m^n > 0$ for each n. Consider a sequence (k_n) in $\mathbb{N}$ such that $r := \lim_{\mathfrak{u}} {s_n}^{k_n} \in (0,1)$ and take, for each n, an $m_n \in \mathbb{N}$ such that $\lim_{\mathfrak{u}} {s_{m_n}^n}^{k_n} = r$. Calling $r_n := s_{m_n}^n$ and $z_n := z_{m_n}^n$, we easily check that $\psi = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$.

Corollary 6.3. *Given $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} \in \mathfrak{M}_0$, there exist a sequence (w_n) in $\mathbb{D}$ with $\lim_{n\to\infty} |w_n| = 1$ and a sequence (s_n) in $(0,1)$ in such a way that all the disks $D^+(w_n, s_n)$ are pairwise disjoint and $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(w_n,s_n)}$.*

Proof. Note that, if $\varphi = \|\cdot\|$, then $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+\left(z_n,|z_n|^2\right)}$, so in all cases we can assume without loss of generality that $D^+(z_n, r_n) \subset C(0, |z_n|)$ for all n. Fix $n_0 \in \mathbb{N}$, and suppose that the set

$$\{n \in \mathbb{N} : |z_n| = |z_{n_0}|\} = \{n_1, \ldots, n_k\}.$$

It is straightforward to prove that, for each $i \in \{1, \ldots, k\}$, there exists $w_{n_i} \in C(z_{n_i}, r_{n_i})$ such that $\left|w_{n_{i_1}} - w_{n_{i_2}}\right| = \max\left\{r_{n_{i_1}}, r_{n_{i_2}}, \left|z_{n_{i_1}} - z_{n_{i_2}}\right|\right\}$ whenever $i_1 \neq i_2$. This implies that the disks $D^-(w_{n_i}, r_{n_i})$ are pairwise disjoint. Of course we can define a sequence (w_n) with the desired properties by putting $w_n := w_{n_i}$ when $z_n = z_{n_i}$. Obviously, $\varphi = \lim_{\mathfrak{u}} \zeta_{D^-(w_n,r_n)}$. Now, if we assume that $\lim_{\mathfrak{u}} r_n > 0$, then the conclusion follows immediately taking into account Corollary 6.2. The case when $\lim_{\mathfrak{u}} r_n = 0$ is similar. □

Remark 6.2. Note that, in Corollary 6.3, if $\mathbf{z}$ is regular with respect to $\mathfrak{u}$, then so is $\mathbf{w}$. Taking into account that each $\varphi = \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}_1$ can be written by $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(w_n,s_n)}$, where all the disks $D^+(w_n, s_n)$ are pairwise disjoint, we conclude that $\varphi \in \mathfrak{M}'_0$. Thus, $\mathfrak{M}_1 \subset \mathfrak{M}'_0$.

Corollary 6.4. *Every $\varphi \in \mathfrak{M}_0$ can be written by $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$, where $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$ and the disks $D^+(z_n, r_n)$ are pairwise disjoint.*

Next we show that the converse of Corollary 6.2 does not hold in general. In fact, very different behavior of the radii along an ultrafilter can lead to the same seminorm (see Example 6.6 and Remark 7.2; see also Theorem 2.10).

Proposition 6.5. *Let $\mathbf{z}$ be a sequence in $\mathbb{D}$ with $\lim_{n\to\infty} |z_n| = 1$, and let $\mathbf{k}$ be a sequence in $\mathbb{N}$. Suppose that $\mathfrak{u}$ is a nonprincipal ultrafilter in $\mathbb{N}$ with the property that, for every $C \in \mathfrak{u}$,*

$$\lim_{\mathfrak{u}} \prod_{\substack{|z_m|=|z_n| \\ m\in C \\ m\neq n}} |z_n - z_m|^{k_m} < 1.$$

Let (r_n) and (s_n) be sequences in $(0,1)$ with $z_m \notin D^-(z_n, r_n), D^-(z_n, s_n)$ whenever $m \neq n$. If there exists $C_0 := \{n_1, n_2, \ldots, n_i, \ldots\} \in \mathfrak{u}$ such that

$$\lim_{i\to\infty} {r_{n_i}}^{k_{n_i}} = 1 = \lim_{i\to\infty} {s_{n_i}}^{k_{n_i}},$$

then

$$\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,s_n)}.$$

Remark 6.3. Note that in Proposition 6.5, if $\mathbf{z}$ is regular and $\mathbf{k}$ belongs to $\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$, then the seminorm $\phi := \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$ satisfies $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} \leq \phi$. In Example 6.9, we will see that the equality does not hold in general.

Remark 6.4. In Example 6.9, we will also see that a weaker assumption in Proposition 6.5 such as that $\lim_{\mathfrak{u}} {r_n}^{k_n} = 1 = \lim_{\mathfrak{u}} {s_n}^{k_n}$ does not imply that $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,s_n)}$.

Proof of Proposition 6.5. Since we are dealing with an ultrafilter, we can assume without loss of generality that $0 < s_n \leq r_n$ for all $n \in C_0$. It is clear that there exists a sequence (l_n) in $\mathbb{N}$ with $\lim_{n\to\infty} l_n = +\infty$ such that $\lim_{\substack{n\in C_0\\ n\to\infty}} {s_n}^{k_n l_n} = 1/2$.

Take $f \in H^\infty$. We have that, if $\mathtt{Z}_n = \mathtt{Z}\left(D^-\left(z_n, r_n\right)\right)$ and

$$\lambda_n := \prod_{\substack{|z_m|=|z_n|\\ m\neq n}} |z_n - z_m|^{\mathtt{Z}_m}$$

and

$$\mu_n := \prod_{\substack{z\in \mathtt{Z}(C(0,|z_n|))\\ |z-z_m|\geq r_m \forall m}} |z_n - z|$$

for $n \in C_0$, then

$$s_n{}^{\mathtt{Z}_n}\lambda_n\mu_n \leq \xi_{D^+(z_n,s_n)}(f) \leq \xi_{D^+(z_n,r_n)}(f) = r_n{}^{\mathtt{Z}_n}\lambda_n\mu_n. \tag{6.1}$$

Let $\alpha := \lim_{\mathfrak{u}} \mathtt{Z}_n/(k_n l_n)$. We easily see that, if $\alpha = 0$, then $\lim_{\mathfrak{u}} {s_n}^{\mathtt{Z}_n} = 1$ and, taking limits in Equation 6.1, $\lim_{\mathfrak{u}} \xi_{D^+(z_n,s_n)}(f) = \lim_{\mathfrak{u}} \xi_{D^+(z_n,r_n)}(f)$.

On the other hand, if $0 < \alpha \leq +\infty$, then there exist $A \in \mathfrak{u}$ with $A \subset C_0$ and $\beta > 0$ such that $\mathtt{Z}_n \geq \beta k_n l_n$ for all $n \in A$. Next, for $n \in A$ we define

$$L_n := \min\left\{l_m : m \in A, |z_m| = |z_n|\right\},$$

and obtain

$$\lambda_n \leq \prod_{\substack{|z_m|=|z_n|\\ m\in A\\ m\neq n}} |z_n - z_m|^{\beta k_m L_n} = \left(\prod_{\substack{|z_m|=|z_n|\\ m\in A\\ m\neq n}} |z_n - z_m|^{k_m}\right)^{\beta L_n}.$$

Since $\lim_{n\to\infty} l_n = +\infty$, $\lim_{\substack{n\in A\\ n\to\infty}} L_n = +\infty$. Also, by hypothesis, there exist $M < 1$ and $A' \in \mathfrak{u}$ with $A' \subset A$ and

$$\prod_{\substack{|z_m|=|z_n|\\ m\in A\\ m\neq n}} |z_n - z_m|^{k_m} \leq M$$

for all $n \in A'$. This gives $\lim_{\substack{n\in A'\\ n\to\infty}} \lambda_n = 0$, and consequently $\lim_{\mathfrak{u}} \lambda_n = 0$.

Finally, it follows from Equation 6.1 that

$$\lim_{\mathfrak{u}} \xi_{D^+(z_n,r_n)}(f) = 0 = \lim_{\mathfrak{u}} \xi_{D^+(z_n,s_n)}(f) = 0,$$

and we are done. □

We next give an example where Proposition 6.5 can be applied.

Example 6.6. Let $\mathbf{z}$ be a sequence in $\mathbb{D}$ with $\prod_{n=1}^{\infty}|z_n|>0$. Let $R_1<R_2<\cdots$ be the absolute values of the z_n and, for each i, suppose that there are $M_i\geq 2$ points z_n of absolute value R_i, and that $\lim_{i\to\infty}M_i=+\infty$. Suppose also that there exists $M\in|\mathbb{K}^\times|\cap(0,1)$ such that, for all $i\in\mathbb{N}$, $|z_n-z_m|=\sqrt[M_i-1]{M}\in(0,R_i)$ whenever $|z_n|=R_i=|z_m|$, $n\neq m$.

Fix a nonprincipal ultrafilter $\mathfrak{v}$ in $\mathbb{N}$ and consider the family $\mathfrak{F}$ of the complements of all sets D in $\mathbb{N}$ with the property that

$$\lim_{\mathfrak{v}}\frac{\operatorname{card}(\{z_n:n\in D\}\cap C(0,R_i))}{M_i}=0.$$

It is easy to check that $\mathfrak{F}$ is a filter in $\mathbb{N}$, and that any ultrafilter $\mathfrak{u}$ containing $\mathfrak{F}$ satisfies the conditions of Proposition 6.5 for $k_n=1$ for all n. Thus, if $r_n:=\sqrt[M_i-1]{M}$ for each n with $|z_n|=R_i$, then $\lim_{\mathfrak{u}}\zeta_{D^+(z_n,r_n)}=\lim_{\mathfrak{u}}\zeta_{D^+(z_n,s_n)}$ for any sequence (s_n) such that $\lim_{\mathfrak{u}}s_n=1$ and $s_n\leq r_n$ for all n.

Proposition 6.7. *Let $\mathbf{z}$ be a sequence in $\mathbb{D}$ with $\lim_{n\to\infty}|z_n|=1$, and let $\mathfrak{u}$ be a nonprincipal ultrafilter in $\mathbb{N}$. Suppose that $(D^-(z_n,r_n))$ is a sequence of pairwise disjoint open disks.*

If $\mathbf{z}$ is not regular with respect to $\mathfrak{u}$, then

$$\lim_{\mathfrak{u}}\zeta_{D^+(z_n,r_n)}=\lim_{\mathfrak{u}}\delta_{z_n}.$$

Proof. For $f\in H^\infty$ with $|f(0)|=1$ fixed, let C be the set of all n such that $D^-(z_n,r_n)$ contains no zeros of f. Suppose first that C belongs to $\mathfrak{u}$. Then $|f|$ takes a constant value on each disk $D^-(z_n,r_n)$, for $n\in C$, and this same value is taken at each z_n, $n\in C$. It is now straightforward to see that $\lim_{\mathfrak{u}}\delta_{z_n}(f)=\lim_{\mathfrak{u}}\zeta_{D^-(z_n,r_n)}(f)$. Suppose next that $C\notin\mathfrak{u}$, and take any $C'\in\mathfrak{u}$ with $C'\subset\mathbb{N}\setminus C$. Since $\prod_{n\in C'}|z_n|\geq 1/\|f\|>0$ and $(z_n)_{n\in C'}$ is not regular, we can assume that, for all $n\in C'$, there exists at least one $m\in C'$, $m\neq n$, with $|z_m|=|z_n|$. Then fix a zero u_n of f in each $D^-(z_n,r_n)$ for all $n\in C'$. It is clear that, for $n\in C'$,

$$\xi_{D^+(z_n,r_n)}(f)\leq\prod_{\substack{|u_m|=|z_n|\\ m\in C'\\ m\neq n}}|z_n-u_m|=\prod_{\substack{|z_m|=|z_n|\\ m\in C'\\ m\neq n}}|z_n-z_m|.$$

Since $(z_n)_{C'}$ is not regular, $\inf_{n\in C'}\xi_{D^+(z_n,r_n)}(f)=0$, and we easily deduce from Corollary 3.4 that $\lim_{\mathfrak{u}}\zeta_{D^+(z_n,r_n)}(f)=\lim_{\mathfrak{u}}\xi_{D^+(z_n,r_n)}(f)=0$. Finally, since $\delta_{z_n}\leq\zeta_{D^+(z_n,r_n)}$, we conclude that $\lim_{\mathfrak{u}}\delta_{z_n}(f)=0$. □

Example 6.8. Let $\mathbf{z}$ be a sequence in $\mathbb{D}$ with $\prod_{n=1}^{\infty}|z_n|>0$. Let $\{R_i:i\in\mathbb{N}\}$ be the set of the absolute values of all z_n, and suppose that $S_i:=|z_n-z_m|$ is constant for all $n,m\in\mathbb{N}$ with $|z_n|=R_i=|z_m|$, $i\in\mathbb{N}$. Suppose also that $\mathfrak{u}$ is a nonprincipal ultrafilter in $\mathbb{N}$ such that $\mathbf{z}$ is not regular with respect to $\mathfrak{u}$. Then Proposition 6.7 gives us

$$\lim_{\mathfrak{u}}\delta_{z_n}=\lim_{\mathfrak{u}}\zeta_{D^+(z_n,S_i)}. \tag{6.2}$$

Obviously, we can define a map $\pi : \mathbb{N} \to \mathbb{N}$ associating each n with the number $\pi(n)$ with $|z_n| = R_{\pi(n)}$. The meaning of $\pi(A)$ for $A \subset \mathbb{N}$ is clear, as well as that of $\pi(\mathfrak{u})$. In fact, $\pi(\mathfrak{u})$ is also a nonprincipal ultrafilter in $\mathbb{N}$. Now, it is easy to check that, by Equality 6.2, if each w_k is any point in $D^+\left(z_n, S_{\pi(n)}\right)$, then

$$\lim_{\mathfrak{u}} \delta_{z_n} = \lim_{\pi(\mathfrak{u})} \zeta_{D^+(w_k,S_k)}.$$

The following example shows that the result in Proposition 6.5 cannot be sharpened (see Remarks 6.3 and 6.4).

Example 6.9. We consider M, (R_i), (M_i), $\mathbf{z}$ and $\mathfrak{u}$ to be the same as in Example 6.6. Suppose that (N_k) is a sequence in $(0,1)$ with $\prod_{k=1}^{\infty} N_k > 0$, where $N_1 := M^2$. Clearly, we can find a sequence (A_k) in $\mathfrak{u}$ with $A_1 = \mathbb{N}$ and $A_{k+1} \subsetneq A_k$ for all k such that each A_k satisfies the following property: Given $i \in \mathbb{N}$, if the cardinal K_k^i of the n in A_k with $z_n \in C(0, R_i)$ is not 0, then $K_k^i \geq 2$ and

$$\left(\sqrt[M_i-1]{M} \right)^{K_k^i} \geq N_k.$$

Now select sequences (r_n) and (δ_n) in $(0,1)$ with $\lim_{n\to\infty} r_n = 1$ and $\lim_{n\to\infty} \delta_n = 0$, and such that $0 < \delta_n < r_n \leq \sqrt[M_i-1]{M}$ whenever $|z_n| = R_i$. Next consider a function $f \in H^\infty$ having exactly $\mathtt{Z}_n$ simple zeros in each $D^+(z_n, \delta_n)$, where $\mathtt{Z}_n := \max\{k \in \mathbb{N} : n \in A_k\}$, and no other zeros in the corresponding $C(0, R_i)$ (see Proposition 5.5). Note that, for $i \in \mathbb{N}$, $\sum_{|z_n|=R_i} \mathtt{Z}_n \leq \sum_{k=1}^{\infty} K_k^i$. Consequently, for each $n \in \mathbb{N}$ with $|z_n| = R_i$, $\xi_{D^+(z_n,r_n)}(f) = {r_n}^{\mathtt{Z}_n} \lambda_n$, where

$$\lambda_n := \prod_{\substack{|z_m|=|z_n| \\ m\neq n}} |z_n - z_m|^{\mathtt{Z}_m} \geq \left(\sqrt[M_i-1]{M} \right)^{K_1^i + K_2^i + K_3^i + \cdots} \geq \prod_{k=1}^{\infty} N_k.$$

Note also that there exists a sequence (l_n) in $\mathbb{N}$ with $\lim_{n\to\infty} {r_n}^{l_n} = 1/2$, and this sequence satisfies $\lim_{n\to\infty} l_n = +\infty$. As in the proof of Proposition 6.5 (with $k_n = 1$ for all n), we see that, if $\lim_{\mathfrak{u}} \mathtt{Z}_n/l_n > 0$, then $\lim_{\mathfrak{u}} \lambda_n = 0$. Since this is not the case, we deduce that $\lim_{\mathfrak{u}} \mathtt{Z}_n/l_n = 0$, and consequently that $\lim_{\mathfrak{u}} {r_n}^{\mathtt{Z}_n} = 1$. By Corollary 3.4, taking into account that $\xi_{D^+(z_n,r_n)}(f) \geq {r_n}^{\mathtt{Z}_n} \prod_{k=1}^{\infty} N_k$ for all n, we conclude that $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}(f) \neq 0$. On the other hand, since $A_k \in \mathfrak{u}$ for all k, $\lim_{\mathfrak{u}} \mathtt{Z}_n = +\infty$, which implies that for all $s \in (0,1)$, $\lim_{\mathfrak{u}} s^{\mathtt{Z}_n} = 0$ and consequently, $\zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{1},s}(f) = 0$. Thus, $\zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{1},1} \neq \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$ (see Remark 6.3).

On the other hand, Example 6.6 tells us that, if the sequence (r_n) is taken as above, then $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} = \lim \zeta_{D^+(z_n,s_n)}$, where $s_n = \sqrt[M_i-1]{M}$ whenever $|z_n| = R_i$. Now, it is easy to see that $\lim \zeta_{D^+(z_n,s_n)} = \lim_{\pi(\mathfrak{u})} \zeta_{D^+\left(w_i, \sqrt[M_i-1]{M}\right)}$,

where each w_i belongs to $D^+\left(z_n,\ \sqrt[M_i-1]{M}\right)$ and π is defined as in Example 6.8. In other words, if $\mathbf{w}=(w_i)$ and $\mathbf{M}=(M_i-1)$, then $\lim_{\mathfrak{u}}\zeta_{D^+(z_n,r_n)}=\zeta^{\mathbf{M},M}_{\mathbf{w},\pi(\mathfrak{u})}$.

We can prove that, for the function f above, if $t\in\left(0,\zeta^{\mathbf{M},M}_{\mathbf{w},\pi(\mathfrak{u})}(f)\right)$, then there is a sequence (t_n) such that $\lim_{\mathfrak{u}}\zeta_{D^+(z_n,t_n)}(f)=t$ (for this fact, see the proof of Theorem 2.1 in Section 7). It is also clear that $t_n\leq\sqrt[M_i-1]{M}$ whenever $|z_n|=R_i$. On the one hand, this implies that, if we put $\phi_t:=\lim_{\mathfrak{u}}\zeta_{D^+(z_n,t_n)}$, then

$$\zeta^{\mathbf{1},1}_{\mathbf{z},\mathfrak{u}}\leq\phi_{t'}\leq\phi_{t''}\leq\zeta^{\mathbf{M},M}_{\mathbf{w},\pi(\mathfrak{u})}$$

and $\phi_{t'}(f)<\phi_{t''}(f)$ whenever $t'<t''$. This means by Proposition 6.5 that there is no set $\{n_k:k\in\mathbb{N}\}\in\mathfrak{u}$ such that $\lim_{k\to\infty}t_{n_k}=1$. Since obviously $\lim_{\mathfrak{u}}t_n=1$, we see that Remark 6.4 is correct.

Proof of Theorem 2.4. It is obvious that each $\delta_{\mathbf{w},\mathfrak{v}}$ belongs to $\mathfrak{M}_0$, because it can be written as $\lim_{\mathfrak{v}}\varphi_{D^+(w_m,1/(2m))}$. On the other hand, we take $\varphi=\lim_{\mathfrak{u}}\varphi_{D^+(z_n,r_n)}$, and assume that $\lim_{\mathfrak{u}}r_n>0$. By Corollary 6.3, we can assume that all the disks $D^+(z_n,r_n)$ are pairwise disjoint and that $r_n<|z_n|$ for every n. We see that that the result follows from Proposition 6.7 if $\mathbf{z}$ is not regular with respect to $\mathfrak{u}$.

More in general, by Corollary 6.2, each r_n can be taken in $|\mathbb{K}^\times|$. Now, for each $n\in\mathbb{N}$, pick $N_n\in\mathbb{N}$ with $N_n\geq n+1$ and such that $\lim_{n\to\infty}{r_n}^{N_n}=0$. Also, consider $A_n:=\left\{w_1^n,\ldots,w_{N_n}^n\right\}\subset C(z_n,r_n)$ with $\left|w_i^n-w_j^n\right|=r_n$ whenever $i\neq j$. We clearly see that all the A_n can be taken in such a way that $D^+(z,r_n)\cap D^+(w,r_m)=\emptyset$ whenever $z\in A_n$ and $w\in A_m$. Using the lexicographic order, define a sequence $\mathbf{w}$ with all the points in $\bigcup_{n=1}^\infty A_n$ (that is, if $m<m'$, then $w_m=w_i^n$ and $w_{m'}=w_j^{n'}$ with $n\leq n'$ and, for $n=n'$, $i<j$).

Next consider the family $\mathfrak{F}$ of the complements of all sets D in $\mathbb{N}$ with the property that

$$\lim_{\mathfrak{u}}\frac{\operatorname{card}(\{w_m:m\in D\}\cap A_n)}{N_n}=0.$$

It is a routine matter to check that $\mathfrak{F}$ is a filter in $\mathbb{N}$ and that, given an ultrafilter $\mathfrak{v}$ containing $\mathfrak{F}$, $\mathbf{w}$ is not regular with respect to $\mathfrak{v}$.

It is also clear that, if $s_m:=r_n$ whenever $w_m\in A_n$, then $\varphi=\lim_{\mathfrak{v}}\zeta_{D^+(w_m,s_m)}$. By Proposition 6.7, $\varphi=\lim_{\mathfrak{v}}\delta_{w_m}$. □

We easily see that a slight modification of the above proof shows that each $\delta_{\mathbf{z},\mathfrak{u}}$ with $\mathbf{z}$ regular with respect to $\mathfrak{u}$ can be written as $\delta_{\mathbf{w},\mathfrak{v}}$ with $\mathbf{w}$ *not* regular with respect to $\mathfrak{v}$.

7. Kernels of seminorms

In this section we prove most of the results stated in Section 2.

Proof of Theorem 2.1. Suppose that $\varphi = \lim_{\mathfrak{u}} \zeta_{D^+(z_n,s_n)}$, where $s_n < |z_n|$ for all n. By Corollary 3.4, $\lim_{\mathfrak{u}} \xi_{D^+(z_n,s_n)}(f) = 0$, so $\xi_{D^+(z_n,s_n)}(f) < r/(2\|f\|)$ for all n in some $C_0 \in \mathfrak{u}$.

Fix $n \in C_0$ and suppose that $w_1, \ldots, w_k$ are the zeros of f in $C(0, |z_n|)$. It is clear that the function $F_n : [0, |z_n|] \to \mathbb{R}$ given by $s \mapsto \prod_{j=1}^{k} \max\{s, |z_n - w_j|\}$, is continuous and increasing. Also $F_n(|z_n|) = |z_n|^{\mathtt{Z}(f, C(0,|z_n|))}$, and, consequently $\lim_{n\in C_0} F_n(|z_n|) = 1$, and there exists $n_r \in C_0$ such that $F_n(|z_n|) > r/\|f\|$ for all $n \in C_0$ with $n \geq n_r$. Since $F_n(0) \leq \xi_{D^+(z_n,s_n)}(f) < r/\|f\|$, for $n \in C_0$ with $n \geq n_r$, we can find $r_n \in (0, |z_n|)$ such that

$$\frac{r}{\|f\|} = F_n(r_n) = {r_n}^{\mathtt{Z}(f, D^+(z_n,r_n))} \prod_{|z_n - w_j| > r_n} |z_n - w_j|.$$

Obviously $\xi_{D^+(z_n,r_n)}(f) = r/\|f\|$ for all n. Consequently, if we define $\psi := \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$, then by Corollary 3.4, $\psi(f) = r$.

Note that any two of the above disks $D^+(z_n, r_n)$ are either equal or disjoint. For each $k \in C_0$, we set $n_k := \min\{n : D^+(z_n, r_n) = D^+(z_k, r_k)\}$, in such a way that the disks $D^+(z_{n_k}, r_{n_k})$ are pairwise disjoint. Put $v_k := z_{n_k}$ and $t_k := r_{n_k}$ for all k. Then define a new ultrafilter $\mathfrak{v}$ in $\mathbb{N}$: A set $C \subset \mathbb{N}$ belongs to $\mathfrak{v}$ if the set of all $n \in C_0$ such that $D^+(z_n, r_n) = D^+(v_k, t_k)$, for some $k \in C$, belongs to $\mathfrak{u}$. It is a routine matter to check that $\psi = \lim_{\mathfrak{v}} \zeta_{D^+(v_k,t_k)}$. On the other hand, by the definition of r_n, we easily see that each $\mathtt{Z}(f, D^+(z_n, r_n)) \geq 1$, which implies that, for $k \in \mathbb{N}$ fixed,

$$\prod_{\substack{|v_l| = |v_k| \\ l \neq k}} |v_k - v_l| \geq \prod_{|z_{n_k} - w_m| > r_{n_k}} |z_{n_k} - w_m| \geq \frac{r}{\|f\|}.$$

The fact that $\mathbf{v}$ is regular with respect to $\mathfrak{v}$ follows easily and, consequently, ψ belongs to $\mathfrak{M}'_0$.

On the other hand, by Proposition 5.5, we can find $g \in H^\infty$ with as many zeros in each $D^+(z_n, r_n)$ as we need so that $\psi(g) = 0$. This shows that ψ is not a norm. □

Proposition 7.1. *Let* $\mathbf{z}$ *be a regular sequence with respect to* $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$, *and let* $\mathbf{k} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$. *Then there exists* $f \in H^\infty$ *with* $\|f\| = 1$ *such that*

$$0 < \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}(f) \leq r$$

for all $r \in (0,1)$ *and* $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}(f) < \zeta^{\mathbf{k},s}_{\mathbf{z},\mathfrak{u}}(f)$ *if* $0 < r < s < 1$.

Proof. We consider $C \in \mathfrak{u}$ such that

$$M := \inf_{n\in C} \prod_{\substack{m\in C \\ m\neq n}} |z_n - z_m|^{k_m} > 0.$$

For $r \in (0,1)$ and $n \in C$, put $r_n := \sqrt[k_n]{r}$. Consider a sequence (δ_n) of positive numbers converging to 0 with the property that the disks $D^+(z_n, \delta_n)$ are pairwise disjoint. Then, since $\prod_{n\in C} |z_n|^{k_n} > 0$, we can use Proposition 5.5

and take $f \in H^\infty$ with $\|f\| = 1$ and $f(0) \neq 0$ having exactly k_n simple zeros in each $D^+(z_n, \delta_n)$ whenever $n \in C$, and no other zeros in the circles $C(0, |z_m|)$.

We put, for each $n \in C$, $T_n := \sum_{\substack{|z_n - z_m| \leq r_n \\ m \in C}} k_m$. Note that if $T := \inf_{n \in C} {r_n}^{T_n} = 0$, then $M = 0$, against our hypothesis. Thus $T > 0$ and

$$\alpha := \lim_{\mathfrak{u}} \frac{T_n}{k_n} \in (1, +\infty).$$

On the other hand, it is clear that, for every $n \in C$,

$$\xi_{D^+(z_n, r_n)}(f) = {r_n}^{T_n} \prod_{\substack{|z_m - z_n| > r_n \\ |z_m| = |z_n| \\ m \in C}} |z_n - z_m|^{k_m}$$

belongs to the interval $\left[M {r_n}^{T_n}, {r_n}^{k_n}\right] \subset [MT, r]$ and, by Corollary 3.4, $MT \leq \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}(f) \leq r$.

Suppose next that $s \in (r, 1)$ and $s_n := \sqrt[k_n]{s}$ for $n \in C$. As above,

$$\xi_{D^+(z_n, s_n)}(f) = {s_n}^{S_n} \prod_{\substack{|z_m - z_n| > s_n \\ |z_m| = |z_n| \\ m \in C}} |z_n - z_m|^{k_m}$$

where $S_n := \sum_{\substack{|z_n - z_m| \leq s_n \\ m \in C}} k_m$. Also, for all $n \in C$,

$${s_n}^{S_n} \geq {s_n}^{T_n} \prod_{\substack{s_n \geq |z_m - z_n| > r_n \\ m \in C}} |z_n - z_m|^{k_m} \geq {r_n}^{T_n} \prod_{\substack{s_n \geq |z_m - z_n| > r_n \\ m \in C}} |z_n - z_m|^{k_m}$$

and, consequently, the fact that $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}(f) = \zeta^{\mathbf{k},s}_{\mathbf{z},\mathfrak{u}}(f)$ implies that $\lim_{\mathfrak{u}} {s_n}^{T_n} = \lim_{\mathfrak{u}} {r_n}^{T_n}$, that is, $s^\alpha = r^\alpha$. We conclude that $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}(f) < \zeta^{\mathbf{k},s}_{\mathbf{z},\mathfrak{u}}(f)$. □

Remark 7.1. In the proof of Proposition 7.1, we see that if the set C can be taken equal to $\mathbb{N}$, then the same function f makes the result hold for all $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$ simultaneously.

Prior to proving Theorem 2.5, we give the following lemma.

Lemma 7.2. *Let $\alpha : (0, 1) \to [0, +\infty]$ be an increasing function. If $r_0 \in (0, 1)$, then there exist $r_1 > r_0$ and $M \in \mathbb{R}$ such that*

$$\left| r^{\alpha(\max\{r_0, r\})} - {r_0}^{\alpha(\max\{r_0, r\})} \right| \leq M\, |r - r_0|$$

for every $r \in (r_0/2, r_1]$.

Proof. Let $\beta := \inf_{r > r_0} \alpha(r)$. If $\beta = +\infty$, then $\alpha(r) = +\infty$ whenever $r \in (r_0, 1)$, so $r^{\alpha(r)} - {r_0}^{\alpha(r)} = 0$. If $\beta < +\infty$, we find $r_1 > r_0$ such that $\beta \leq \alpha(r_1) < +\infty$. By the Mean Value Theorem, for each $r \in (r_0, r_1]$, there exists $c \in (r_0, r)$ with $r^{\alpha(r)} - {r_0}^{\alpha(r)} = \alpha(r)\, c^{\alpha(r)-1} (r - r_0)$. Now, if $\beta < 1$, then r_1 can be taken with $a(r_1) < 1$, giving $c^{\alpha(r)-1} \leq {r_0}^{\beta - 1}$ and

$r^{\alpha(r)} - {r_0}^{\alpha(r)} \leq \alpha(r_1)\ {r_0}^{\beta-1}\,(r - r_0)$. On the other hand, if $\beta \geq 1$, then $c^{\alpha(r)-1} \leq 1$ and $r^{\alpha(r)} - {r_0}^{\alpha(r)} \leq \alpha(r_1)\,(r - r_0)$.

We next consider the case $0 < r < r_0$. First, if $\alpha(r_0) = +\infty$, then ${r_0}^{\alpha(r_0)} - r^{\alpha(r_0)} = 0$. On the other hand, if $\alpha(r_0) < +\infty$, then there exists $c \in (r, r_0)$ with ${r_0}^{\alpha(r_0)} - r^{\alpha(r_0)} = \alpha(r_0)\ c^{\alpha(r_0)-1}\,(r_0 - r)$. This implies that, when $\alpha(r_0) \geq 1$,

$$ {r_0}^{\alpha(r_0)} - r^{\alpha(r_0)} \leq \alpha(r_0)\,(r_0 - r) $$

for all $r \in (0, r_0)$, whereas when $\alpha(r_0) < 1$

$$ {r_0}^{\alpha(r_0)} - r^{\alpha(r_0)} \leq \alpha(r_0)\ \left(\frac{r_0}{2}\right)^{\alpha(r_0)-1} (r_0 - r) $$

for $r \in (r_0/2, r_0)$.

The conclusion follows easily. □

Proof of Theorem 2.5. We write $\zeta_r := \zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{k},r}$, for short. We deduce from Proposition 7.1 that the map $\Phi : (0,1) \to \mathfrak{M}$, $r \mapsto \zeta_r$, is injective. Let us next see that it is continuous. Fix $f \in H^\infty$ with $0 < \|f\| \leq 1$ and, for $0 < r < 1$ and $n \in \mathbb{N}$, put $\mathtt{Z}_n(r) := \mathtt{Z}\left(f, D^+\left(z_n,\ \sqrt[k_n]{r}\right)\right)$ and

$$ \alpha(r) := \lim_{\mathfrak{u}} \frac{\mathtt{Z}_n(r)}{k_n}. $$

It is easy to see that the function $\alpha : (0,1) \to [0, +\infty]$ is increasing.

Now, consider $0 < s < r < 1$. Since there exists $C \in \mathfrak{u}$ such that $\lim_{\substack{n\in C\\ n\to\infty}} |z_n|^{k_n} = 1$ and we are dealing with an ultrafilter, there is no loss of generality if we assume that $\sqrt[k_n]{r} < |z_n|$ for every $n \in C$. By Lemma 6.1,

$$ \left|\zeta_{D^+\left(z_n,\ \sqrt[k_n]{r}\right)}(f) - \zeta_{D^+\left(z_n,\ \sqrt[k_n]{s}\right)}(f)\right| \leq \left(\sqrt[k_n]{r}\right)^{\mathtt{Z}_n(r)} - \left(\sqrt[k_n]{s}\right)^{\mathtt{Z}_n(r)} $$

for all $n \in C$, so

$$ \begin{aligned} |\zeta_r(f) - \zeta_s(f)| &\leq \lim_{\mathfrak{u}} \left(\sqrt[k_n]{r}\right)^{\mathtt{Z}_n(r)} - \lim_{\mathfrak{u}} \left(\sqrt[k_n]{s}\right)^{\mathtt{Z}_n(r)} \\ &= r^{\alpha(r)} - s^{\alpha(r)}. \end{aligned} $$

The fact that Φ is continuous is now easy by Lemma 7.2.

Let us next study whether there exist $\lim_{r\to 0} \zeta_r$ and $\lim_{r\to 1} \zeta_r$. Note that, given $f \in H^\infty$, the map $\Psi_f : (0,1) \to \mathbb{R}$, $r \mapsto \zeta_r(f)$ is increasing and bounded, so there exist $\zeta_0(f) := \lim_{r\to 0} \Psi_f(r)$ and $\zeta_1(f) := \lim_{r\to 1} \Psi_f(r)$. It is clear that the maps ζ_0 and ζ_1 defined in this way belong to $\mathfrak{M}$. Also, since, $\zeta_r \neq \zeta_s$ for every $r \neq s$, we conclude that the the natural extension of Φ to a new map (call it also Φ) $\Phi : [0,1] \to \mathfrak{M}$ is indeed injective and continuous, so it is a homeomorphism onto its image. The fact that $\Phi[0,1] = \mathrm{cl}_{\mathfrak{M}}\left(\zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{k},0}, \zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{k},1}\right)$ is now easy.

We finally prove that $\zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{k},0}, \zeta_{\mathbf{z},\mathfrak{u}}^{\mathbf{k},0} \in \mathfrak{M}_0$. We are going to see that there exist a sequence $\mathbf{w}$ in $\mathbb{D}$ with $\lim_{m\to\infty} |w_m| = 1$, a nonprincipal ultrafilter $\mathfrak{v}$ in $\mathbb{N}$,

and sequences (r_m) and (t_m) in $(0,1)$ such that $\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} = \lim_{\mathfrak{v}} \zeta_{D^+(w_m,r_m)}$ and $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} = \lim_{\mathfrak{v}} \zeta_{D^+(w_m,t_m)}$.

We fix $s \in (0,1)$. For each $n \in \mathbb{N}$ and $j = 1, \dots, n$, let $r_n^j := \sqrt[k_n]{s/j}$. We write $A_n := \left\{ r_n^j : 1 \leq j \leq n \right\}$, and consider $A := \bigcup_{n=1}^{\infty} A_n$. Then rename the $r_n^j \in A$ by $r_1 := r_1^1, r_2 := r_2^1, r_3 := r_2^2, r_4 := r_3^1, \dots$ We also put $w_m := z_n$ when $r_m = r_n^j$.

For each $N \in \mathbb{N}$, each $D \in \mathfrak{u}$, and each sequence $\mathbf{l}$ in $\mathbb{N}$ such that $\lim_{\mathfrak{u}} l_n = +\infty$ and $l_n \leq n$ for all n, consider the set $D^N_{\mathbf{l}}$ of all $m \in \mathbb{N}$ satisfying $r_m = r_n^j$, for $N \leq j \leq l_n$ and $n \in D$. It is easy to check that the family $\mathcal{F}$ of all sets $D^N_{\mathbf{l}}$ is the basis for a filter $\mathfrak{F}$ in $\mathbb{N}$. Fix an ultrafilter $\mathfrak{v}$ containing $\mathfrak{F}$. Since, for $N \in \mathbb{N}$ fixed, the set C of all m such that $r_m = r_n^j$ with $j \geq N$ belongs to $\mathfrak{F}$, $\zeta^{\mathbf{k},s/N}_{\mathbf{z},\mathfrak{u}} \geq \psi := \lim_{\mathfrak{v}} \zeta_{D^+(w_m,r_m)}$, and consequently $\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} \geq \psi$.

On the other hand, for $f \in H^\infty$ and $\epsilon > 0$, there exists $C \in \mathfrak{v}$ such that $\zeta_{D^+(w_m,r_m)}(f) < \psi(f) + \epsilon$ for all $m \in C$. Consider the set $M_n := \left\{ j : r_n^j = r_m, m \in C \right\}$ for each $n \in \mathbb{N}$, and note that the family D of all n with $M_n \neq \emptyset$ belongs to $\mathfrak{u}$. Also, for $n \in D$, define $m_n := \min M_n$. By the construction of $\mathcal{F}$, $N := \lim_{\mathfrak{u}} m_n$ belongs to $\mathbb{N}$, and consequently the set of all $n \in D$ with $m_n = N$ belongs to $\mathfrak{u}$. Then $\zeta_{D^+(z_n,r_n^N)}(f) < \psi(f) + \epsilon$ for all $n \in D$ and $\zeta^{\mathbf{k},s/N}_{\mathbf{z},\mathfrak{u}}(f) \leq \psi(f) + \epsilon$. It is a routine matter to check that $\zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} \leq \psi$.

As for $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}$, we define $t_n^j := \sqrt[k_n]{1 - s/j}$ for each $n \in \mathbb{N}$ and $j = 1, \dots, n$, and set $t_m := t_n^j$ in a similar way as above. Consider also the same sequence (w_m) and the same ultrafilter $\mathfrak{v}$ as above. The fact that $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} = \lim_{\mathfrak{v}} \zeta_{D^+(w_m,t_m)}$ follows easily. □

Proof of Corollary 2.6. The fact that $\ker \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} = \ker \zeta^{\mathbf{k},s}_{\mathbf{z},\mathfrak{u}}$, for $r, s \in (0,1)$, follows easily from Lemma 3.5 and Corollary 3.4. Also, if $r \in (0,1)$, then $\zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \leq \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}}$. Since $\zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} = \lim_{r \to 1} \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$, $\ker \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} = \ker \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$ for all $r \in (0,1)$. □

Proof of Corollary 2.7. Obviously, for every $r \in (0,1)$, $\zeta^{\mathbf{l},r}_{\mathbf{z},\mathfrak{u}} \leq \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$, so $\ker \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} \subset \ker \zeta^{\mathbf{l},1}_{\mathbf{z},\mathfrak{u}}$. Now, $\ker \zeta^{\mathbf{k},1}_{\mathbf{z},\mathfrak{u}} \subsetneq \ker \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$ by Proposition 7.1, and we are done. □

Proof of Corollary 2.8. Fix $\varphi = \ker \zeta^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}} \in \mathfrak{M}_1$. By Proposition 7.1, $\ker \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$ strictly contains $\ker \varphi$, so $\ker \varphi$ is not maximal. On the other hand, by Remark 1.2 (assumming without loss of generality that $C = \mathbb{N}$), we fix $r_0 \in (0,1)$ such that all the disks $D^+\left(z_i, \sqrt[k_i]{r_0}\right)$ are pairwise disjoint. Next, taking into account that $\prod_{n=1}^{\infty} |z_n|^{k_n} > 0$, it is easy to see that there exists a sequence (l_n) in $\mathbb{N}$ with $\lim_{n\to\infty} l_n = +\infty$ such that $\prod_{n=1}^{\infty} |z_n|^{l_n k_n} > 0$. Now, we can use Proposition 5.5 to construct $f \in H^\infty$ having $l_n k_n$ zeros in each $D^+\left(z_n, \sqrt[k_n]{r_0}\right)$. Obviously, $\zeta^{\mathbf{k},r_0}_{\mathbf{z},\mathfrak{u}}(f) = 0$. By Corollary 2.6, $\ker \varphi = \ker \zeta^{\mathbf{k},r_0}_{\mathbf{z},\mathfrak{u}}$, so φ is not a norm. □

Proof of Corollary 2.9. Clearly, if $\lim_{\mathfrak{u}} k_n = +\infty$, then $\zeta^{\mathbf{1},1/2}_{\mathbf{z},\mathfrak{u}} \leq \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$, and $\ker \zeta^{\mathbf{k},1/2}_{\mathbf{z},\mathfrak{u}} \subsetneq \ker \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}} \subset \ker \zeta^{\mathbf{1},1/2}_{\mathbf{z},\mathfrak{u}}$. It follows from Corollary 2.8 that $\ker \zeta^{\mathbf{k},0}_{\mathbf{z},\mathfrak{u}}$ is nonzero and nonmaximal. The converse is easy. □

Proof of Theorem 2.3. Let $[\delta_{\mathbf{z},\mathfrak{u}}, \| \|]_{\mathfrak{M}_0}$ be the family of all seminorms in $\mathfrak{M}_0$ of the form $\lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}$. It is immediate to see that $[\delta_{\mathbf{z},\mathfrak{u}}, \| \|]_{\mathfrak{M}_0}$ is linearly ordered with respect to the usual order $\leq$. We next prove that $A^{\mathfrak{u}}_{\mathbf{z}} := \mathrm{cl}_{\mathfrak{M}} [\delta_{\mathbf{z},\mathfrak{u}}, \| \|]_{\mathfrak{M}_0}$ is also linearly ordered.

Given different $\varphi_1, \varphi_2 \in A^{\mathfrak{u}}_{\mathbf{z}}$, there exists $f \in H^\infty$ which separates them. We can assume without loss of generality that $\varphi_1(f) < \varphi_2(f)$. Next, as in the proof of Theorem 2.1, for $r \in (\varphi_1(f), \varphi_2(f))$ we can find $r_n \in (0,1)$ such that $\zeta_{D^+(z_n,r_n)}(f) = r$ for all n in a certain $C \in \mathfrak{u}$. Obviously $\psi := \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)} \in [\delta_{\mathbf{z},\mathfrak{u}}, \| \|]_{\mathfrak{M}_0}$ satisfies

$$\varphi_1(f) < \psi(f) < \varphi_2(f).$$

Now, let $\left(\varphi^{\mathbf{k}^\lambda, r_\lambda}_{\mathbf{z},\mathfrak{u}}\right)_{\lambda\in\Lambda}$ be a net in $[\delta_{\mathbf{z},\mathfrak{u}}, \| \|]_{\mathfrak{M}_0}$ converging to φ_1. Then there exists $\lambda_0 \in \Lambda$ such that $\varphi^{\mathbf{k}^\lambda, r_\lambda}_{\mathbf{z},\mathfrak{u}}(f) < \psi(f)$ for all $\lambda \geq \lambda_0$, $\lambda \in \Lambda$. In particular, for each $\lambda \geq \lambda_0$,

$$\lim_{\mathfrak{u}} \zeta_{D^+\left(z_n,\, \sqrt[k^\lambda]{r_\lambda}\right)}(f) < \lim_{\mathfrak{u}} \zeta_{D^+(z_n,r_n)}(f),$$

and consequently there exists $E_\lambda \in \mathfrak{u}$ such that

$$r_\lambda{}^{1/k^\lambda{}_n} < r_n$$

for all $n \in E_\lambda$. This obviously implies that, for $g \in H^\infty$, $\varphi^{\mathbf{k}^\lambda, r_\lambda}_{\mathbf{z},\mathfrak{u}}(g) \leq \psi(g)$ whenever $\lambda \geq \lambda_0$. We conclude that $\varphi_1 \leq \psi$. Similarly $\psi \leq \varphi_2$. The fact that the compact set $A^{\mathfrak{u}}_{\mathbf{z}}$ is linearly ordered follows.

We next see that $A^{\mathfrak{u}}_{\mathbf{z}}$ is connected. Suppose to the contrary that $A^{\mathfrak{u}}_{\mathbf{z}}$ is the union of two disjoint (nonempty) clopen subsets U, V (with respect to the induced topology). Suppose also that $\varphi_1 \in U$ and $\varphi_2 \in V$ satisfy $\varphi_1 \leq \varphi_2$. We define

$$\psi_1 := \sup \{\varphi \in U : \varphi \leq \varphi_2\}.$$

Obviously $\psi_1 \in U$ and $\psi_1 \leq \varphi_2$. Similarly,

$$\psi_2 := \inf \{\varphi \in V : \psi_1 \leq \varphi\}$$

belongs to V, and $\psi_1 \leq \psi_2$. As we showed above there exists $\psi \in A^{\mathfrak{u}}_{\mathbf{z}}$, different from ψ_1 and ψ_2 such that $\psi_1 \leq \psi \leq \psi_2$. It is clear that $\psi \notin U \cup V$, which is impossible.

Now suppose that $\varphi \in A^{\mathfrak{u}}_{\mathbf{z}}$, $\varphi \neq \delta_{\mathbf{z},\mathfrak{u}}, \| \|$. Then there exists $r \in (0,1)$ such that $\zeta^{\mathbf{1},r}_{\mathbf{z},\mathfrak{u}} \leq \varphi$ and, by Corollary 2.8, $\ker \varphi$ is not maximal. On the other hand, since $\varphi \neq \| \|$, $\ker \varphi \neq \{0\}$, as follows from Proposition 4.1. □

Proof of Theorem 2.10. Suppose that $\varphi \notin \mathfrak{M}_{\mathbf{z},\mathfrak{u}}$, that is, for all $C \in \mathfrak{u}$, $\inf_{n\in C} \prod_{\substack{m\in C\\ m\neq n}} |z_n - z_m|^{k_m} = 0$. We deduce that, if $\mathbf{m} \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$, then $\lim_{\mathfrak{u}} m_n/k_n = 0$, and $\zeta^{\mathbf{m},1}_{\mathbf{z},\mathfrak{u}} \leq \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$. Thus, $\sup_{\mathbf{m}\in\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})} \zeta^{\mathbf{m},1}_{\mathbf{z},\mathfrak{u}} \leq \varphi$.

To finish the proof, it is enough to see that for each $f \in H^\infty$, there exists $\mathbf{m}(f) \in \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$ such that $\varphi(f) = \zeta^{\mathbf{m}(f),1}_{\mathbf{z},\mathfrak{u}}(f)$. Consider $f \in H^\infty$. If $\varphi(f) = 0$, then $\zeta^{\mathbf{1},r/2}_{\mathbf{z},\mathfrak{u}}(f) = 0$ and, by Corollary 2.6, $\zeta^{\mathbf{1},1}_{\mathbf{z},\mathfrak{u}}(f) = 0$, so we can take $\mathbf{m}(f) = \mathbf{1}$. Next, suppose that $f \notin \ker\varphi$.

For each $n \in \mathbb{N}$, put $r_n = \sqrt[k_n]{r}$ and $m_n := \mathtt{Z}(f, D^-(z_n, r_n))$. If there exists $C \in \mathfrak{u}$ such that $m_n = 0$ for all $n \in C$, then by Corollary 3.3 $|f(z_n)| = \zeta_{D^+(z_n,r_n)}(f)$ for every $n \in C$. It follows easily that $\delta_{\mathbf{z},\mathfrak{u}}(f) = \zeta^{\mathbf{1},M}_{\mathbf{z},\mathfrak{u}}(f) = \varphi(f)$ for all $M \in (0,1)$, so $\varphi(f) = \zeta^{\mathbf{1},1}_{\mathbf{z},\mathfrak{u}}(f)$. On the other hand, if the above set C does not belong to $\mathfrak{u}$, then for $n \in \mathbb{N}$

$$\prod_{\substack{|z_j|=|z_n|\\ j\in\mathbb{N}\setminus C\\ j\neq n}} |z_n - z_j|^{m_j} \geq \xi_{D^+(z_n,r_n)}(f).$$

Also, by Corollary 3.4, there is $D \in \mathfrak{u}$ with $D \subset \mathbb{N}\setminus C$ such that $\xi_{D^+(z_n,r_n)}(f) \geq \varphi(f)/2\,\|f\|$ for all $n \in D$. Therefore $\mathbf{m}(f) := (\max\{m_n, 1\})$ belongs to $\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$. On the other hand, it is a routine matter to check that, for $M \in (0,1)$ fixed, the set of all n with $\sqrt[m_n]{M} < r_n$ belongs to $\mathfrak{u}$ and, by Lemma 3.5,

$$M\xi_{D^+(z_n,r_n)}(f) \leq \xi_{D^+\left(z_n,\,\sqrt[m_n]{M}\right)}(f) \leq \xi_{D^+(z_n,r_n)}(f).$$

Again by Corollary 3.4, this implies that $M\varphi(f) \leq \zeta^{\mathbf{m}(f),M}_{\mathbf{z},\mathfrak{u}}(f) \leq \varphi(f)$ for all $M \in (0,1)$, and consequently $\varphi(f) = \zeta^{\mathbf{m}(f),1}_{\mathbf{z},\mathfrak{u}}(f)$. □

Remark 7.2. The following should be compared with Proposition 6.5. Let $\mathbf{z}$ be a regular sequence with respect to $\mathfrak{u} \in \beta\mathbb{N}\setminus\mathbb{N}$, and let $\mathbf{k}$ be a sequence in $\mathbb{N}$. Let $r \in (0,1)$ be such that $z_m \notin D^+(z_n, \sqrt[k_n]{r})$ whenever $m \neq n$. We see in the proof of Theorem 2.10 that, if $\mathbf{k} \notin \mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$, then $\varphi^{\mathbf{k},s}_{\mathbf{z},\mathfrak{u}} = \varphi^{\mathbf{k},r}_{\mathbf{z},\mathfrak{u}}$ for all $s \in (0,r]$.

Proof of Corollary 2.11. Set $r_n := \sqrt[k_n]{r}$ for all n. Suppose that there exists $f \in \ker\varphi$, $f \neq 0$, and put $\mathtt{Z}_n := \mathtt{Z}(f, C(0,|z_n|))$ for all $n \in \mathbb{N}$. By Corollary 3.4, $\lim_{\mathfrak{u}} \xi_{D^+(z_n,r_n)}(f) = 0$, and consequently $\lim_{\mathfrak{u}} r_n^{\mathtt{Z}_n} = 0$. This implies that $\lim_{\mathfrak{u}} \mathtt{Z}_n/k_n = +\infty$, so there exists $C \in \mathfrak{u}$ with $\mathtt{Z}_n \geq k_n$ for all $n \in C$. Since $\|f\| \geq 1/\prod_{n\in C} |z_n|^{\mathtt{Z}_n}$, we conclude that $\prod_{n\in C} |z_n|^{k_n} > 0$. Now the fact that $\mathbf{k}$ belongs to $\mathtt{Comp}_{\mathfrak{u}}(\mathbf{z})$ is easy.

On the other hand, if $\ker\varphi = \{0\}$, then the fact that $\varphi = \|\ \|$ follows from Proposition 4.1. □

Corollary 7.3. *Given* $\varphi := \varphi_{\mathbf{z},\mathrm{u}}^{\mathbf{k},r} \in \mathfrak{M}_0'$. *If* $\varphi \notin \mathfrak{M}_1$, *then*

$$\ker \varphi = \bigcap_{\mathbf{m} \in \mathtt{Comp}_{\mathrm{u}}(\mathbf{z})} \ker \zeta_{\mathbf{z},\mathrm{u}}^{\mathbf{m},1}.$$

We end the paper by listing some questions for which we do not have an answer.

1. Does there exist $\varphi \in \mathfrak{M}_0$ with nonmaximal kernel such that $\ker \psi \neq \ker \varphi$ for all $\psi \in \mathfrak{M}_0 \setminus \{\varphi\}$?
2. More generally, does there exist $\varphi \in \mathfrak{M}$ with *unique* nonmaximal kernel, that is, such that $\ker \psi \neq \ker \varphi$ whenever $\psi \in \mathfrak{M}$ and $\psi \neq \varphi$?
3. Does there exist $\varphi \in \mathfrak{M}$ with *nonmaximal* kernel such that $f \in \ker \varphi$ and $f' \notin \ker \varphi$ for some $f \in H^\infty$?
4. Does there exist $\varphi \in \mathfrak{M}$ with *maximal* kernel such that $f' \in \ker \varphi$ whenever $f \in \ker \varphi$? (stated in [13])

8. Acknowledgements

The author wishes to thank the referee for his/her interesting remarks.

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